Class:

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2011

MATHEMATICS EXTENSION 2

Time Allowed – 3 Hours
(Plus 5 minutes Reading Time)

- *All* questions may be attempted
- *All* questions are of equal value
- Department of Education approved calculators and templates are permitted
- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged work
- No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate *stapled* bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate Number.

Question 1 (15 Marks) Marks

(a) Find:

(i) $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx$ 2

(ii) $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$ 2

(iii) $\displaystyle\int \frac{d\theta}{2+\cos\theta}$ 3

(b) Evaluate: $\displaystyle\int_{-1}^{1} \frac{x}{x^2+2x+5}\,dx$ 4

(c) If $I=\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+2\sin x}\,dx$ and $J=\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x+2\sin x}\,dx$,

(i) Show that $2I-J=\ln 2$. 1

(ii) Evaluate $I+2J$. 1

(iii) Hence, find the exact values of I and J. 2

Question 2 (15 Marks) START A NEW PAGE

(a) Plot neatly on an Argand diagram the points A, B and C corresponding to the complex numbers w, w^2 and $w\overline{w}$ respectively where $w=\sqrt{3}+i$. 3

(b) Let $z=x+iy$ be a complex number satisfying the inequality 4

$$z\overline{z}+(1-2i)z+(1+2i)\overline{z}\le 4 \qquad \text{where } x \text{ and } y \text{ are real.}$$

Sketch the locus of z on an Argand diagram.

(c) (i) Solve the equation for w: 2

$$w^2=-11-60i.$$

Write your answer in the form $w=x+yi$, where x and $y\in\mathbb{R}$

(ii) Hence, or otherwise, solve the equation: 3

$$z^2-(1+4i)z-(1-17i)=0$$

(d) Five girls and three boys are seated at random around a circular table. What is the probability that at least two boys are sitting next to each other? 3

Question 3 (15 Marks) START A NEW PAGE **Marks**

(a) $ABCD$ is a cyclic quadrilateral. Chords BE and DF bisect $\angle ABC$ and $\angle ADC$ respectively.

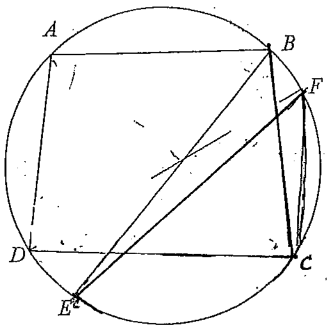


Diagram not to scale

Copy the diagram and prove that EF is a diameter of the circle. **3**

(b) (i) Show whether the function $f(x)=2\,|x-1|-|x|+2\,|x+1|$ is even, odd or neither, giving reasons. **2**

(ii) Sketch the graph of the function $f(x)=2\,|x-1|-|x|+2\,|x+1|$, clearly showing all intercepts with the coordinate axes and critical points. Label all branches with the relevant equations. **3**

(c) $P(x_1, y_1)$ is a point on the rectangular hyperbola $xy=9$.

(i) Show that the Cartesian equation of the tangent at P is $y_1x+x_1y=18$. **2**

(ii) Hence, or otherwise, derive the equation of the chord of contact from an external point $T(x_0, y_0)$ to the hyperbola $xy=9$. **2**

(iii) Prove that the chord of contact is a focal chord when T is a point on the directrix. **3**

Question 4 (15 Marks) START A NEW PAGE **Marks**

(a) (i) Find all stationary points for the curve $y^2=x(3-x)^2$. **3**

(ii) Sketch the curve $y^2=x(3-x)^2$, showing all stationary points and the intercepts with the coordinate axes. **3**

(b) A particle of mass 2kg is projected vertically upwards with a velocity of U ms^{-1} in a medium which exerts a resistive force of $\frac{v}{10}$ Newtons.

(i) Show that the maximum height H metres reached by the particle is given by: **3**

$$H=20U+4000\ln\left(\frac{200}{200+U}\right) \qquad (\text{take } g=10 \text{ ms}^{-2})$$

(ii) Find the time taken for the particle to reach the maximum height H. **3**

(iii) If $U=400$, show that the average speed during the ascent is: **3**

$$200\left(\frac{2}{\ln 3}-1\right) \text{ ms}^{-1}.$$

Question 5 (15 Marks) START A NEW PAGE

(a) A block of mass 5 kg is to be moved along a rough horizontal surface by a force (F Newtons) inclined at an angle of θ with the direction of motion where $0\le\theta\le\frac{\pi}{2}$.

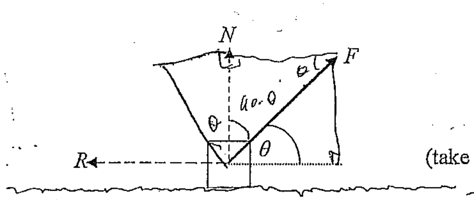


Diagram not to scale

(take $g=10$ ms^{-2})

The motion is resisted by a frictional force (R Newtons) which is proportional to the normal reaction force (N Newtons) exerted on the block by the surface, such that $R=0{\cdot}2N$.

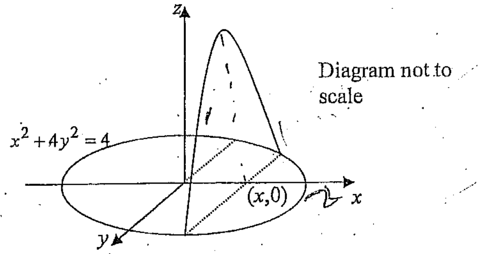
(i) Show that $F=\dfrac{50}{5\cos\theta+\sin\theta}$ Newtons, when the block is about to move. **4**

(ii) Calculate the minimum value of F needed to overcome the frictional resistance between the block and the surface. **2**

Question 5 continued over page

(b) (i) A parabola has the equation $x^2 = 4ay$. Show that the area bounded by this parabola and the focal chord perpendicular to the axis is equal to $\frac{8a^2}{3}$ units^2. 3

(ii) A solid has an elliptical base whose equation is $x^2 + 4y^2 = 4$ and each cross-section perpendicular to the major axis of the base is a parabola with its focus on the major axis.



(α) Show that the area of the parabolic cross-section, x units from the origin, is given by the formula

$$A(x) = \frac{4 - x^2}{6}$$ 3

(β) Hence, find the volume of the resultant solid. 3

Question 6 (15 Marks) START A NEW PAGE

(a) The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } \phi > \theta \text{ and } a > b.$$

The points $P'(a\cos\theta, a\sin\theta)$ and $Q'(a\cos\phi, a\sin\phi)$ lie on the auxiliary circle and subtend a right angle at the origin.

(i) Draw a neat sketch of the above information showing the relative positions of the points P, Q, P' and Q'. 2

(ii) Express the coordinates of Q in terms of θ. 1

(iii) The tangents at P and Q meet in point R. Find the coordinates of R in terms of θ. 4

(iv) Show that R lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ 1

Question 6 continued over page

(b) (i) If $\tan(x)\tan(\theta - x) = k$ prove that: 4

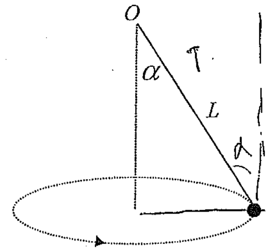
$$\frac{1+k}{1-k} = \frac{\cos(2x-\theta)}{\cos\theta}$$

(ii) Hence, or otherwise, solve the equation for all x. 3

$$\tan x \tan\left(\frac{\pi}{3} - x\right) = 2 + \sqrt{3}$$

Question 7 (15 Marks) START A NEW PAGE

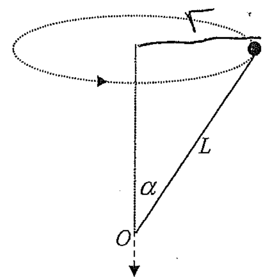
(a) A particle of mass m kg is fastened to one end of a light inextensible string of length L metres and the other end is attached to a fixed point O. The particle rotates with a uniform angular velocity ω rad/s about a vertical line through O.



(i) Show that if α is the angle of inclination of the string to the downward vertical, then $\alpha = \cos^{-1}\left(\frac{g}{L\omega^2}\right)$. 4

(ii) Explain why steady circular motion is only possible when $\omega^2 > \frac{g}{L}$. 2

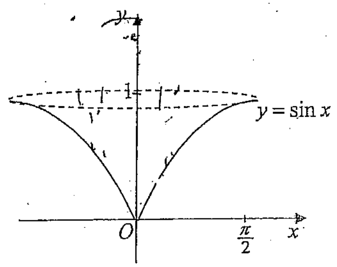
(iii) The point O is now made to descend with a uniform acceleration of f ms^{-2}, whilst the particle continues to rotate with uniform angular velocity ω.



Find f so that the string makes an angle of α with the upward vertical. 3

Question 7 continued over page

(b) The area between the curve $y = \sin x$, from $x = 0$ to $x = \frac{\pi}{2}$, the y-axis and the line $y = 1$ is rotated about the y-axis.



(i) Show that the volume of the solid formed can be found by using the formula **3**

$$V = \pi \int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$$

(ii) Hence, calculate the volume of the solid. **3**

Question 8 (15 Marks) START A NEW PAGE

(a) The total number of different groups with 4 members which can be chosen from a group of n people is five times as many as the total number of different groups with 3 members which can be chosen from a group of $n-2$ people. **3**

Find all possible values of n.

(b) Prove that $\tan^{-1}(5) + \tan^{-1}(3) + \tan^{-1}\left(\frac{4}{7}\right) = \pi$ **4**

(c) A curve, defined by the equation $x^2 + 2xy + y^5 = 4$, has a horizontal tangent at the point $P(X, Y)$.

(i) Show that X is a root to the equation $x^5 + x^2 + 4 = 0$. **3**

(ii) Show that the value of X is between -2 and -1. **1**

(iii) With the use of a graph, or otherwise, show that X is the only real root to the equation $x^5 + x^2 + 4 = 0$. **4**

End of Examination

TRIAL 2011

MATHEMATICS Extension 2 : Question 1

Suggested Solutions	Marks	Marker's Comments
a) i) Let $I=\int \frac{e^x\,dx}{\sqrt{e^{2x}-1}}$ Let $u=e^x$ "$du=e^x\,dx$"		
$I=\int\frac{du}{\sqrt{u^2-1}}$	1	
$=\ln\lvert u+\sqrt{u^2-1}\rvert+k$ From tables		
$=\ln\left(e^x+\sqrt{e^{2x}-1}\right)+k$	1	
$(e^x>0)$		
ii) Let $I=\int\frac{dx}{x^2-5x+6}$		
$\frac{A}{x-3}+\frac{B}{x-2}=\frac{1}{(x-3)(x-2)}$		
$=\int\frac{dx}{(x-3)(x-2)}$		
$A(x-2)+B(x-3)=1$		
$x=2\rightarrow B=-1$	1	
$x=3\rightarrow A=1$		
$=\int\left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx$		½ mark deducted if no absolute value signs
$=\ln\lvert x-3\rvert-\ln\lvert x-2\rvert+c$		
$=\ln\left\lvert\frac{x-3}{x-2}\right\rvert+c$	1	
iii) Let $I=\int\frac{d\theta}{2+\cos\theta}$		
Let $t=\tan\frac{\theta}{2}$, "$dt=\frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$"		
$=\int\frac{2\,dt}{(1+t^2)\left(2+\frac{1-t^2}{1+t^2}\right)}$ "$\frac{2\,dt}{1+t^2}=d\theta$"	1	
Also $\cos\theta=\frac{1-t^2}{1+t^2}$		
$=\int\frac{2\,dt}{2+2t^2+1-t^2}$		
$=\int\frac{2\,dt}{t^2+3}$	1	
$=\frac{2}{\sqrt3}\tan^{-1}\frac{t}{\sqrt3}+c$		
$=\frac{2}{\sqrt3}\tan^{-1}\left(\frac{1}{\sqrt3}\tan\frac{\theta}{2}\right)+c$	1	

MATHEMATICS Extension 2 : Question 1 (cont)

Suggested Solutions	Marks	Marker's Comments
b) $\int_{-1}^{1}\frac{x\,dx}{x^2+2x+5}=\frac{1}{2}\left\{\int_{-1}^{1}\frac{2x+2-2}{x^2+2x+5}\,dx\right\}$		
$=\frac{1}{2}\int_{-1}^{1}\frac{2x+2}{x^2+2x+5}\,dx-\int_{-1}^{1}\frac{dx}{(x+1)^2+4}$	1, 1	
$=\frac{1}{2}\left[\ln(x^2+2x+5)\right]_{-1}^{1}-\left[\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1}$		
$=\frac{1}{2}\ln 8-\frac{1}{2}\ln 4-\frac{1}{2}\tan^{-1}1$		
$=\frac{1}{2}\ln 2-\frac{\pi}{8}$	1, 1	
c) i) $2I-J=\int_0^{\pi/2}\frac{2\cos x-\sin x}{\cos x+2\sin x}\,dx$		
$=\left[\ln(\cos x+2\sin x)\right]_0^{\pi/2}$		
$=\ln(0+2)-\ln(1)=\ln 2-0=\ln 2$	1	
ii) $I+2J=\int_0^{\pi/2}\frac{\cos x+2\sin x}{\cos x+2\sin x}\,dx$		
$=\int_0^{\pi/2}dx=\left[x\right]_0^{\pi/2}=\pi/2$	1	
iii) $2I-J=\ln 2$ —(1)		
$I+2J=\pi/2$ —(2)		
2×(2) $2I+4J=\pi$ —(3)		
(3)−(1) $5J=\pi-\ln 2$		Generally I knocked ½ mark for trivial numerical errors.
$J=\frac{\pi-\ln 2}{5}$		However in c, iii solving simultaneous equations requires accuracy.
Substitute into (2)		1 or 0 usually.
$I=\frac{\pi}{2}-2\left(\frac{\pi-\ln 2}{5}\right)=\frac{5\pi-4\pi+4\ln 2}{10}$		
$I=\frac{\pi+4\ln 2}{10}$, $J=\frac{\pi-\ln 2}{5}$	1, 1	

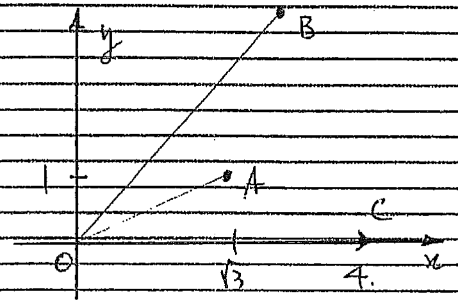
MATHEMATICS Extension 2: Question (2).

Suggested Solutions | **Marks** | **Marker's Comments**

(a) $w = \sqrt{3}+i$ $\left(2\,\text{cis}\frac{\pi}{6}\right)$.

$w^2 = (\sqrt{3}+i)^2 = 2(1+\sqrt{3}i)$ $\left(4\,\text{cis}\frac{\pi}{3}\right)$.

$w\bar{w} = |w|^2 = 4$



[3].

1 mark for each point.

(b) $z\bar{z} = |z|^2 = x^2+y^2$

Let $z = x+iy$.

$(1-2i)(x+iy) = x+iy-2ix+2y$.

$(1+2i)(x-iy) = x-iy+2ix+2y$

$\therefore z\bar{z}+(1-2i)z+(1+2i)\bar{z} \le 4$

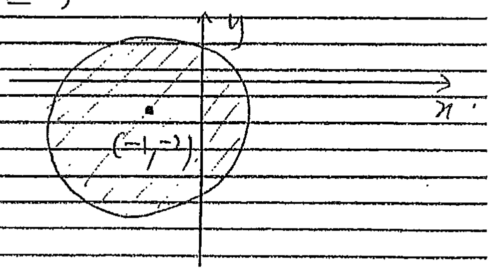
reduces to

$x^2+y^2+2x+4y \le 4$.

$(x+1)^2+(y+2)^2 \le 9$

circle centre $(-1,-2)$

$r = 3$



1 mark substituting and simplifying $z\bar{z}$.

1 mark completing the square.

1 mark centre, radius of the circle.

1 Correct diagram for the locus.

MATHEMATICS Extension 2: Question........

Suggested Solutions | **Marks** | **Marker's Comments**

(c) (i) Let $\sqrt{-11-60i} = a+ib$.

$-11-60i = (a^2-b^2)+i\,2ab$.

Equate real and imaginary parts

$a^2-b^2 = -11$ $\quad 2ab = -60$

$(a^2-b^2)^2 = (a^2+b^2)^2-4a^2b^2$

$(a^2+b^2)^2 = 3721$, $\quad a^2+b^2 = 61$

$2a^2 = 50$ $\quad a^2 = 25$ $\quad a = \pm 5$

$b = \mp 6$.

$\therefore w = \pm(5-6i)$

[2]

1 Correct quadratic expression or solving a quartic equation

1 Correct solution for w.

(ii) $z^2-(1+4i)z-(1-17i) = 0$

$z = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$z = \dfrac{(1+4i)\pm\sqrt{-11-60i}}{2}$

$= \dfrac{1+4i \pm (5-6i)}{2}$

$z = 3-i$, $\quad -2+5i$

[3].

1 use quadratic formula.

1 Apply c(i) into quad. formula.

1 Correct solution.

Extension (2).

MATHEMATICS: Question……..

Suggested Solutions	Marks	Marker's Comments
(d)		
(i) Number of ways that 5 girls seated around a circular table $= 4!$		(i) or (iii) 1 mark.
(ii) Number of ways that 3 boys seated between the girls. $= (5 \times 4 \times 3) \times 4!$ 5 ← 1st boy, 4 ← 2nd boy, 3 ← 3rd boy		(ii) or (iv) equivalent or merit 1 mark.
(iii) 8 people can seat around a circular table $7!$ ways.		
(iv) No of ways that at least 2 boys are sitting next to each other $= 7! - (5 \times 4 \times 3) \times 4!$ $= 3600$		
$\therefore P(E) = \frac{3600}{5040} = \frac{5}{7}$.	1	mark correct prob.

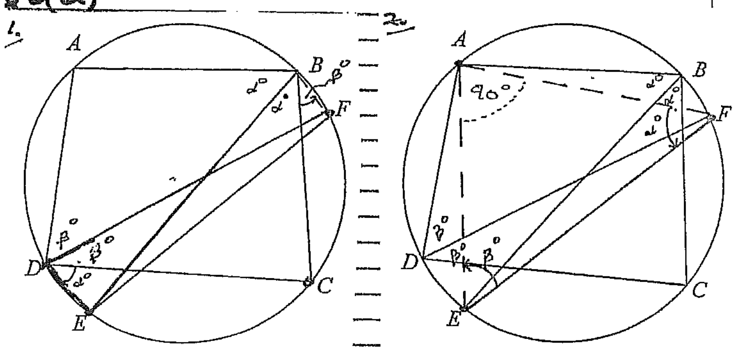
MATH. EXT 2 TRIAL 2011

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments

Q3(a)

i.



Method 2

i. ✓

2. $\angle DAF + \beta^\circ + \alpha^\circ = 180^\circ$ (angle sum of $\triangle DAF$ is 180°)

i.e. $\angle DAF = 90^\circ$

Conclusion – REASON

Let $\angle ABE = \angle CBE = \alpha^\circ$ $\quad \angle ABC$ is bisected by EB

$\angle ADF = \angle CDF = \beta^\circ$ $\quad \angle ADC$ " " by FD.

1. $2\alpha + 2\beta = 180$ (Opposite angles of cyclic quadrilateral ABCD is 180°)

$\therefore \alpha + \beta = 90$ — 1

2. $\angle EDC = \alpha^\circ$ (Angles subtended on circumference standing on the same arc EC are equal) — 1

3. $\therefore \angle EDF = \alpha^\circ + \beta^\circ = 90^\circ$ (adjacent angle addition)

E, D and F exist on circumcircle EDF

A right angle at the circumference subtends a diameter

$\therefore$ EF is a diameter (converse of — $\boxed{3}$

(b)(i) $f(x) = 2|x-1| - |x| + 2|x+1|$ — justification needed.

$f(-x) = 2|-x-1| - |-x| + 2|-x+1|$ — $\frac{1}{2}$

$= 2|-1(x+1)| - |-x| + 2|-1(x-1)|$ — $\frac{1}{2}$

$= 2|x+1| - |x| + 2|x-1|$ — $\frac{1}{2}$

$|-a| = |a|$

$|b-a| = |-1(a-b)|$ ✓

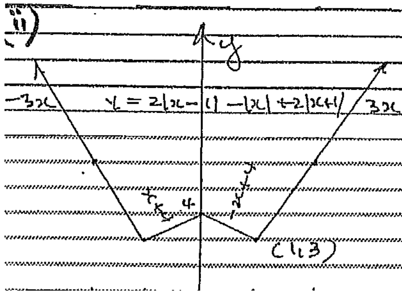
$= |-1|\,|a-b|$

$\therefore |b-a| = |a-b|$

i.e. $f(-x) = f(x)$

$\therefore f(x)$ is an EVEN fn — $\frac{1}{2}$ — $\boxed{2}$

(ii)



$y = 2|x-1| - |x| + 2|x+1|$

$-3x$ $\quad 3x$ $\quad -x+4$ $\quad x+4$ $\quad 4$ $\quad (1,3)$

$x \geq 1 \quad x - 1 \geq 0$

$2|x-1| = 2(x-1)$

$|x| = x$

$2|x+1| = 2(x+1)$

$\therefore y = f(x) = 3x$

$0 \leq x < 1$

$2|x-1| = -2(x-1)$

$|x| = x$

$2|x+1| = 2(x+1)$

$\therefore y = f(x) = -x+4$

1 For $y = 3x, -3x$

1 For $y = -x+4$, $x+4$

$\frac{1}{2}$ For $(0,4)$

$\frac{1}{2}$ For $(1,3)$ — $\boxed{3}$

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments

Q3(c) (i) $xy = 9$ $\quad y = \frac{9}{x} = 9x^{-1}$

$\therefore y + x\frac{dy}{dx} = 0$

i.e. $\frac{dy}{dx} = \frac{-y}{x}$ $\qquad \frac{dy}{dx} = \frac{-9}{x^2}$

$\therefore$ Gradient of tangent at $P(x_1, y_1)$: $m_T = \frac{-y_1}{x_1}$ (or $-\frac{9}{x_1^2}$) — 1 For m_T = and equ. of Tangent;

$\therefore$ Equ. of Tangent at P

$y - y_1 = \frac{-y_1}{x_1}(x - x_1)$ *

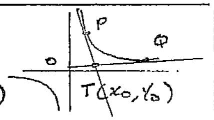
$x_1 y - x_1 y_1 = -y_1 x + x_1 y_1$

so $y_1 x + x_1 y = 2x_1 y_1$, but $x_1 y_1 = 9$ as pt $P(x_1, y_1)$ lies on $xy = 9$

$\therefore y_1 x + x_1 y = 2 \times 9 = 18$ — $\frac{1}{2} + \frac{1}{2}$ For explaining why $x_1 y_1 = 9$ — $\boxed{2}$

(ii) Tangent at $P(x_1, y_1)$ is $y_1 x + x_1 y = 18$ — $\frac{1}{2}$

" " $Q(x_2, y_2)$ is $y_2 x + x_2 y = 18$



Let PQ be the chord of contact now from $T(x_0, y_0)$

these tangents above pass through $T(x_0, y_0)$

$\therefore y_1 x_0 + x_1 y_0 = y_0 x_1 + x_0 y_1 = 18$ — (1) — $\frac{1}{2}$

and $y_2 x_0 + x_2 y_0 = y_0 x_2 + x_0 y_2 = 18$ — (2) — $\boxed{2}$

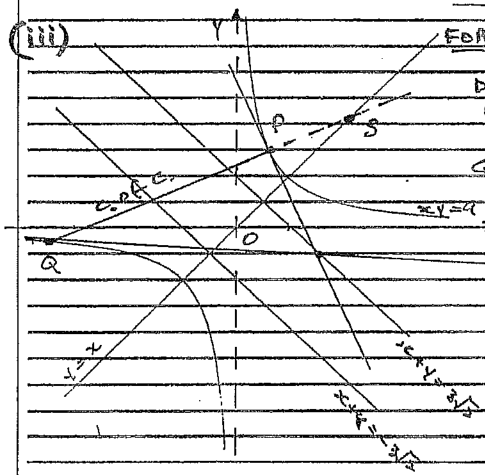
So points P and Q lie on the equation $y_0 x + x_0 y = 18$

which is a line — $\frac{1}{2}$ For justifying linking ...

$\therefore$ PQ uniquely determine the "chord of contact"

which is of the form $y_0 x + x_0 y = 18$ — $\frac{1}{2}$

(iii)



FOR $xy = 9 \quad \therefore c^2 = 9 \Rightarrow c = 3$

Directrices: $x + y = \pm a = \pm c\sqrt{2} = \pm 3\sqrt{2}$ — $\frac{1}{2}$

Foci $(\pm 3\sqrt{2}, \pm 3\sqrt{2})$

Given $T(x_0, y_0)$ is on a Directrix

$\therefore x_0 + y_0 = \pm 3\sqrt{2}$ — 1

Now Chord of Contact is $y_0 x + x_0 y = 18$ — (1)

This line is to intersect with transverse axis $y = x$ — (2)

$\therefore x(y_0 + x_0) = 18$

$x = \frac{18}{x_0 + y_0} = \frac{18}{\pm 3\sqrt{2}} = \pm 3\sqrt{2}$ — $\frac{1}{2}$

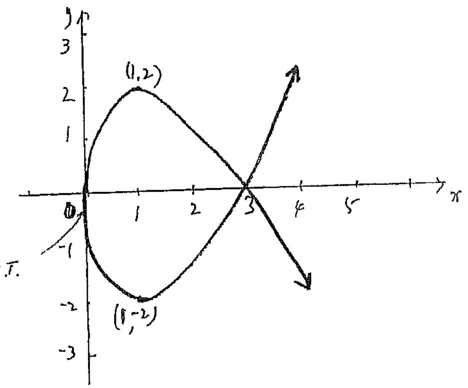
$\therefore y = \pm 3\sqrt{2}$

but this is a Focus S or S' — $\frac{1}{2}$ — $\boxed{3}$

$\therefore$ when $T(x_0, y_0)$ is on a directrix — $\frac{1}{2}$

the chord of contact of Chord becomes a focal chord.

Y.12 TRIAL 2011 **MATHEMATICS Extension 1: Question 4a**

Suggested Solutions	Marks	Marker's Comments
a) $y^2 = x(3-x)^2$		
method 1:		
$y = \pm\sqrt{x(3-x)^2}$		FORGOT $\pm$ $-\frac{1}{2}$m
$y' = \frac{3}{2\sqrt{x}} - \frac{3}{2}\sqrt{x}$ (for $y>0$)	1	
s.p $y'=0$ when $0 = \frac{3}{2\sqrt{x}}(1-x)$ $(x \neq 0)$		
$\therefore x=1$ $y=2$	1	many forget y values. $-\frac{1}{2}$m
By symmetry s.p $(1,2)$ or $(1,-2)$	1	
Method 2 implicit differentiation		
$y^2 = x(3-x)^2 = 9x - 6x^2 + x^3$		
$2yy' = 9 - 12x + 3x^2$	1	
s.p $0 = \frac{3(x-3)(x-1)}{2y} = \frac{3(x-3)(x-1)}{\pm 2\sqrt{x}\,\lvert 3-x\rvert}$		many forget $\pm$ $-\frac{1}{2}$m
$\therefore x = 3$ or 1 but $x \neq 3$ or 0	1	if $x=3$ is included max 2 m.
$\therefore x=1$ only		
s.p $(1,2)$ or $(1,-2)$	1	
b) 	3	(1) Stationary pts (½) Vertical tangent at $x=0$ (½) slope at $x=3$ (1) shape, scale curvature

Y12 TRIAL 2011 **MATHEMATICS Extension 1: Question 4b**

Suggested Solutions	Marks	Marker's Comments
4b i) $m\ddot{x} = -mg - \frac{v}{10}$ $\uparrow +$ $\downarrow mg$ $\downarrow \frac{v}{10}$		
$2\ddot{x} = -20 - \frac{v}{10}$		
$\ddot{x} = -\left(10 + \frac{v}{20}\right)$	1 m	
$v\frac{dv}{dx} = -\left(\frac{v+200}{20}\right)$		
$\int_0^H dx = \int_u^0 \frac{-20v}{200+v}\,dv$ At max ht H, $v=0$		many forget max ht H, $v=0$
$H = -20\int_u^0 1 - \frac{200}{200+v}\,dv = 20\int_0^u 1 - \frac{200\,dv}{200+v}$	½ m	SHOW Question.
$H = 20\Big[v - 200\ln(200+v)\Big]_0^u = 20\left[u - 200\ln\left(\frac{200+u}{200}\right)\right]$	½ m	must show $-\ln\left(\frac{200+u}{200}\right) = +\ln\left(\frac{200}{200+u}\right)$
$H = 20u + 4000\ln\left(\frac{200}{200+u}\right)$	1 m	
ii) $\frac{dv}{dt} = -\left(\frac{200+v}{20}\right)$		
$\int_0^T dt = \int_u^0 \frac{-20\,dv}{200+v}$ max ht H, $v=0$	1 m	did well
$T = \Big[-20\ln(200+v)\Big]_u^0$	1 m	
$T = -20\ln\left(\frac{200}{200+u}\right) = +20\ln\left(\frac{200+u}{200}\right)$ sec #	1 m	many forget unit $-\frac{1}{2}$m
iii) Ave Speed $= \frac{D}{T} =$ $T=$		"SHOW" Q.
$T = 20\ln\left(\frac{200+400}{200}\right) = 20\ln 3$	½ m	must show detail.
$D = 400\times 20 + 4000\ln\left(\frac{200}{200+400}\right) = 8000 + 4000\ln\left(\frac{200}{600}\right) = 8000 + 4000\ln\frac{1}{3}$	1 m	must show $\ln\left(\frac{200+400}{200}\right) = \ln\left(\frac{600}{200}\right) = \ln 3$ $-\frac{1}{2}$m
Ave Speed $= \frac{D}{T} = \frac{8000 + 4000\ln\frac{1}{3}}{20\ln 3}$	½ m	must show $\ln\frac{1}{3} = -\ln 3$ $-\frac{1}{2}$m
$= \frac{400\left(2 + \ln\frac{1}{3}\right)}{20\ln 3} = 20\left(\frac{2}{\ln 3} + \frac{\ln\frac{1}{3}}{\ln 3}\right)$	½m + ½m	must show factorization $-\frac{1}{2}$m
$= 20\left(\frac{2}{\ln 3} - 1\right)$ m/s #		

MATHEMATICS Extension 1: Question 5a

Suggested Solutions	Marks	Marker's Comments
i) Vert. $N + F\sin\theta = mg$	$\frac{1}{2}$	
$N + F\sin\theta = 5\times 10$ ①	$\frac{1}{2}$	must show where 50 comes from. $-\frac{1}{2}$m.
Horiz. $R - F\cos\theta = 0$	$\frac{1}{2}$	
$0.2N = F\cos\theta$	$\frac{1}{2}$	This is 'SHOW' question.
$N = 5F\cos\theta$ ②		
① + ② $5F\cos\theta + F\sin\theta = 50$		
$F(5\cos\theta + \sin\theta) = 50$	1	
$F = \dfrac{50}{5\cos\theta + \sin\theta}$ #		$\boxed{3}$
ii) F is min when $5\cos\theta + \sin\theta$ is max	1	Pretty well done.
$5\cos\theta + \sin\theta = R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$		
$\therefore R\sin\alpha = 1$, $R\cos\alpha = 5$		
$\therefore R = \sqrt{1^2+5^2} = \sqrt{26}$		
$\therefore 5\cos\theta + \sin\theta = \sqrt{26}\sin(\theta+\alpha)$	1	
$\lvert\sin(\theta+\alpha)\rvert \le 1$		many forgot max $\lvert\sin(\theta+\alpha)\rvert = 1$ $-\frac{1}{2}$m.
$\therefore \max(5\cos\theta + \sin\theta) = \sqrt{26}$		
$\therefore \min F = \dfrac{50}{\frac{5\times5}{\sqrt{26}} + \frac{1}{\sqrt{26}}} = \dfrac{50}{\frac{26}{\sqrt{26}}} = \dfrac{50}{\sqrt{26}}$ N.	1	$\boxed{3}$
Altern. $F' = \dfrac{-50}{(5\cos\theta+\sin\theta)^2}(-5\sin\theta + \cos\theta) = 0$	$\frac{1}{2}$	many forgot to justify min/max -1 m.
when $5\sin\theta = \cos\theta$ $\therefore \tan\theta = \frac{1}{5}$	$\frac{1}{2}$	
$\therefore \sin\theta = \frac{1}{\sqrt{26}}$, $\cos\theta = \frac{5}{\sqrt{26}}$ $(0 \le \theta \le 90°)$		many forgot $0 \le \theta \le 90°$ $-\frac{1}{2}$m
Justify min F	1	
$\min F = \dfrac{50}{\frac{5\times5}{\sqrt{26}} + \frac{1}{\sqrt{26}}} = \dfrac{50}{\sqrt{26}}$ N.	1	

MATHEMATICS Extension 2: Question 5b

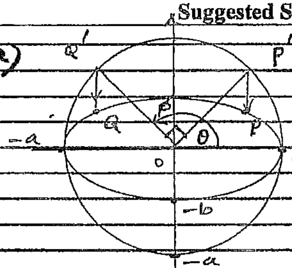
Suggested Solutions	Marks	Marker's Comments
i) $4ay = x^2$ $\therefore y = \frac{x^2}{4a}$; when $y = a$ $4a^2 = x^2$ $\therefore x = \pm 2a$	1 m	
$A = \int_{-2a}^{2a} a - y\,dx = 2\times\int_0^{2a} a - \frac{x^2}{4a}\,dx$ (even function)	1 m	Many forgot even $-\frac{1}{2}$m
$= 2\times\left[ax - \frac{x^3}{12a}\right]_0^{2a} = 2\times\left[2a^2 - \frac{8a^3}{12a}\right]$		$2\int_{-2a}^{2a}\frac{x^2}{4a}dx$ is the wrong area max $1\frac{1}{2}$m
$= 2\times\left[2a^2 - \frac{2}{3}a^2\right] = 2 \times \frac{4}{3}a^2 = \frac{8a^2}{3}$ unit²	1 m	
or $x = \pm 2\sqrt{ay}$	1 m	
$A = 2\int_0^a x\,dy$ (even) $= 2\int_0^a 2\sqrt{ay}\,dy$	1 m	many fudging since answer given
$A = 4\sqrt{a}\left[y^{3/2}\cdot\frac{2}{3}\right]_0^a = \frac{8}{3}\sqrt{a}\,a^{3/2} = \frac{8}{3}a^2$ unit²	1 m	
		'SHOW' question: must show $a = \frac{y}{2}$
ii α) length of latus rectum $= 2y = 4a$, $a = \frac{y}{2}$	1 m	Some use Simpson's Rule (must mention) $-\frac{1}{2}$m
from (i) $A = \frac{8}{3}a^2 = \frac{8}{3}\left(\frac{y}{2}\right)^2 = \frac{8}{3}\cdot\frac{y^2}{4}$		
$A = \frac{2}{3}y^2$, but $x^2 + 4y^2 = 4$ $\therefore y^2 = \frac{4-x^2}{4}$	1 m	$\frac{4a}{6}(0 + 4a + 0)$ 1m
$A = \frac{2}{3}\cdot\frac{4-x^2}{4} = \frac{4-x^2}{6}$	1 m	$= \frac{2y}{6}\times 2y = \frac{2}{3}y^2 = \frac{4-x^2}{6}$ 1m
ii β) $V = \lim_{\Delta x\to 0}\sum_{x=-2}^{2} A(x)\,\Delta x$	$\frac{1}{2}$ m	Some forgot the limit statement $-\frac{1}{2}$m
$= \int_{-2}^{2}\frac{4-x^2}{6}dx = 2\int_0^2\frac{4-x^2}{6}dx = \frac{1}{3}\left[4x - \frac{x^3}{3}\right]_0^2$	1 m + $\frac{1}{2}$ m	
$= \frac{1}{3}\left(8 - \frac{8}{3}\right) = \frac{16}{9}$ unit³ #	1	

MATH. EXT 2 TRIAL 2011

MATHEMATICS Extension 2: Question 6

Suggested Solutions	Marks	Marker's Comments

Q6(a)



(i) $P(a\cos\theta,\ a\sin\theta)$

$\therefore P(a\cos\theta,\ b\sin\theta)$

$Q'(a\cos\phi,\ a\sin\phi)$

Now $\phi > \theta > 0$ and $a > b$

$\phi = \theta + \frac{\pi}{2}$

Marker's comments: $\frac{1}{2}$ For each correct P'; P Q', Q with $\angle P'OQ' = \frac{\pi}{2}$ $\boxed{2}$

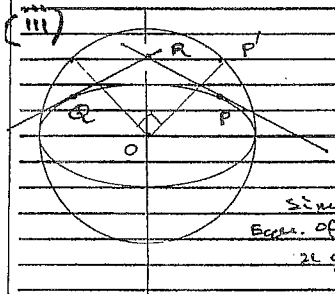
(ii) $Q = (a\cos\phi,\ b\sin\phi)$

$= \left(a\cos(\theta+\frac{\pi}{2}),\ b\sin(\theta+\frac{\pi}{2})\right)$

$Q = (-a\sin\theta,\ b\cos\theta)$

Marker's comments: $\frac{1}{2}$ For each correct ordinate. $\boxed{1}$

(iii)



$E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$P(a\cos\theta,\ b\sin\theta)$

Eqn. of tangent at P

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ——(1)

ie $bx\cos\theta + ay\sin\theta = ab$

similarly:

Eqn. of tangent at Q

$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1$ ($\frac{1}{2}$)

ie $-\frac{x\sin\theta}{a} + \frac{y\cos\theta}{b} = 1$ ——(2)

ie $-bx\sin\theta + ay\cos\theta = ab$

Solving simultaneously

(1) $\times\sin\theta$ $\quad bx\sin\theta\cos\theta + ay\sin^2\theta = ab\sin\theta$ ----(1a)

(2) $\times\cos\theta$ $\quad -bx\sin\theta\cos\theta + ay\cos^2\theta = ab\cos\theta$ ---(2a)

ADD $\quad 0 \quad ay = ab(\cos\theta + \sin\theta)$

$\therefore y = b(\cos\theta + \sin\theta) = b\sqrt{2}\cos(\theta - \frac{\pi}{4})$

subst in (1) $\quad bx\cos\theta + ab(\cos\theta+\sin\theta).\sin\theta = ab$

$\Rightarrow x = a(\cos\theta - \sin\theta) = a\sqrt{2}\cos(\theta + \frac{\pi}{4})$

$\therefore R = \left(a(\cos\theta - \sin\theta),\ b(\cos\theta + \sin\theta)\right)$

Marker's comments: 1 For getting Eqn. of tangent at P $m_T = -\frac{b\cos\theta}{a\sin\theta}$; 1 For tangent at Q; $\frac{1}{2}$ For correct method that leads to result; 1 For getting $y_R =$ or $x_R =$; $\frac{1}{2}$ For $R = \ldots\ldots$ $\boxed{4}$

(iv) LHS $= \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2(\cos\theta - \sin\theta)^2}{a^2} + \frac{b^2(\cos\theta + \sin\theta)^2}{b^2}$

$= c^2 + s^2 - 2cs + c^2 + s^2 + 2cs$

$= 2(c^2 + s^2)$

$= 2 \times 1 = 2$

$=$ RHS

$\therefore$ R is a parametric point on $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$. q.e.d.

Marker's comments: $\frac{1}{2}$ For a subst. LHS to test; $\frac{1}{2}$ For getting '2' correctly $\boxed{1}$

MATHEMATICS Extension 2: Question 6

Q6(b) (i) LHS: $\frac{1+k}{1-k} = \frac{1 + \tan x\tan(\theta - x)}{1 - \tan x\tan(\theta - x)}$ — using $\tan A = \frac{\sin A}{\cos A}$

$= \frac{\cos x\cos(\theta - x) + \sin x\sin(\theta - x)}{\cos x\cos(\theta - x) - \sin x\sin(\theta - x)}$ — 1+1

$= \frac{\cos[x - (\theta - x)]}{\cos[x + (\theta - x)]}$ — 1, 1 — using $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$= \frac{\cos(2x - \theta)}{\cos\theta}$

$=$ RHS

$\therefore \frac{1+k}{1-k} = \frac{\cos(2x - \theta)}{\cos\theta}$ — (4)

APPROACH II: $\frac{1+k}{1-k} = \frac{1 + \tan x\left(\frac{\tan\theta - \tan x}{1 + \tan\theta\tan x}\right)}{1 - \tan x\left(\frac{\tan\theta - \tan x}{1 + \tan\theta\tan x}\right)}$ — 1

$= \frac{1 + \tan\theta\tan x + \tan x\tan\theta - \tan^2 x}{1 + \tan\theta\tan x - \tan x\tan\theta + \tan^2 x}$

$= \frac{1 + 2\tan\theta\tan x - \tan^2 x}{1 + \tan^2 x}$ — 1

$= \frac{1 + 2\tan\theta\tan x - (\sec^2 x - 1)}{\sec^2 x}$

$= \cos^2 x\left[2 - \sec^2 x + 2\tan\theta.\frac{\sin x}{\cos x}\right]$ — 1

$= 2\cos^2 x - 1 + 2\sin x\cos x.\frac{\sin\theta}{\cos\theta}$

$= \cos 2x + \sin 2x.\frac{\sin\theta}{\cos\theta}$ — 1

$= \frac{\cos 2x\cos\theta + \sin 2x\sin\theta}{\cos\theta} \equiv \frac{\cos(2x - \theta)}{\cos\theta}$ q.e.d.

(ii) $\theta = \pi/3$ so $\frac{\cos(2x - \pi/3)}{\cos\pi/3} = \frac{1 + 2 + \sqrt{3}}{1 - (2 + \sqrt{3})} = \frac{3 + \sqrt{3}}{-1 - \sqrt{3}}$ — 1 For subst + simplifying

$k = 2 + \sqrt{3}$

$\therefore \frac{\cos(2x - \frac{\pi}{3})}{\frac{1}{2}} = -\frac{(3 + \sqrt{3})}{1 + \sqrt{3}} = -\frac{(3 + \sqrt{3})}{\sqrt{3} + 1} = -\frac{\sqrt{3}(\sqrt{3}+1)}{\sqrt{3}+1} = -\sqrt{3}$ * ($\frac{1}{2}$)

$\cos(2x - \frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

$\therefore 2x - \frac{\pi}{3} = 2m\pi \pm \left(\frac{5\pi}{6}\right)$, $m \in \mathbb{Z}$ — 1

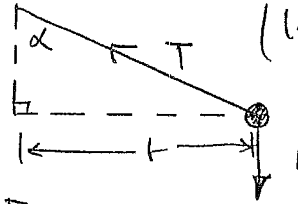
$x = \begin{cases} m\pi + 7\pi/12 & (12m + 7)\frac{\pi}{12} \\ m\pi - \pi/4 & (4m - 1)\frac{\pi}{4} \end{cases}$ — 1 SOLUTIONS $\boxed{3}$

$2x = 2m\pi \pm \frac{5\pi}{6} + \frac{\pi}{3}$

$x = m\pi \pm \frac{5\pi}{12} + \frac{\pi}{6}$

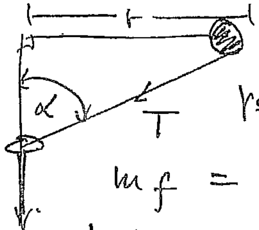
Extension (2)

MATHEMATICS: Question……. (7)

Suggested Solutions	Marks	Marker's Comments
(a) (i)  (length = L)		[4]
$\sum F_y = 0$		1 mark each for the resolution of T.
$T\cos\alpha - mg = 0$ —— (1)	1	
$T\sin\alpha = mr\omega^2$ —— (2)	1	
$\therefore \sin\alpha = \frac{r}{L}$ —— (3)		
$\therefore T\cancel{r}/L = m\cancel{r}\omega^2$ —— (4)	1	$T = mL\omega^2$.
Substitute (4) into (1)		
$\therefore \cancel{m}L\omega^2\cos\alpha = \cancel{m}g$		
$\Longrightarrow \cos\alpha = \frac{g}{L\omega^2}$.	1	$\cos\alpha = \frac{g}{L\omega^2}$.
(ii) $0 \le \cos\alpha \le 1$		
i.e $0 \le \frac{g}{L\omega^2} \le 1 \Rightarrow g \le L\omega^2$	1	Implication for $\cos\alpha > 1$ $\therefore$ motion impossible.
If $g > L\omega^2 \Rightarrow \cos\alpha > 1$ i.e circular motion is impossible.		
Also, if ω increased $\cos\alpha\left(\frac{g}{L\omega^2}\right)$ decreased	1	α increases
$\Longrightarrow \alpha$ is increased i.e when $\cos\alpha \to 0$ $\alpha \to \frac{\pi}{2}$.		

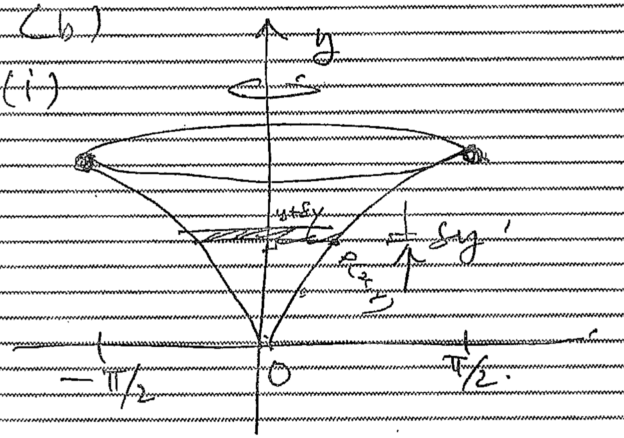
Extension (2)

MATHEMATICS: Question…… (7)

Suggested Solutions	Marks	Marker's Comments
(iii)  length = L		
Vertically: $mf = mg + T\cos\alpha$ —— (1)	1	resolve the forces correctly in vert.
Horizontally: $T\sin\alpha = mr\omega^2$, $\sin\alpha = \frac{r}{L}$		
$T\frac{\cancel{r}}{L} = m\cancel{r}\omega^2$		
$\therefore T = mL\omega^2$ —— (2)	1	correctly subst. (2) & (3) into (1)
but $\cos\alpha = \frac{g}{L\omega^2}$ —— (3)		
Substitute (2) & (3) into (1)		
$mf = mg + \left(m\cancel{L\omega^2} \times \frac{g}{\cancel{L\omega^2}}\right)$	1	Correct solution.
$mf = 2mg$		
$\therefore f = 2g$.		

(8)

MATHEMATICS Extension 2: Question.....7 (b)

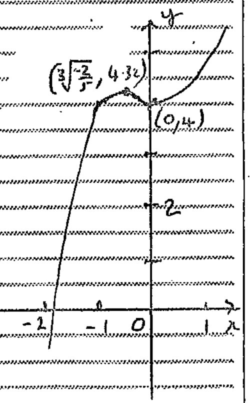
Suggested Solutions	Marks	Marker's Comments
(b) (i)		[3 mar
		
$\delta V = \pi x^2 \delta y$	1	
$V = \lim_{\delta y \to 0} \pi \sum_0^1 x^2 \delta y$		
$\therefore V = \pi \int_0^1 x^2 \, dy$	1	
$y = \sin x$, $\frac{dy}{dx} = \cos x$		
$\therefore V = \pi \int_0^{\pi/2} x^2 \cos x \, dx$	1	
(ii) $V = \pi \int_0^{\pi/2} x^2 \frac{d}{dx}(\sin x \, dx)$		
$= \pi \left[x^2 \sin x \right]_0^{\pi/2} - 2\int_0^{\pi/2} x \sin x \, dx$	1	
$2\int_0^{\pi/2} x \frac{d}{dx}(-\cos x) \, dx$	1	
$= \left[-2x\cos x \right]_0^{\pi/2} + 2\int_0^{\pi/2} \cos x \, dx$		
$\therefore V = \pi \left[\frac{\pi^2}{4} - 2 \right]$	1	

TRIAL 2011

MATHEMATICS Extension 1: Question 8

Suggested Solutions	Marks	Marker's Comments
a) ${}^{n}C_4 = 5\,{}^{n-2}C_3$	1	
$\frac{n!}{(n-4)!\,4!} = \frac{5\,(n-2)!}{(n-5)!\,3!}$		
$\frac{n(n-1)}{(n-4)\,4} = 5$		
$n(n-1) = 20(n-4)$ $n^2 - n = 20n - 80$ $n^2 - 21n + 80 = 0$	1	
$(n-16)(n-5) = 0$		
$n = 5$ or 16	1	
b) $0 < \tan^{-1}5 < \pi/2$ $0 < \tan^{-1}3 < \pi/2$		Many people proved only that tan(LHS) = tan(RHS) without any mention of restrictions.
$\therefore 0 < \tan^{-1}3 + \tan^{-1}5 < \pi$	1	
$\tan(\tan^{-1}3 + \tan^{-1}5) = \frac{3+5}{1-15} = -\frac{4}{7}$	1	Maximum of 2 in that case
$\therefore \tan^{-1}3 + \tan^{-1}5 = \tan^{-1}\left(-\frac{4}{7}\right) + n\pi$		(i.e. 1 mark for general solution or equivalent.
But as shown above $0 < \tan^{-1}3 + \tan^{-1}5 < \pi$		
$\therefore n = 1$	1	1 mark for earlier restriction
$\tan^{-1}3 + \tan^{-1}5 = \tan^{-1}\left(-\frac{4}{7}\right) + \pi$		
$\tan^{-1}3 + \tan^{-1}5 = -\tan^{-1}\left(\frac{4}{7}\right) + \pi$		
$\therefore \tan^{-1}3 + \tan^{-1}5 + \tan^{-1}\left(\frac{4}{7}\right) = \pi$	1	
c) i) Differentiate implicitly with respect to x: $2x + 2y + 2x\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0$	1	
$\frac{dy}{dx}(2x + 5y^4) = -2x - 2y$		
$\frac{dy}{dx} = \frac{-2(x+y)}{2x + 5y^4}$	½	

MATHEMATICS Extension 1: Question 8 (cont)

Suggested Solutions	Marks	Marker's Comments
For horizontal tangent $\frac{dy}{dx} = 0$	½	
$\Rightarrow x = -y$.		
i.e. at P $Y = -X$		
But P is on the curve so		
$X^2 + 2XY + Y^5 = 4$		
Substitute $Y = -X$ $\quad X^2 - 2X^2 - X^5 = 4$ $X^5 + X^2 + 4 = 0$	½	
i.e. X is a root of $x^5 + x^2 + 4 = 0$	½	
ii) Let $f(x) = x^5 + x^2 + 4$		
$f(-2) = -32 + 4 + 4 = -24$ $f(-1) = -1 + 1 + 4 = 4$	½	
f changes sign and since f is continuous between $-2 < -1$, there must be a root in that domain.	½	½ reserved for word continuous
iii) $f'(x) = 5x^4 + 2x = 0$ at S. Point $x(5x^3 + 2) = 0$ $x = 0$ or $x = \sqrt[3]{\frac{-2}{5}}$ (≈ -0.74)	1	turning points
$f''(x) = 20x^3 + 2$ $f''(0) = 2 > 0$ $\therefore$ Concave up $\therefore$ Local min at $(0,4)$		
$f''\left(\sqrt[3]{\frac{-2}{5}}\right) = -6 \quad < 0$ $\therefore$ Concave down $\therefore$ Local max at $\left(\sqrt[3]{\frac{-2}{5}}, 4.32\right)$	1	nature of turning point
 	1	graph
As there is no other minimum than that at $x=0$ and $f(0) > 0$, there can be no further roots for $x > 0$. As there are no other turning points, the only root is that between $x=-2$ and $x=-1$ so this must be X.	1	conclusion mentioning only one minimum with that value positive.