J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION 2

MATHEMATICS

2013

Weighting: 15% (HSC Assessment Mark)

NAME: ____________________

MARK: / 37

Time Allowed: 50 minutes

Topics: Integration, Volume, Conics & Harder 3 Unit.

Directions:

- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks <u>may not</u> be awarded for careless or badly arranged work

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TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

Note: $\ln x = \log_e x, \quad x > 0$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Marks

Question 1 (15 marks) **[START A NEW PAGE]**

(a) By completing the square, find $\int \frac{1}{x^2-2x+10}dx$ **2**

(b) Find real numbers a, b and c such that

i.

$$\frac{x^2-11}{(3x-1)(x+2)^2} \equiv \frac{a}{3x-1}+\frac{b}{x+2}+\frac{c}{(x+2)^2}.$$ **3**

ii. Hence, or otherwise find $\int \frac{x^2-11}{(3x-1)(x+2)^2}dx$ **2**

(c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{13\sin\theta+5}d\theta$ using the substitution $t=\tan\frac{\theta}{2}$. **4**

(d)

i. Let $I_n=\int_0^x \cos^n t\,dt$, where $0\le x\le \frac{\pi}{2}$.

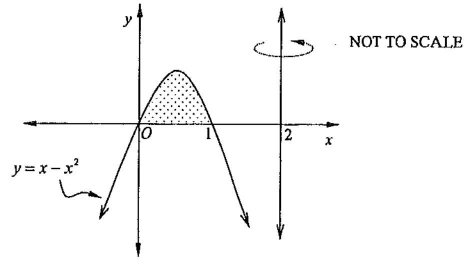
Show that $I_n=\left(\frac{n-1}{n}\right)I_{n-2}$ with $n\ge 2$. **2**

ii. Hence, otherwise, find the exact value I_4. **2**

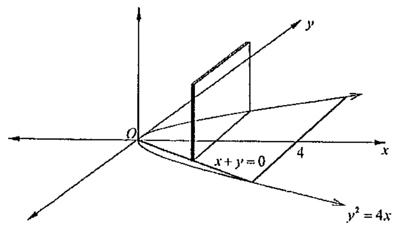
Marks

Question 2 (13 marks) **[START A NEW PAGE]**

(a) Using the method of cylindrical shells, find the volume of the solid
3 formed by rotating the region bounded by $y=x-x^2$ and $y=0$ about the line $x=2$.



(b) The base of the solid is the region bounded by the curve $y^2=4x$ and the lines $x+y=0$ and $x=4$. Every cross-sectional slice perpendicular to the x axis is a square having a side with one end-point on the line $x+y=0$ and the other on the curve $y^2=4x$.



i. Show that the area of the cross-sectional is given by $A(x)=4x+x^2-4x^{\frac{3}{2}}$. **2**

ii. Hence find the volume of the solid formed. **3**

Marks

Question 2 continued

(c)

i. Show that if $a, b > 0, \frac{a}{b} + \frac{b}{a} \geq 2$. **1**

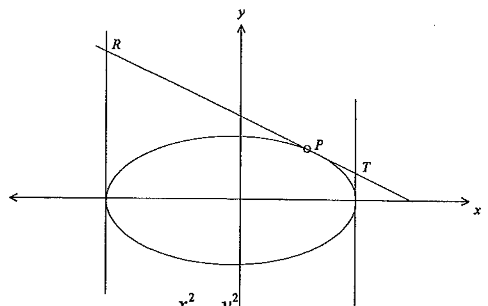
ii. Show that if $a, b > 0, (a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$. **1**

iii. Hence, or otherwise, show that if $a, b, c > 0, (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$. **3**

Question 3 (9 marks) **[START A NEW PAGE**

Marks

(a)



The point P lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$. The tangent at P meets the tangents at the ends of the major axis at R and T.

(i) Use the parametric representation of an ellipse to show the equation of the tangent is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. **2**

(ii) Show that RT subtends a right angle at either focus. **3**

(b) For the hyperbola H with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, Q and R are the points of intersection between the x-axis and the directrices. The x coordinate of Q is positive.

(i) Show that the equation of the tangent to H at the point $P(a\sec\theta, b\tan\theta)$ is **2**

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

(ii) Find the equations, in terms of θ and e, of the tangents to H at P that pass through Q and R. **2**

Ext 2 HSC Task 3 solutions 13

a)

$$\int \frac{1}{x^2-2x+10}dx = \int \frac{dx}{(x-1)^2+3^2}$$

$$= \ln\left((x-1)+\sqrt{(x-1)^2+3^2}\right)+c$$

$$= \ln\left((x-1)+\sqrt{x^2-2x+10}\right)+c$$

b) (i) By partial fractions

$$\frac{x^2-11}{(3x-1)(x+2)^2} \equiv \frac{a}{(3x-1)}+\frac{b}{(x+2)}+\frac{c}{(x+2)^2} = \frac{a(x+2)^2+b(3x-1)(x+2)+c(3x-1)}{(3x-1)(x+2)^2}$$

$$x^2-11 = a(x+2)^2+b(3x-1)(x+2)+c(3x-1)$$

Let $x=-2$

$$\therefore (-2)^2-11 = c(3(-2)-1)$$

$$\therefore c=1$$

Let $x=\frac{1}{3}$

$$\therefore \left(\frac{1}{3}\right)^2-11 = a\left(\frac{1}{3}+2\right)^2$$

$$\therefore \frac{1}{9}-11 = a\left(\frac{7}{3}\right)^2$$

$$-\frac{98}{9} = \frac{49}{9}a$$

$$-2=a$$

Let $x=0$

$$-11=-2(2)^2+b(-1)(2)+1(-1)$$

$$\therefore\ 1=b$$

(ii) $\int \frac{x^2-11}{(3x-1)(x+2)^2}\,dx$

$$= \int \frac{-2}{3x-1}+\frac{1}{x+2}+\frac{1}{(x+2)^2}\,dx$$

$$= -\frac{2}{3}\ln(3x-1)+\ln(x+2)+\frac{-1}{x+2}+c$$

C)hih

$t=\tan\frac{\theta}{2}$

$$dt=\frac{1}{2}\sec^2\frac{\theta}{2}d\theta$$

$$dt=\frac{1}{2}(1+t^2)d\theta$$

$$d\theta=\frac{2}{1+t^2}dt$$

When $\theta=0$ then $t=0$ and when $\theta=\frac{\pi}{2}$ then $t=1$

$$13\sin\theta+5 = 13\left[\frac{2t}{(1+t^2)}\right]+5$$

$$= \frac{26t+5+5t^2}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{13\sin\theta+5}d\theta = \int_0^1 \frac{1}{\frac{26t+5+5t^2}{1+t^2}}\times\frac{2}{1+t^2}dt$$

$$= \int_0^1 \frac{1+t^2}{26t+5+5t^2}\times\frac{2}{1+t^2}dt$$

$$= \int_0^1 \frac{2}{(5t+1)(t+5)}dt$$

$$= \int_0^1 \frac{a}{(5t+1)}+\frac{b}{(t+5)}dt$$

To find the values of a and b.

$$a(t+5)+b(5t+1)=2$$

$$(a+5b)t+(5a+b)=2$$

Hence $a+5b=0$ - (1)

$5a+b=2$ - (2)

Eqn (1) × 5

$5a+25b=0$ - (3)

Eqn (3)-(2)

$$24b=-2,\ b=-\frac{1}{12} \text{ and } a=\frac{5}{12}$$

$$\int_0^1 \frac{a}{(5t+1)} + \frac{b}{(t+5)}\,dt = \int_0^1 \frac{\frac{5}{12}}{(5t+1)} - \frac{\frac{1}{12}}{(t+5)}\,dt$$

$$= \frac{1}{12}\left[\ln(5t+1) - \ln(t+5)\right]_0^1$$

$$= \frac{1}{12}\left[\ln 6 - \ln 1 - \ln 6 + \ln 5\right]$$

$$= \frac{1}{12}\ln 5$$

$$I_n = \int_0^{\frac{\pi}{2}} \cos^{n-1} t \cos t dt$$

$$= \left[\cos^{n-1} t \sin t\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} t \sin^2 t dt$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} t \sin^2 t dt$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} t(1-\cos^2 t)dt$$

$$= (n-1)\int_0^{\frac{\pi}{2}} (\cos^{n-2} t - \cos^n t)dt$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} tdt - (n-1)\int_0^{\frac{\pi}{2}} \cos^n tdt$$

Using the original integral

$$\int_0^{\frac{\pi}{2}} \cos^n tdt = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} tdt - n\int_0^{\frac{\pi}{2}} \cos^n tdt + \int_0^{\frac{\pi}{2}} \cos^n tdt$$

$$n\int_0^{\frac{\pi}{2}} \cos^n tdt = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} tdt$$

$$\int_0^{\frac{\pi}{2}} \cos^n tdt = \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} tdt$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

$$I_4 = \frac{(4-1)}{4} I_{4-2}$$

$$= \frac{3}{4}\int_0^{\frac{\pi}{2}} \cos^2 tdt$$

$$= \frac{3}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2t)dt$$

$$= \frac{3}{8}\left[(x + \frac{\sin 2t}{2})\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{8}\left[(\frac{\pi}{2} + \frac{\sin 0}{2}) - (0 + \frac{\sin 0}{2})\right]$$

$$= \frac{3\pi}{16}$$

Question 2

Consider a slice parallel to the line $x = 2$ with thickness Δx

The slice is rotated about $x = 2$ to form a cylindrical shell with radius $2 - x$ and height $x - x^2$

Volume of shell $\Delta V = 2\pi(2-x)(x-x^2)\Delta x$

$$\therefore V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi(2-x)(x-x^2)\Delta x$$

$$= 2\pi\int_0^1 \left(2x - 3x^2 + x^3\right)dx$$

$$= 2\pi\left[x^2 - x^3 + \frac{x^4}{4}\right]_0^1$$

$$= 2\pi\left(1 - 1 + \frac{1}{4} - (0 - 0 + 0)\right)$$

$$= \frac{\pi}{2} \text{ cubic units}$$

iswer.

The cross-section is a square with side length $y_1 + y_2$ where y_1 is on the curve $y^2 = 4x$ and y_2 is on the line $x + y = 0$.

$$A(x) = (y_1 + y_2)^2$$

$$= (2\sqrt{x} - x)^2$$

$$= 4x + x^2 - 4x^{\frac{3}{2}}$$

Volume of slice $\Delta V = \left(4x + x^2 - 4x^{\frac{3}{2}}\right)\Delta x$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{4} \Delta V$$

$$= \lim_{\Delta x \to 0} \sum_{x=0}^{4} \left(x^2 + 4x - 4x^{\frac{3}{2}}\right)\Delta x$$

$$= \int_0^4 \left(x^2 + 4x - 4x^{\frac{3}{2}}\right) dx$$

$$= \left[\frac{x^3}{3} + 2x^2 - \frac{8x^{\frac{5}{2}}}{5}\right]_0^4$$

$$= \left(\frac{64}{3} + 32 - \frac{8 \times 32}{5}\right) - (0)$$

$$= \frac{32}{15} \text{ cubic units}$$

Assume $\frac{a}{b} + \frac{b}{a} < 2$

$\therefore a^2 + b^2 < 2ab \Rightarrow a^2 - 2ab + b^2 < 0$

ie $(a-b)^2 < 0$ which is not true

$\therefore \frac{a}{b} + \frac{b}{a} \geq 2$

$$\text{LHS} = (a+b)\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$= a \times \frac{1}{a} + a \times \frac{1}{b} + b \times \frac{1}{a} + b \times \frac{1}{b}$$

$$= 2 + \frac{a}{b} + \frac{b}{a}$$

≥ 4 using (i)

Using (i) three times: $\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} \geq 2 + 2 + 2$

$$\therefore \frac{a+c}{b} + \frac{b+c}{a} + \frac{a+b}{c} \geq 6$$

$$1 + \frac{a+c}{b} + 1 + \frac{b+c}{a} + 1 + \frac{a+b}{c} \geq 9$$

$$\frac{a+b+c}{b} + \frac{a+b+c}{a} + \frac{a+b+c}{c} \geq 9$$

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$$

Question 3

To find the equation of tangent through *P*

$x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= b\cos\theta \times \frac{1}{-a\sin\theta}$$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

Equation of the tangent

$$y - y_1 = m(x - x_1)$$

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

At *T* $x = a$ then $\frac{a}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$$\frac{y}{b}\sin\theta = 1 - \cos\theta$$

$$y = \frac{b(1-\cos\theta)}{\sin\theta}$$

At *R* $x = -a$ then similarly $y = \frac{b(1+\cos\theta)}{\sin\theta}$

Gradients of lines at the focus *S*(*ae*,0)

Gradient RS $\times$ Gradient TS

$$= \frac{\frac{b(1+\cos\theta)}{\sin\theta} - 0}{-a - ae} \times \frac{\frac{b(1-\cos\theta)}{\sin\theta} - 0}{a - ae}$$

$$= \frac{b(1+\cos\theta)}{-a(1+e)\sin\theta} \times \frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$$

$$= \frac{b^2(1-\cos^2\theta)}{-a^2(1-e^2)\sin^2\theta}$$

$$= -1$$

Gradients of lines at the focus S'(-*ae*,0)

Gradient RS' $\times$ Gradient TS'

$$= \frac{\frac{b(1+\cos\theta)}{\sin\theta} - 0}{-a + ae} \times \frac{\frac{b(1-\cos\theta)}{\sin\theta} - 0}{a + ae}$$

$$= \frac{b(1+\cos\theta)}{-a(1-e)\sin\theta} \times \frac{b(1-\cos\theta)}{a(1+e)\sin\theta}$$

$$= \frac{b^2(1-\cos^2\theta)}{-a^2(1-e^2)\sin^2\theta}$$

$$= -1$$

(b) (i) At P, $x = a\sec\theta$

$$\frac{dx}{d\theta} = -a(\cos\theta)^{-2} \times -\sin\theta$$

$$= a\sec\theta\tan\theta$$

$$y = b\tan\theta$$

$$\frac{dy}{d\theta} = b\sec^2\theta$$

So, $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \cdot \dfrac{d\theta}{dx}$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$

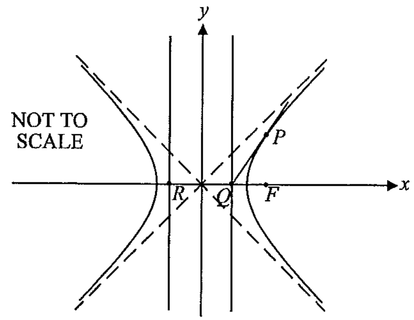
The equation of the tangent at P is

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

$$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$$

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \text{ as required}$$



The equation of the tangent to H at P through $Q\left(\frac{a}{e}, 0\right)$ is

$$y - 0 = \frac{b\sec\theta}{a\tan\theta}\left(x - \frac{a}{e}\right)$$

$$ay\tan\theta = bx\sec\theta - \frac{ab}{e}\sec\theta$$

$$bx\sec\theta - ay\tan\theta = \frac{ab}{e}\sec\theta$$

$$bx - ay\sin\theta = \frac{ab}{e}$$

The equation of the tangent to H at P through $R\left(-\frac{a}{e}, 0\right)$ is

$$y - 0 = \frac{b\sec\theta}{a\tan\theta}\left(x + \frac{a}{e}\right)$$

$$ay\tan\theta = bx\sec\theta + \frac{ab}{e}\sec\theta$$

$$bx\sec\theta - ay\tan\theta = -\frac{ab}{e}\sec\theta$$

$$bx - ay\sin\theta = \frac{-ab}{e}$$