NSW INDEPENDENT SCHOOLS

2010
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Student name / number ....

Marks

Question 1 **Begin a new booklet**

(a) Find $\int \left(e^{x} + e^{-\frac{x}{2}}\right)^{2} dx$. **2**

(b) Use the substitution $u = 1 + \sin^2 x$ to find $\int \frac{\sin 2x}{\sqrt{1+\sin^2 x}} dx$. **2**

(c) Evaluate in simplest exact form $\int_0^{\frac{\pi}{4}} \frac{\sec x + \tan x}{\cos x} dx$. **3**

(d) Evaluate in simplest exact form $\int_0^{4} \frac{x-9}{(x+1)(x^2+9)} dx$. **4**

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate in simplest exact form **4**

$\int_0^{\frac{\pi}{2}} \frac{1}{3 - \cos x - 2\sin x} dx$.

Student name / number ....

Marks

Question 2 **Begin a new booklet**

(a) If $z = 3 - i$ and $w = 1 + 2i$, find in the form $a + ib$, where a and b are real, the values of

(i) $z - 2w$. **1**

(ii) $z\overline{w}$. **1**

(iii) $\frac{z}{w}$. **1**

(b)(i) Show that $\tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$. **1**

(ii) Express $z = \left(\sqrt{3}+1\right) + \left(\sqrt{3}-1\right)i$ in modulus argument form. **2**

(iii) Express z^6 in the form $a + ib$, where a and b are real. **1**

(c)(i) On an Argand diagram, shade the region where both $|z - 1 - i| \le \sqrt{2}$ and $0 \le \arg z \le \frac{\pi}{4}$. **2**

(ii) Find in simplest exact form the area of the shaded region. **2**

(d)(i) If $z_1 = r(\cos\alpha + i\sin\alpha)$ and $z_2 = r(\cos\beta + i\sin\beta)$ show that $\left|\sqrt{z_1 z_2}\right| = r$ and $\arg\left(\sqrt{z_1 z_2}\right) = \frac{1}{2}(\alpha+\beta) + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$ **2**

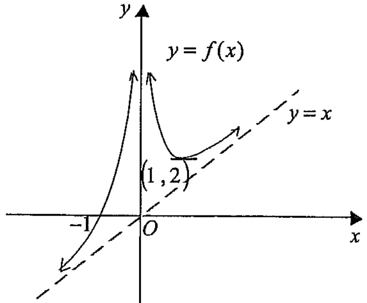
(ii) If $0 < \alpha < \beta < \frac{\pi}{2}$, show on an Argand diagram the points A, B, C, D and E such that vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ represent z_1, z_2, $z_1 + z_2$ respectively, and $\overrightarrow{OD}$, $\overrightarrow{OE}$ represent the two square roots of $z_1 z_2$. **2**

Marks

Question 3 **Begin a new booklet**

(a) The polynomial $P(x)=x^3-6x^2+9x+c$ has a double zero. Find any possible values of the real number c. 3

(b) The graph below shows the curve $y=f(x)$ with asymptotes $x=0$ and $y=x$.



On separate diagrams, sketch the following graphs showing clearly any intercepts and asymptotes:

(i) $y=|f(x)|$. 1

(ii) $y=f(|x|)$. 1

(iii) $y=f(x)-x$. 2

(iv) $y=\dfrac{1}{f(x)}$. 2

(c) $P(x)=x^4-2x^3+3x^2-4x+1$ and the equation $P(x)=0$ has roots α, β, γ and δ.

(i) Show that the equation $P(x)=0$ has no integer roots. 1

(ii) Show that $P(x)=0$ has a real root between 0 and 1. 1

(iii) Show that $\alpha^2+\beta^2+\gamma^2+\delta^2=-2$. 2

(iv) Hence find the number of real roots of the equation $P(x)=0$, giving reasons. 2

Student name / number ....

Marks

Question 4 **Begin a new booklet**

(a) For the hyperbola $\dfrac{x^2}{9}-\dfrac{y^2}{72}=1$ find

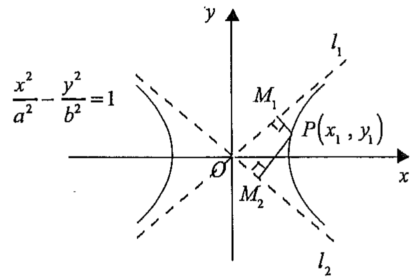
(i) the eccentricity. 1
(ii) the coordinates of the foci. 1
(iii) the equations of the directrices. 1

(b) For the curve $y^3+2xy+x^2+2=0$

(i) show that $\dfrac{dy}{dx}=\dfrac{-2(y+x)}{3y^2+2x}$. 2

(ii) find the coordinates of any stationary points on the curve. 3

(c)



$P(x_1, y_1)$ is a point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, $a>b>0$, with asymptotes l_1 and l_2. M_1 and M_2 are the feet of the perpendiculars from P to l_1 and l_2 respectively.

(i) Show that $PM_1\times PM_2=\dfrac{a^2b^2}{a^2+b^2}$. 3

(ii) Show that $\tan\angle M_1OM_2 = \dfrac{2ab}{a^2-b^2}$. 1

(iii) Hence find the area of ΔPM_1M_2 in terms of a and b. 3

Student name / number ....

Marks

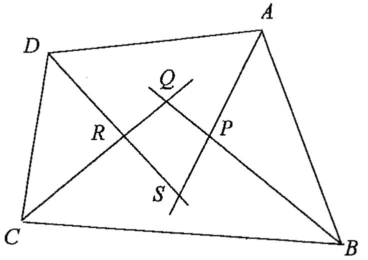
Question 5 **Begin a new booklet**

(a) The numerals 1, 2, 4, 5, 7, 8 are marked one on each side of three counters so that the sum of the numerals on any particular counter is 9. The counters are drawn at random one-by-one from a box, each counter being tossed after it is drawn, then with the uppermost faces unchanged, placed side-by-side on a table (in the order in which they were drawn) to form a three-digit number.

(i) Show that the probability the three-digit number formed is a multiple of 3 is $\frac{1}{4}$. **2**

(ii) If the random trial described above is performed several times, find the probability the second three-digit multiple of 3 occurs on the 5^{th} trial. **2**

(b)

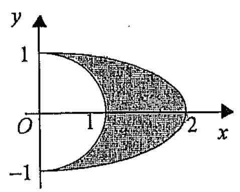


In the quadrilateral $ABCD$ shown above, APS, BPQ, CRQ and DRS are the bisectors of the vertex angles at A, B, C and D respectively.

(i) Show that $PQRS$ is a cyclic quadrilateral. **2**

(ii) If $ABCD$ is a trapezium, deduce that one of the diagonals of $PQRS$ is a diameter of the circle through P, Q, R and S. **2**

(c)



The base of a solid is the shaded region between the circle $x^2 + y^2 = 1$ and the ellipse $\frac{x^2}{4} + y^2 = 1$ for $x \geq 0$. Vertical cross-sections taken parallel to the x-axis are rectangles with heights equal to the squares of their base lengths.

(i) Show that the volume V of the solid is given by $V = \int_{-1}^{1} \left(1 - y^2\right)^{\frac{3}{2}}\, dy$. **2**

(ii) Show that $\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$. **2**

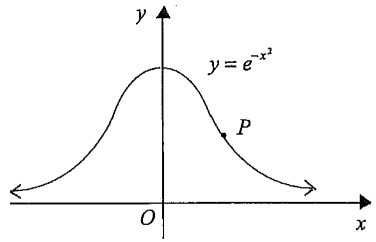
(iii) Use the substitution $y = \sin u$ and the result from (ii) to find the value of V. **3**

Student name / number ....

Marks

Question 6 **Begin a new booklet**

(a) **4**



P is a variable point on the curve $y = e^{-x^2}$. Find the minimum value of OP^2.

(b) Use Mathematical Induction to show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all positive integers $n \geq 2$. **4**

(c) A particle of mass m kg falls from rest in a medium where the resistance to motion is proportional to the square of its speed and its terminal velocity is 20 ms^{-1}. The value of g, the acceleration due to gravity, is 10 ms^{-2}. At time t seconds, the particle has fallen x metres and acquired a velocity v ms^{-1}.

(i) Explain why $\ddot{x} = \frac{1}{40}(400 - v^2)$. **2**

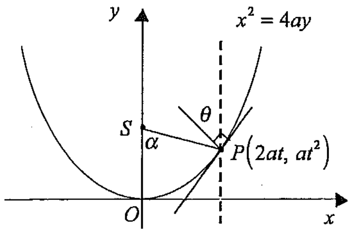
(ii) Find t as a function of v by integration, then show $\frac{1}{40}v = \frac{\frac{1}{2}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right)}{\left(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}\right)}$. **3**

(iii) Find x as a function of t. **2**

Student name / number ....

Question 7 **Begin a new booklet** **Marks**

(a)



P is the point $(2at, at^2)$, $0 < t < 1$, on the parabola $x^2 = 4ay$ with focus S.
The normal to the parabola at P makes an angle θ with the vertical through P, while the focal chord PS makes an angle α with the vertical.

(i) Explain why $\tan\theta = t$ and show that $\alpha = 2\theta$. **2**

(ii) If l is the focal distance PS, show that $l\cos^2\theta = a$. **1**

(b) A particle P of mass m is travelling in a horizontal circle with constant angular velocity ω around the inside of a parabolic bowl of focal length a, formed by rotating the arc of the above parabola which lies below the focus around the y-axis. The particle P is suspended from the focus S of the parabola by a string of length l. The tension in the string is T and the surface of the bowl exerts a force N on the particle.

(i) If the normal to the surface at P makes an angle θ with the vertical, using the result from 7(a)(i), explain why $T\cos 2\theta + N\cos\theta = mg$ **3**

and $2T\cos\theta + N = 2ml\cos\theta\,\omega^2$

(ii) Using 7(a)(ii), show that $T = 2ma\omega^2 - mg$, and find N in terms of m, l, a, g and ω^2. **2**

(iii) Deduce that $\dfrac{g}{2a} \le \omega^2 \le \dfrac{g}{2a - l}$. **2**

(c) For positive real numbers $a, b, c, a_1, a_2, \ldots, a_n$:

(i) Show that $a + \dfrac{1}{a} \ge 2$. **1**

(ii) Hence show that $(a+b)\left(\dfrac{1}{a} + \dfrac{1}{b}\right) \ge 4$ and $(a+b+c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \ge 9$. **2**

(iii) Show that $(a_1 + a_2 + \ldots + a_n)\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_n}\right) \ge n^2$. **2**

Student name / number ....

Question 8 **Begin a new booklet** **Marks**

(a) $I_n = \displaystyle\int_0^1 x^n \ln(1+x)\,dx$, $n = 0, 1, 2, \ldots$

(i) Show that $\displaystyle\int \ln(1+x)\,dx = (1+x)\ln(1+x) - x + c$. **1**

(ii) Show that $(n+1)I_n = 2\ln 2 - \dfrac{1}{n+1} - nI_{n-1}$, $n = 1, 2, \ldots$ **2**

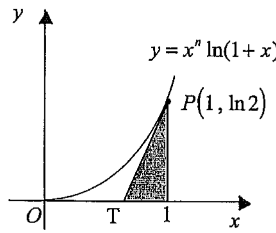
(iii) Evaluate $3I_2$ and $4I_3$. **2**

(iv) Show that $(n+1)I_n = \begin{cases} \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{n+1}, & n \text{ odd} \\ 2\ln 2 - \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n+1}\right), & n \text{ even} \end{cases}$ **2**

(b) Consider the function $y = x^n \ln(1+x)$, $x \ge 0$ for n a positive integer.

(i) Show that for $x > 0$, the function is increasing and its graph is concave up. **2**

(ii) **2**



The tangent at $P(1, \ln 2)$ on the curve $y = x^n \ln(1+x)$ meets the x-axis at T.

Considering the area of the shaded region, show that if $I_n = \displaystyle\int_0^1 x^n \ln(1+x)\,dx$,

then $(n+1)I_n > \dfrac{(1+\frac{1}{n})(\ln 2)^2}{\frac{1}{n} + 2\ln 2}$.

(iii) Hence, using 8(a), show that for even $n > 0$, $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n+1} < \frac{3}{2}\ln 2$. **2**

(iv) Deduce that for all positive integers n, $\frac{1}{2} \le \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1}\frac{1}{n} < \frac{3}{2}\ln 2$. **2**

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• rearranges integrand into appropriate sum of terms	1
• finds primitive	1

Answer

$$\int\left(e^x+e^{-\frac{x}{2}}\right)^2 dx=\int\left(e^{2x}+2e^{\frac{x}{2}}+e^{-x}\right)dx$$
$$=\tfrac{1}{2}e^{2x}+4e^{\frac{x}{2}}-e^{-x}+c$$

b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• performs substitution to write integral in terms of u	1
• writes primitive in terms of u then in terms of x	1

Answer

$u=1+\sin^2 x$

$du=2\sin x\cos x\, dx$

$\quad =\sin 2x\, dx$

$$\int\frac{\sin 2x}{\sqrt{1+\sin^2 x}}dx=\int\frac{1}{\sqrt{u}}du$$
$$=2\sqrt{u}+c$$
$$=2\sqrt{1+\sin^2 x}+c$$

c. Outcomes assessed : H8

Marking Guidelines

Criteria	Marks
• rearranges integrand into appropriate form	1
• writes primitive function	1
• evaluates in simplest surd form	1

Answer

$$\int_0^{\frac{\pi}{4}}\frac{\sec x+\tan x}{\cos x}dx=\int_0^{\frac{\pi}{4}}\left(\sec^2 x+\sec x\tan x\right)dx$$
$$=\left[\tan x+\sec x\right]_0^{\frac{\pi}{4}}$$
$$=\tan\tfrac{\pi}{4}-\tan 0+\sec\tfrac{\pi}{4}-\sec 0$$
$$=\sqrt{2}$$

1d. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• expresses integrand as sum of partial fractions	1
• finds the primitive function	1
• substitutes limits	1
• uses log laws to simplify	1

1d Answer

Let $\dfrac{x-9}{(x+1)(x^2+9)}\equiv\dfrac{a}{(x+1)}+\dfrac{bx+c}{(x^2+9)}$ $\quad a,\ b,\ c$ constant.

Then $\quad x-9\equiv a(x^2+9)+(bx+c)(x+1)$

put $x=-1$: $\quad -10=10a \quad \therefore a=-1$

put $x=0$: $\quad -9=9a+c \quad \therefore c=0$

equate coeffs of x: $\quad 1=b+c \quad \therefore b=1$

$$\int_0^4\frac{x-9}{(x+1)(x^2+9)}dx=\int_0^4\left(\frac{-1}{x+1}+\frac{x}{x^2+9}\right)dx$$
$$=\left[-\ln(x+1)+\tfrac{1}{2}\ln(x^2+9)\right]_0^4$$
$$=-\ln 5+\tfrac{1}{2}(\ln 25-\ln 9)$$
$$=-\ln 5+\ln 5-\ln 3$$
$$=-\ln 3$$

e. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes dx in terms of dt and converts x limits to t limits	1
• uses t-formulae to write given integrand in terms of t	1
• simplifies and rearranges new integrand to obtain primitive function	1
• evaluates integral in simplest exact form	1

Answer

$t=\tan\frac{x}{2}$

$dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dx=\dfrac{2}{1+t^2}dt$

$x=0\ \Rightarrow\ t=0$

$x=\frac{\pi}{2}\ \Rightarrow\ t=1$

$$3-\cos x-2\sin x$$
$$=\frac{3(1+t^2)-(1-t^2)-4t}{1+t^2}$$
$$=\frac{2(2t^2-2t+1)}{1+t^2}$$
$$=2\left\{(t-\tfrac{1}{2})^2+\tfrac{1}{4}\right\}.\frac{2}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{3-\cos x-2\sin x}dx=\int_0^1\frac{(\frac{1}{2})}{(t-\frac{1}{2})^2+\frac{1}{4}}dt$$
$$=\left[\tan^{-1}2(t-\tfrac{1}{2})\right]_0^1$$
$$=\tan^{-1}1-\tan^{-1}(-1)$$
$$=\tfrac{\pi}{4}-(-\tfrac{\pi}{4})$$
$$=\tfrac{\pi}{2}$$

Question 2

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes value of $z-2w$	1
ii • writes value of $z\overline{w}$	1
iii • writes value of $\frac{z}{w}$	1

Answer

$z=3-i$ and $w=1+2i$

i. $z-2w=(3-i)-2(1+2i)=1-5i$

ii. $z\overline{w}=(3-i)(1-2i)=1-7i$

iii. $\frac{z}{w}=\frac{(3-i)(1-2i)}{(1+2i)(1-2i)}=\frac{1-7i}{1^2+2^2}=\frac{1}{5}-\frac{7}{5}i$

b. Outcomes assessed : H5, E3

Marking Guidelines

Criteria	Marks
i • applies result for tan of a difference	1
ii • finds the modulus of z	1
• deduces z has argument $\frac{\pi}{12}$ and writes z in modulus argument form	1
iii • finds the 6th power in required form	1

Answer

i. $\tan\frac{\pi}{12}=\tan(\frac{\pi}{3}-\frac{\pi}{4})=\frac{\tan\frac{\pi}{3}-\tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}=\frac{\sqrt{3}-1}{1+\sqrt{3}.1}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$

ii. $z=(\sqrt{3}+1)+(\sqrt{3}-1)i \Rightarrow |z|=\sqrt{8}=2\sqrt{2}$ and $\arg z=\alpha$ where $\tan\alpha=\frac{\sqrt{3}-1}{\sqrt{3}+1}$, $0<\alpha<\frac{\pi}{2}$.

$\therefore\ z=2\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

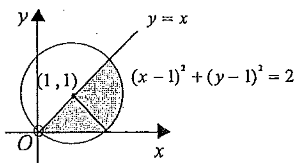
iii. $z^6=2^9\left(\cos\frac{6\pi}{12}+i\sin\frac{6\pi}{12}\right)=512i$

c. Outcomes assessed : P4, E3

Marking Guidelines

Criteria	Marks
i • realises region lies inside circle, centre (1, 1) and radius $\sqrt{2}$	1
• sketches region bounded by circle, x-axis and line $y=x$, excluding origin O	1
ii • realises region comprises a right triangle and a quarter circle, finding the area of one part	1
• adds second part to give exact area	1

Answer i.



ii.

Area is $\frac{1}{2}\sqrt{2}.\sqrt{2}+\frac{1}{4}\pi\left(\sqrt{2}\right)^2=1+\frac{\pi}{2}$

2d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • uses de Moivre's theorem to write down the product z_1z_2 and deduce that $\left|\sqrt{z_1z_2}\right|=r$.	1
• deduces the possible values of $\arg(z_1z_2)$	1
ii • shows A, B, C on an Argand diagram, with $OACB$ forming a rhombus.	1
• shows D, E collinear with O and C, and $OA=OB=OD=OE=r$	1

Answer

i. $z_1=r\left(\cos\alpha+i\sin\alpha\right)$, $z_2=r\left(\cos\beta+i\sin\beta\right)$

Using de Moivre's theorem,

$z_1z_2=r^2\left(\cos(\alpha+\beta)+i\sin(\alpha+\beta)\right)$

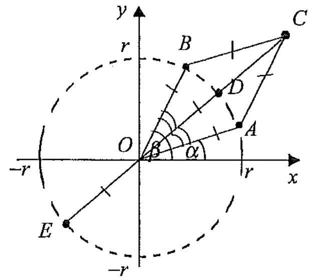
$=r^2\left\{\cos\left(\frac{\alpha+\beta+2n\pi}{2}\right)+i\sin\left(\frac{\alpha+\beta+2n\pi}{2}\right)\right\}^2$

where $n=0,\ \pm1,\ \pm2,\ldots$

Now $|z_1z_2|=r^2$, hence $\left|\sqrt{z_1z_2}\right|=r$. Also

$\arg\left(\sqrt{z_1z_2}\right)=\frac{1}{2}(\alpha+\beta)+n\pi,\quad n=0,\pm1,\pm2,\ldots$

ii. If $0<\alpha<\beta<\frac{\pi}{2}$, the two square roots of z_1z_2 have principal arguments $\frac{1}{2}(\alpha+\beta)$, $\frac{1}{2}(\alpha+\beta)-\pi$.



Question 3

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• solves $P'(x)=0$ to find the possible double zeros of $P(x)$	1
• finds c if 3 is a double zero of $P(x)$	1
• finds c if 1 is a double zero of $P(x)$	1

Answer

$P(x)=x^3-6x^2+9x+c$

$P'(x)=3x^2-12x+9$

$=3\left(x-3\right)\left(x-1\right)$

$\therefore P'(x)=0$ for $x=3$ or $x=1$

$\therefore P'(3)=P(3)=0 \Leftrightarrow 27-54+27+c=0 \Leftrightarrow c=0$

and $P'(1)=P(1)=0 \Leftrightarrow 1-6+9+c=0 \Leftrightarrow c=-4$

$\therefore P(x)$ has a double zero if and only if $c=0$ or $c=-4$.

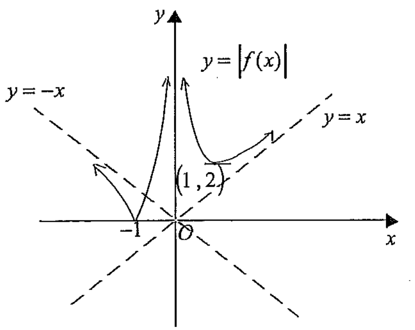
3b. Outcomes assessed : E6

Marking Guidelines

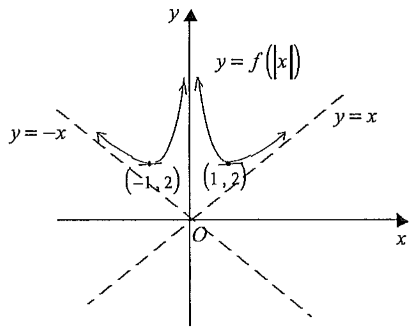
Criteria	Marks
i • reflects section of curve below x-axis in x-axis, showing asymptote as $x \to -\infty$	1
ii • reflects branch of curve to right of y-axis in y-axis, showing asymptote as $x \to -\infty$	1
iii • shows first quadrant branch through $(1, 1)$ with x- and y-axes as asymptotes	1
• shows second quadrant branch through $(-1, 1)$ with x- and y-axes as asymptotes	1
iv • shows first quadrant branch with turning point, x-axis as asymptote and origin excluded	1
• shows second and third quadrant branches with asymptotes and behaviour near origin	1

Answer

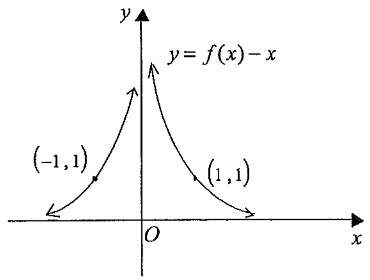
i.



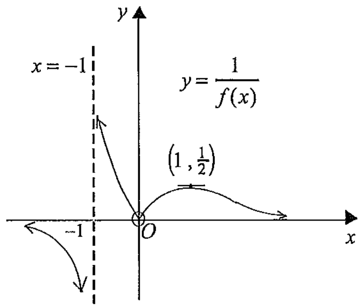
ii.



iii.



iv.



3c. Outcomes assessed : E2, E3, E4

Marking Guidelines

Criteria	Marks
i • tests the only possibilities and deduces there are no integer roots	1
ii • notes that $P(x)$ is continuous and shows $P(0)$ and $P(1)$ have opposite signs	1
iii • expresses sum of squares in terms of sums of products of roots taken one or two at a time.	1
• uses the relationships between roots and coefficients to evaluate the sum of squares	1
iv • explains why there must be at least two non-real roots	1
• deduces that the fourth root cannot be non-real and hence that there are exactly 2 real roots	1

Answer

$P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$. $\alpha,\ \beta,\ \gamma$ and δ are roots of $P(x) = 0$

i. Only possible integer roots are ± 1. But $P(1) = -1 \neq 0$ and $P(-1) = 11 \neq 0$. Hence there are no integer roots.

ii. $P(x)$ is a continuous, real function and $P(0) = 1 > 0$ while $P(1) = -1 < 0$. Hence, considering the graph of $y = P(x)$, there is a real root of $P(x) = 0$ between 0 and 1.

iii. $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) = 2^2 - 2 \times 3 = -2$

iv. Since $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2$, at least one of these squares must be negative. Hence $P(x) = 0$ has a non-real root. Then its complex conjugate is a second non-real root, since the coefficients of $P(x)$ are real. We know there is a real root between 0 and 1. Since the non-real roots come in complex conjugate pairs, the remaining fourth root cannot be non-real.
Hence the equation $P(x) = 0$ has two real roots and two non-real roots.

Question 4

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • finds e	1
ii • writes the coordinates of both foci	1
iii • writes the equations of both directrices	1

Answer

i. $\dfrac{x^2}{9} - \dfrac{y^2}{72} = 1$ $b^2 = a^2(e^2 - 1) \Rightarrow e^2 = 1 + \dfrac{b^2}{a^2} = 1 + \dfrac{72}{9} = 9$ $\therefore e = 3$

ii. Foci have coordinates $(9, 0)$ and $(-9, 0)$

iii. Directrices have equations $x = 1$ and $x = -1$

4b. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
i • derives implicitly with respect to x	1
• rearranges to obtain required expression for the derivative	1
ii • finds an equation for the x coordinate of any stationary point	1
• factors the cubic expression	1
• notes quadratic factor has negative discriminant and writes coordinates of stationary point	1

Answer

i. $y^3+2xy+x^2+2=0$

$$3y^2\frac{dy}{dx}+2\left(1.y+x\frac{dy}{dx}\right)+2x=0$$

$$\frac{dy}{dx}\left(3y^2+2x\right)=-2\left(y+x\right)$$

$$\frac{dy}{dx}=\frac{-2\left(y+x\right)}{\left(3y^2+2x\right)}$$

ii. $\frac{dy}{dx}=0$ for $y=-x$ and $y^3+2xy+x^2+2=0$

$$-x^3-2x^2+x^2+2=0$$

$$x^3+x^2-2=0$$

$$(x-1)(x^2+2x+2)=0$$

Hence $\left(1,-1\right)$ is the only stationary point since quadratic factor has $\Delta<0$.

c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes equations of asymptotes in general form	1
• writes the product of distances from P to the asymptotes	1
• simplifies the product using the equation of the hyperbola	1
ii • writes and simplifies expression for tangent of angle using the gradients of the asymptotes	1
iii • finds area in terms of a, b, and sine of angle between asymptotes	1
• uses expression for tan angle and Pythagorean triad to express sine angle in terms of a, b	1
• writes area of triangle in terms of a, b	1

Answer

l_1, l_2 have equations $bx-ay=0$, $bx+ay=0$ respectively.

i. $$PM_1\times PM_2=\frac{\left|bx_1-ay_1\right|}{\sqrt{b^2+a^2}}\times\frac{\left|bx_1+ay_1\right|}{\sqrt{b^2+a^2}}=\frac{\left|b^2x_1{}^2-a^2y_1{}^2\right|}{b^2+a^2}$$

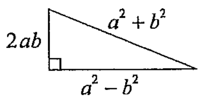
$$\therefore PM_1\times PM_2=\frac{a^2b^2}{a^2+b^2}\cdot\left|\frac{x_1{}^2}{a^2}-\frac{y_1{}^2}{b^2}\right|=\frac{a^2b^2}{a^2+b^2}$$

since P on the hyperbola $\Rightarrow \frac{x_1{}^2}{a^2}-\frac{y_1{}^2}{b^2}=1$.

ii. $$\tan\angle M_1OM_2=\left|\frac{\frac{b}{a}-(-\frac{b}{a})}{1+\frac{b}{a}(-\frac{b}{a})}\right|=\left|\frac{2ab}{a^2-b^2}\right|=\frac{2ab}{a^2-b^2}$$

since $a>b>0$.

iii. $$Area\ \Delta PM_1M_2=\tfrac{1}{2}PM_1\times PM_2\sin\left(180°-\angle M_1OM_2\right)=\frac{a^2b^2}{2(a^2+b^2)}\sin\left(\angle M_1OM_2\right)$$



$\tan\theta=\frac{2ab}{a^2-b^2}\Rightarrow\sin\theta=\frac{2ab}{a^2+b^2}$ $\therefore$ Area of ΔPM_1M_2 is $\frac{a^3b^3}{\left(a^2+b^2\right)^2}$

Question 5

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • counts possible 3-digit numbers	1
• counts multiples of 3 and finds required probability	1
ii • uses binomial distribution (or counts possible orders) to write expression for probability	1
• calculates the probability	1

Answer

i. There are $6\times4\times2=48$ possible three-digit numbers.
A number is a multiple of 3 if and only if the sum of its digits is a multiple of 3.
Hence for the number to be a multiple of 3, the numerals uppermost on the counters must be 1, 4 and 7 or 2, 5 and 8. $\therefore$ there are $3!+3!=12$ multiples of 3.

$\therefore$ $P(\textit{number is a multiple of } 3)=\frac{12}{48}=\frac{1}{4}$

ii. Consider the outcome of a trial to be a *success* if the number formed is a multiple of 3, and a *failure* otherwise. We then have a sequence of identical, independent random trials where the probability of *success* is $\frac{1}{4}$ and the probability of *failure* is $\frac{3}{4}$ for each trial.

$P(2^{nd}\textit{ success on } 5^{th}\textit{ trial})=P(\textit{exactly } 1\textit{ success in first } 4\textit{ trials})\times P(\textit{success on } 5^{th}\textit{ trial})$

$\therefore P(2^{nd}\textit{ success on } 5^{th}\textit{ trial})={}^4C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3\times\frac{1}{4}=\frac{27}{256}$

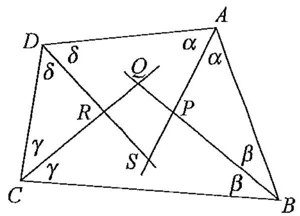
b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • uses the angle sum of a quadrilateral to show $\alpha+\beta+\gamma+\delta=180°$	1
• applies test for cyclic quadrilateral	1
ii • uses supplementary cointerior angles to show $\angle QPS=90°$ if $AD\|BC$	1
• shows $\angle RQP=90°$ if $AB\|DC$	1

Answer

i. Let $\angle A=2\alpha$, $\angle B=2\beta$, $\angle C=2\gamma$, $\angle D=2\delta$



$2\alpha+2\beta+2\gamma+2\delta=360°$ ($\angle$ *sum of quad. is* 360°)

$\therefore\alpha+\beta+\gamma+\delta=180°$

$\angle QPS=\angle APB$ (*vert. opp.* $\angle$*'s are equal*)

$\therefore\angle QPS=180°-(\alpha+\beta)$ ($\angle$ *sum of* Δ *is* 180°) *

Similarly $\angle QRS=180°-(\gamma+\delta)$

$\therefore\angle QPS+\angle QRS=360°-(\alpha+\beta+\gamma+\delta)$
$=180°$

Hence $PQRS$ is a cyclic quadrilateral.

ii. If $AD\parallel BC$, then $2\alpha+2\beta=180°$ (*cointerior* $\angle$*'s between parallel lines are supplementary*)
$\alpha+\beta=90°$

Then $\angle QPS=90°$ (from *) and diagonal QS must be a diameter of the circle $PQRS$.
If $AB\parallel DC$, then $2\beta+2\gamma=180°\Rightarrow\beta+\gamma=90°$ and hence $\angle RQP=90°$ ($\angle$ *sum of* Δ *is* 180°) and RP is a diameter of the circle.

5c. Outcomes assessed : H5, HE6, E1, E2, E7, E9

Marking Guidelines

Criteria	Marks
i • finds dimensions of vertical, rectangular cross-section and hence volume of slice	1
• takes limiting sum of slice volumes to express V as a definite integral	1
ii • shows $z^n + z^{-n} = 2\cos n\theta$ if $z = \cos\theta + i\sin\theta$ (or $\cos^4\theta = \frac{1}{4}(1+\cos 2\theta)^2$ *)	1
• uses binomial expansion of $\left(z^n + z^{-n}\right)^4$ (or expands and simplifies *) to obtain result	1
iii • performs substitution to obtain definite integral in terms of u	1
• finds primitive	1
• substitutes limits to evaluate	1

Answer

i. Rectangular cross-section has base length $2\sqrt{1-y^2} - \sqrt{1-y^2} = \sqrt{1-y^2}$ and height $1-y^2$.

Hence typical vertical slice parallel to x-axis has volume $\delta V = (1-y^2)^{\frac{3}{2}}\,\delta y$.

$$\therefore V = \lim_{\delta y \to 0} \sum_{y=-1}^{y=1} (1-y^2)^{\frac{3}{2}}\,\delta y = \int_{-1}^{1} \left(1-y^2\right)^{\frac{3}{2}}\,dy$$

ii. $z = \cos\theta + i\sin\theta \;\Rightarrow\; z^n + \dfrac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$

$$\left(z+\frac{1}{z}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 \;\Rightarrow\; 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\therefore \cos^4\theta = \tfrac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$$

iii. Using the symmetry of an even function, $V = 2\displaystyle\int_0^1 \left(1-y^2\right)^{\frac{3}{2}}\,dy$.

$y = \sin u,\quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$dy = \cos u\,du$

$y = 0 \Rightarrow u = 0$

$y = 1 \Rightarrow u = \frac{\pi}{2}$

$\left(1-y^2\right)^{\frac{3}{2}} = \cos^3 u$

$$\begin{aligned} V &= 2\int_0^{\frac{\pi}{2}} \cos^4 u\,du \\ &= \tfrac{1}{4}\int_0^{\frac{\pi}{2}} \left(\cos 4u + 4\cos 2u + 3\right) du \\ &= \tfrac{1}{4}\left[\tfrac{1}{4}\sin 4u + 2\sin 2u + 3u\right]_0^{\frac{\pi}{2}} \\ &= \tfrac{1}{4}\left(0 + 0 + 3\tfrac{\pi}{2}\right) \\ &= \tfrac{3\pi}{8} \end{aligned}$$

Question 6

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• writes OP^2 in terms of x and finds first derivative with respect to x	1
• solves $\dfrac{d}{dx}(OP^2) = 0$	1
• uses first or second derivative tests to show min. value for $2x^2 = \ln 2$	1
• substitutes to find minimum value	1

Answer

$P\left(x,\, e^{-x^2}\right) \qquad \therefore OP^2 = x^2 + e^{-2x^2} \qquad$ Let $S = x^2 + e^{-2x^2}$

$$\begin{aligned} \frac{dS}{dx} &= 2x - 4xe^{-2x^2} \\ &= 2x\left(1 - 2e^{-2x^2}\right) \\ &= 2x\,e^{-2x^2}\left(e^{2x^2} - 2\right) \end{aligned}$$

$$\begin{aligned} \frac{d^2S}{dx^2} &= 2 - 4\left(e^{-2x^2} - 4x^2e^{-2x^2}\right) \\ &= 2e^{-2x^2}\left\{e^{2x^2} - 2 + 8x^2\right\} \end{aligned}$$

$\dfrac{dS}{dx} = 0 \;\Rightarrow\; x = 0 \;$ or $\; e^{2x^2} = 2$

$2x^2 = \ln 2$

$\therefore\; x = 0 \;$ or $\; x = \pm\sqrt{\tfrac{1}{2}\ln 2}$

$x = 0 \;\Rightarrow\; \dfrac{d^2S}{dx^2} = -2 < 0 \quad \therefore$ local max. value

$2x^2 = \ln 2 \;\Rightarrow\; \dfrac{d^2S}{dx^2} = 1 \times 4\ln 2 > 0 \quad \therefore$ local min. value

Minimum value of OP^2 is $\tfrac{1}{2}\ln 2 + e^{-\ln 2} = \tfrac{1}{2}(\ln 2 + 1)$

b. Outcomes assessed : HE2, E9

Marking Guidelines

Criteria	Marks
• defines a sequence of statements to be tested by Mathematical induction and shows first is true	1
• writes an inequality for the sum of the first k+1 terms conditional on the truth of $S(k)$	1
• rearranges the RHS of this inequality into an appropriate form	1
• deduces $S(k+1)$ is true if $S(k)$ is true, and hence $S(n)$ is true for $n = 2, 3, 4, \ldots$	1

Answer

Let $S(n)$, $n = 2, 3, \ldots$ be the sequence of statements defined by $S(n): \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{n^2} < 2 - \dfrac{1}{n}$

Consider $S(2)$: $\quad \dfrac{1}{1^2} + \dfrac{1}{2^2} = \tfrac{5}{4} < \tfrac{6}{4} = 2 - \tfrac{1}{2} \qquad$ Hence $S(2)$ is true.

Consider $S(k+1)$: $\quad \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{k^2} + \dfrac{1}{(k+1)^2} < 2 - \dfrac{1}{k} + \dfrac{1}{(k+1)^2} \qquad$ if $S(k)$ is true, using *

$$\begin{aligned} &= 2 - \frac{1}{k+1}\left(\frac{k+1}{k} - \frac{1}{k+1}\right) \\ &= 2 - \frac{1}{k+1}\left(1 + \frac{1}{k(k+1)}\right) \\ &< 2 - \frac{1}{k+1} \qquad \text{since } k(k+1) > 0 \end{aligned}$$

$\therefore\; S(k+1)$ is true if $S(k)$ is true. But $S(2)$ is true, $\therefore S(3)$ is true … . Hence $S(n)$ is true for $n = 2, 3, 4, \ldots$

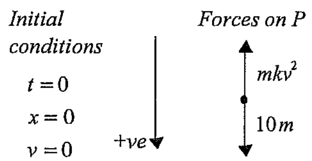
6c. Outcomes assessed : H5, E1, E5, E8

Marking Guidelines

Criteria	Marks
i • considers forces on P to explain why $\ddot{x} = 10 - kv^2$ for some constant k	1
• uses the given terminal velocity to evaluate k	1
ii • expresses $\frac{dt}{dv}$ as sum of partial fractions	1
• finds primitive function and uses initial conditions to evaluate constant of integration	1
• rearranges to find required expression for v	1
iii • writes v as $\frac{dx}{dt}$ then finds primitive function for x in terms of t	1
• evaluates constant of integration and simplifies expression for x in terms of t	1

Answer

i.



By Newton's 2nd Law, $m\ddot{x} = 10m - mkv^2$

$\therefore\ \ddot{x} = 10 - kv^2$

$\ddot{x} \to 0$ as $v^2 \to \frac{10}{k}$. Hence terminal velocity is $\sqrt{\frac{10}{k}}$

$\therefore 20^2 = \frac{10}{k}$ $\quad \therefore k = \frac{1}{40}$ and $\ddot{x} = \frac{1}{40}(400 - v^2)$

ii.

$$\frac{dv}{dt} = \tfrac{1}{40}(400 - v^2)$$

$$\frac{dt}{dv} = \frac{40}{(20+v)(20-v)} = \frac{1}{(20+v)} + \frac{1}{(20-v)}$$

$$\therefore t = \ln\left(\frac{20+v}{20-v}\right) + c$$

$$\left.\begin{matrix} t=0 \\ v=0 \end{matrix}\right\} \Rightarrow \quad c = 0, \quad t = \ln\left(\frac{20+v}{20-v}\right)$$

$$e^t = \frac{20+v}{20-v}$$

$$20e^t - ve^t = 20 + v$$

$$20(e^t - 1) = v(e^t + 1)$$

$$20e^{\frac{1}{2}t}(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}) = ve^{\frac{1}{2}t}(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t})$$

$$v = \frac{20(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})}{(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t})}$$

$$\tfrac{1}{40}v = \frac{\frac{1}{2}(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})}{(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t})}$$

iii.

$$\frac{1}{40}\frac{dx}{dt} = \frac{\frac{1}{2}(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})}{(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t})}$$

$$\therefore \tfrac{1}{40}x = \ln(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}) + c_1$$

$$t = 0,\ x = 0 \Rightarrow c = -\ln 2$$

$$\therefore \tfrac{1}{40}x = \ln(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}) - \ln 2$$

$$x = 40\ln\left\{\tfrac{1}{2}(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t})\right\}$$

Question 7

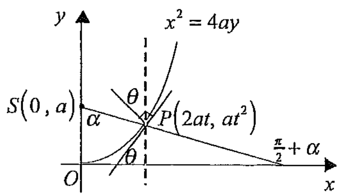
a. Outcomes assessed : PE4

Marking Guidelines

Criteria	Marks
i • explains why $\tan\theta = t$	1
• explains why $\alpha = 2\theta$	1
ii • uses the locus definition of the parabola and appropriate trigonometric identities	1

Answer

i. The normal at P makes angle θ with the vertical, $\therefore$ tangent at P makes an angle θ with the horizontal.



Hence $\tan\theta = \frac{dy}{dx}$ at P. But $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2at}{2a} = t$.

$\therefore \tan\theta = t$

Also gradient $PS = \tan(\frac{\pi}{2} + \alpha) = \frac{a(1-t^2)}{-2at}$

$\therefore -\cot\alpha = -\frac{1-t^2}{2t}$

$\therefore \tan\alpha = \frac{2t}{1-t^2} = \tan 2\theta$ and $\alpha = 2\theta$

ii. PS is equal to the distance from P to the directrix $y = -a$.

$\therefore PS = a(1+t^2) = a(1+\tan^2\theta) = a\sec^2\theta$ $\qquad \therefore l\cos^2\theta = a$

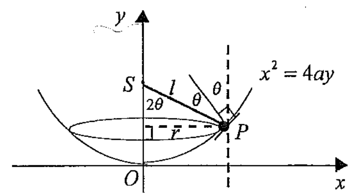
b. Outcomes assessed : E5

Marking Guidelines

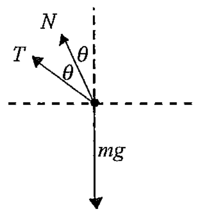
Criteria	Marks
i • shows the forces on P as vectors, and states magnitude and direction of resultant	1
• applies Newton's 2nd Law, resolving vertically and horizontally, to write two equations	1
• uses trigonometric results to simplify equation from horizontal resolution	1
ii • solves simultaneously and substitutes $l\cos^2\theta = a$ to obtain expression for T	1
• finds appropriate expression for N	1
iii • uses the fact that the tension cannot be negative to find one inequality	1
• uses the fact that the normal reaction cannot be negative to find the second inequality	1

Answer

i.



Forces on P



The tension acts along PS making an angle 2θ with the vertical. The resultant force on particle P is a vector of magnitude $mr\omega^2 = ml\sin 2\theta.\omega^2$ directed horizontally toward the centre of the circle of motion.

7bi. (cont)

Using Newton's 2nd Law and resolving vertically and horizontally

vertically: $T\cos 2\theta + N\cos\theta = mg$ (1) *horizontally*: $T\sin 2\theta + N\sin\theta = ml\sin 2\theta.\omega^2$

$$2T\sin\theta\cos\theta + N\sin\theta = 2ml\sin\theta\cos\theta.\omega^2$$

$$2T\cos\theta + N = 2ml\cos\theta.\omega^2 \quad (2)$$

ii. $(2)\times\cos\theta - (1) \Rightarrow T\left(2\cos^2\theta - \cos 2\theta\right) = 2ml\cos^2\theta.\omega^2 - mg$

$$T = 2ma\omega^2 - mg \qquad \text{since } l\cos^2\theta = a \text{ from 7(a)(ii)}$$

Then from (2) $N = 2ml\cos\theta.\omega^2 - 2\cos\theta\left(2ma\omega^2 - mg\right)$

$= 2m\cos\theta\left\{g - \omega^2(2a-l)\right\}$ where $2a = l.2\cos^2\theta = l(1+\cos 2\theta) \ge l$

$= 2m\sqrt{\frac{a}{l}}\left\{g - \omega^2(2a-l)\right\}$

iii. $T \ge 0 \Rightarrow 2ma\omega^2 \ge mg \quad \therefore \omega^2 \ge \dfrac{g}{2a}$. Also $N \ge 0 \Rightarrow \omega^2(2a-l) \le g \quad \therefore \omega^2 \le \dfrac{g}{2a-l}$

$$\therefore \frac{g}{2a} \le \omega^2 \le \frac{g}{2a-l}$$

c. Outcomes assessed : PE3, E9

Marking Guidelines

Criteria	Marks
i • use the fact that the square of a real number is non-negative to prove the result	1
ii • expands the given binomial product then uses the result from i.	1
• expands the trinomial product then uses the result from i.	1
iii • expands and regroups into $n\times 1$ plus a sum of terms of the form (number + reciprocal)	1
• counts the terms in the sum and applies the inequality from i. to obtain result	1

Answer

i. $\left(a+\frac{1}{a}\right)^2 = \left(a-\frac{1}{a}\right)^2 + 4 \ge 4$, since $\left(a-\frac{1}{a}\right)$ real $\Rightarrow \left(a-\frac{1}{a}\right)^2 \ge 0$. $\therefore \left(a+\frac{1}{a}\right) \ge 2$ for $a>0$.

ii. $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 1+1+\frac{a}{b}+\frac{b}{a} = 2+\left(\frac{a}{b}+\frac{b}{a}\right) \ge 4$, using i. with $a \to \frac{a}{b}$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 1+1+1+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right) \ge 3+3\times 2 = 9$$

iii. $(a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) = \sum_{i=1}^{n}\frac{a_i}{a_i} + \sum_{\substack{i=1\\ j>i}}^{n}\left(\frac{a_i}{a_j}+\frac{a_j}{a_i}\right) = n\times 1 + \sum_{\substack{i=1\\ j>i}}^{n}\left(\frac{a_i}{a_j}+\frac{a_j}{a_i}\right)$

There are ${}^nC_2$ ways of selecting two different integers from $1, 2, 3, \ldots, n$

Hence there are ${}^nC_2$ terms of the form $\frac{a_i}{a_j}+\frac{a_j}{a_i}$ where $i<j$.

$$\therefore (a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) \ge n + {}^nC_2\times 2 = n + n(n-1) = n^2$$

Question 8

a. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
i • shows the result by differentiation or integration	1
ii • applies the procedure of integration by parts	1
• rearranges and simplifies to obtain required reduction formula	1
iii • applies reduction formula to evaluate $3I_2$	1
• applies reduction formula to evaluate $4I_3$	1
iv • generalises pattern to write expression for $(n+1)I_n$	1
• simplifies expressions for n odd, n even	1

Answer

i. $\frac{d}{dx}\left\{(1+x)\ln(1+x) - x\right\} = 1.\ln(1+x) + (1+x).\frac{1}{1+x} - 1 = \ln(1+x)$

$$\therefore \int \ln(1+x)\,dx = (1+x)\ln(1+x) - x + c$$

ii. $I_n = \int_0^1 x^n \ln(1+x)\,dx, \quad n = 0, 1, 2, \ldots$

$= \left[x^n\left\{(1+x)\ln(1+x) - x\right\}\right]_0^1 - n\int_0^1 x^{n-1}\left\{(1+x)\ln(1+x) - x\right\}dx, \quad n = 1, 2, \ldots$

$= 2\ln 2 - 1 - n\int_0^1 \left\{x^{n-1}\ln(1+x) + x^n\ln(1+x) - x^n\right\}dx$

$= 2\ln 2 - 1 - nI_{n-1} - nI_n + \frac{n}{n+1}\left[x^{n+1}\right]_0^1$

$= 2\ln 2 - 1 - nI_{n-1} - nI_n + \frac{n}{n+1}$

$= 2\ln 2 - nI_{n-1} - nI_n - \frac{1}{n+1}$

$\therefore (n+1)I_n = 2\ln 2 - \frac{1}{n+1} - nI_{n-1}, \quad n = 1, 2, \ldots$

iii. $I_0 = \int_0^1 \ln(1+x)\,dx = \left[(1+x)\ln(1+x) - x\right]_0^1 = 2\ln 2 - 1$

$3I_2 = 2\ln 2 - \frac{1}{3} - 2I_1 = 2\ln 2 - \frac{1}{3} - \left(2\ln 2 - \frac{1}{2} - I_0\right)$

$\therefore 3I_2 = \left(2\ln 2 - \frac{1}{3}\right) - \left(2\ln 2 - \frac{1}{2}\right) + \left(2\ln 2 - 1\right) = 2\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3}\right) = 2\ln 2 - \frac{5}{6}$

$4I_3 = 2\ln 2 - \frac{1}{4} - 3I_2$

$\therefore 4I_3 = \left(2\ln 2 - \frac{1}{4}\right) - \left(2\ln 2 - \frac{1}{3}\right) + \left(2\ln 2 - \frac{1}{2}\right) - \left(2\ln 2 - 1\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

iv. $(n+1)I_n = \left(2\ln 2 - \frac{1}{n+1}\right) - \left(2\ln 2 - \frac{1}{n}\right) + \ldots + (-1)^{n-1}\left(2\ln 2 - \frac{1}{2}\right) + (-1)^n\left(2\ln 2 - \frac{1}{1}\right)$, with $n+1$ terms.

Hence if n is odd, there is an even number of terms $2\ln 2 - 2\ln 2 + 2\ln 2 - 2\ln 2 + \ldots = 0$

and $(n+1)I_n = -\frac{1}{n+1} + \frac{1}{n} + \ldots - \frac{1}{2} + \frac{1}{1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{n+1}$.

Also if n is even, there is an odd number of terms $2\ln 2 - 2\ln 2 + 2\ln 2 - 2\ln 2 + 2\ln 2 - \ldots = 2\ln 2$

and $(n+1)I_n = 2\ln 2 - \frac{1}{n+1} + \frac{1}{n} + \ldots + \frac{1}{2} - \frac{1}{1} = 2\ln 2 - \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n+1}\right)$.

8b. Outcomes assessed : H5, PE3, E2

Marking Guidelines

Criteria	Marks
i • considers the first derivative to show function is increasing	1
• considers the second derivative to show graph is concave up	1
ii • finds the gradient of the tangent at P and hence the base length of the shaded triangle	1
• finds the area of the triangle and compares areas to obtain required inequality	1
iii • shows RHS of inequality in ii. is a sequence decreasing to limiting value $\frac{1}{2}\ln 2$ as $n\to\infty$	1
• uses 8(a)(iv) and this limiting value to deduce required result	1
iv • deduces inequality for even $n>0$	1
• deduces inequality for odd $n>0$	1

Answer

i. $y=x^n\ln(1+x),\ x>0$ for $n=1,2,3,\dots$

$\frac{dy}{dx}=nx^{n-1}\ln(1+x)+\frac{x^n}{1+x}>0$ for $n=2,3,\dots$, and $\frac{dy}{dx}=\ln(1+x)+\frac{x}{1+x}>0$ for $n=1$

Hence function is increasing for all positive integers n.

$$\frac{d^2y}{dx^2}=n(n-1)x^{n-2}\ln(1+x)+\frac{nx^{n-1}}{1+x}+\frac{nx^{n-1}(1+x)-x^n}{(1+x)^2},\quad n=2,3,\dots$$
$$=n(n-1)x^{n-2}\ln(1+x)+\frac{nx^{n-1}}{(1+x)}+\frac{nx^{n-1}+(n-1)x^n}{(1+x)^2}$$
>0 for $x>0$ and $n=2,3,\dots$

and
$$\frac{d^2y}{dx^2}=\frac{1}{1+x}+\frac{(1+x)-x}{(1+x)^2},\quad n=1$$
$$=\frac{2+x}{(1+x)^2}$$
>0 for $x>0$

Hence graph is concave up for all positive integers n.

ii. Gradient of tangent PT is $n\ln 2+\frac{1}{2}$, hence shaded triangle has base b given by $\frac{\ln 2}{b}=n\ln 2+\frac{1}{2}$

Hence shaded triangle has area $\frac{1}{2}\cdot\frac{\ln 2}{n\ln 2+\frac{1}{2}}\cdot\ln 2=\frac{(\ln 2)^2}{2n\ln 2+1}$

Comparing the area under the curve between O and P, and the area of the shaded triangle:

$$(n+1)I_n>\frac{(n+1)(\ln 2)^2}{2n\ln 2+1}=\frac{(1+\frac{1}{n})(\ln 2)^2}{2\ln 2+\frac{1}{n}}$$

iii. Using 8(a)(iv), for even $n>0$, $2\ln 2-\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{n+1}\right)>\frac{(1+\frac{1}{n})(\ln 2)^2}{2\ln 2+\frac{1}{n}}$

Consider the function $f(u)=\frac{(u+1)(\ln 2)^2}{2u\ln 2+1},\ u>0$.

Then $$f'(u)=(\ln 2)^2\cdot\frac{2u\ln 2+1-(u+1)2\ln 2}{(2u\ln 2+1)^2}=\frac{(\ln 2)^2(1-2\ln 2)}{(2u\ln 2+1)^2}<0$$

Hence $\frac{(n+1)(\ln 2)^2}{2n\ln 2+1}=\frac{(1+\frac{1}{n})(\ln 2)^2}{2\ln 2+\frac{1}{n}}$, $n=1,2,3,\dots$ decreases to a limiting value of $\frac{\ln 2}{2}$ as $n\to\infty$.

$\therefore$ for even $n>0$, $2\ln 2-\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{n+1}\right)>\frac{1}{2}\ln 2$ and hence $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{n+1}<\frac{3}{2}\ln 2$

8b. iv. For even $n>0$, $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots-\frac{1}{n}+\frac{1}{n+1}<\frac{3}{2}\ln 2$ and hence $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots-\frac{1}{n}<\frac{3}{2}\ln 2$.

Also $\frac{1}{k}-\frac{1}{k+1}=\frac{1}{k(k+1)}>0$ for $k>0$, hence $\frac{1}{1}-\frac{1}{2}+(\frac{1}{3}-\frac{1}{4})+\dots+(\frac{1}{n-1}-\frac{1}{n})\geq 1-\frac{1}{2}=\frac{1}{2}$

$\therefore \frac{1}{2}\leq\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots-\frac{1}{n}<\frac{3}{2}\ln 2$ for even $n>0$

For $n=1$, $\frac{1}{2}\leq 1<\frac{3}{2}\ln 2$

If n is odd, $n\geq 3$, then $n-1>0$ is even and $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\dots-\frac{1}{n-1}+\frac{1}{n}<\frac{3}{2}\ln 2$, using iii.

Also $\frac{1}{1}-\frac{1}{2}+(\frac{1}{3}-\frac{1}{4})+\dots+(\frac{1}{n-2}-\frac{1}{n-1})+\frac{1}{n}>1-\frac{1}{2}=\frac{1}{2}$

$\therefore \frac{1}{2}\leq\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{n}<\frac{3}{2}\ln 2$ for odd $n>0$

Hence for all positive integers n, $\frac{1}{2}\leq\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+(-1)^{n-1}\frac{1}{n}<\frac{3}{2}\ln 2$