Neap:

HSC Trial Examination 2010

Mathematics Extension 1

This paper must be kept under strict security and may only be used on or after the afternoon of Thursday 12 August, 2010 as specified in the Neap Examination Timetable.

General Instructions

Reading time – 5 minutes

Working time – 2 hours

Write using black or blue pen

Board-approved calculators may be used

A table of standard integrals is provided at the back of this paper

All necessary working should be shown in every question

Total marks – 84

Attempt questions 1–7

All questions are of equal value

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2010 HSC Mathematics Extension 1 Examination.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

Total marks 84
Attempt Questions 1–7
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 Marks) Use a SEPARATE writing booklet.

(a) A is the point $(-2, 4)$ and B is the point $(2, 1)$.

Find the point P which divides the interval AB externally in the ratio $3 : 2$. **2**

(b) Solve for x, $\dfrac{2}{x+1} \geq 3$. **2**

(c) Evaluate $\displaystyle\int_0^4 \frac{dx}{x^2+16}$. **2**

(d) Find the acute angle between the curves $y = x^2 - 1$ and $y = 2x - 1$ at their point of intersection in the first quadrant. Answer to the nearest degree. **3**

(e) Solve for x, $\left(x+\dfrac{1}{x}\right)^2 - 5\left(x+\dfrac{1}{x}\right) + 6 = 0$. **3**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\lim_{\theta \to 0} \frac{\tan 2\theta}{8\theta}$. **1**

(b) Given that the velocity of a particle is $v = e^{-x}$, find the expression for the acceleration of the particle. **2**

(c) (i) Sketch the graph of $y = |x-2|$. **1**

(ii) For what values of x is $|x-2| < \frac{1}{2}x$? **2**

(d) Evaluate the following integral $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 3x \, dx$, leaving your answer as an exact value. **3**

(e) By writing $y = \tan^{-1}\sqrt{x}$ in the form $x = f(y)$, show that $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}(1+x)}$. **3**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Prove that $\frac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A$. **3**

(b) Find the first derivative of $y = 2x \ln \sqrt{\cos x}$. **3**

(c) Thirteen students are to represent Year 12 on a social committee.

This committee consists of six boys and seven girls. A sub-committe of four girls and three boys is to be formed to organise the food for the social.

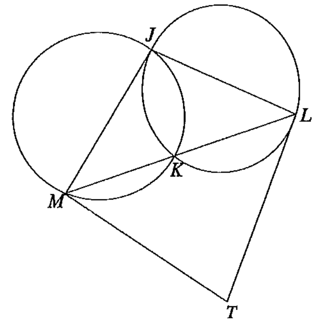
What is the probability that one girl, Jill, is **not** included and one boy, Jack, **will** be included on this sub-committee? **3**

(d) Find the exact vale of $\int_2^3 \frac{x}{(x-1)^2} dx$ using the substitution $u = x - 1$. **3**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) The circles intersect at J and K; LKM is a straight line; TL and TM are tangents.



Prove that $TMJL$ is a cyclic quadrilateral. **3**

(b) Find the term independent of x in the expansion of $\left(x^2 - \frac{2}{x^4}\right)^6$. **3**

(c) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal to the curve at the point P is: $x + py = 2ap + ap^3$. **2**

(ii) Find the coordinates of the point Q where the normal at P meets the y-axis. **1**

(iii) Determine the coordinates of R, the midpoint of PQ. **1**

(iv) Find the Cartesian equation of the locus of R and describe the locus. **2**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Sketch the graph of $P(x) = x^3 - 4x^2$. **1**

(ii) Find the domain of $\ln(x^3 - 4x^2)$. **1**

(b) Use one application of Newton's method to find an approximation to the solution of $\cos x = x$ which lies near 0.5 (answer to 2 decimal places). **2**

(c) A hard-boiled egg cools according to Newton's Law of Cooling $\frac{dT}{dt} = k(T - A)$, where A is the surrounding temperature and T is the temperature of the egg.

(i) Show that $T = A + Be^{kt}$ is a solution to $\frac{dT}{dt} = k(T - A)$. **1**

(ii) A hard-boiled egg at 95°C is placed into water to cool. The water temperature is 15°C. After 5 minutes the egg's temperature is 40°C.

Assuming that the water remains at the same temperature, how long will it take for the egg to cool to 25°C? Give your answer to the nearest minute. **3**

(d) Prove by the method of Mathematical Induction that $n(n+3)$ is a multiple of 2 for all values of n, where n is a positive integer. **4**

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Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) The roots of the polynomial equation $x^3 - 4x^2 - 11x + 30 = 0$, are α, β and γ.

Given that $\alpha = \beta + \gamma$, solve the equation $P(x) = 0$. **3**

(b) A particle is moving in a straight line so that its displacement x metres from a fixed point on the line at any time t seconds is given by $x = \frac{3}{2}\sin 2t + 2\cos 2t$.

(i) Show that $\ddot{x} = -n^2 x$, and describe the motion of the particle. **2**

(ii) Find the maximum displacement of the particle. **2**

(c) (i) State the domain and range for $y = \cos^{-1} 4x$. **1**

(ii) Sketch the graph of $y = \cos^{-1} 4x$. **1**

(iii) Find the equation of the tangent to $y = \cos^{-1} 4x$ at $x = -\frac{1}{8}$. Leave your answer in exact form. **3**

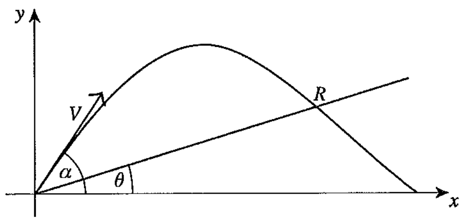
Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) A spherical balloon is deflating so that its volume is decreasing at the constant rate of 20 mm^3 per second.

What is the rate of decrease of its surface area when the radius is just 10 mm? **5**

(b)



A rocket launcher fires missiles from O with velocity V at an angle α above the horizontal, where $\theta > \alpha$. The missile strikes the road at R. Air resistance is ignored, and the acceleration due to gravity is g.

(i) If the missile is at some point (x, y) at time t, write expressions for x and y in terms of t. Hence, show that the equation of the trajectory of the missile is $y = x\tan\alpha - \dfrac{gx^2\sec^2\alpha}{2V^2}$. **2**

(ii) If R is the point (X, Y), express X and Y in terms of OR and θ. Hence, show that the range OR of the missile up the track is given by $OR = \dfrac{2V^2\cos\alpha \times \sin(\alpha - \theta)}{g\cos^2\theta}$. **2**

(iii) Show that OR is a maximum when $\alpha = \dfrac{\theta}{2} + \dfrac{\pi}{4}$. **2**

(iv) Hence show that the maximum value of OR is $\dfrac{V^2}{g(1 + \sin\theta)}$. **1**

End of paper

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Mathematics Extension 1

Solutions and marking guidelines

Question 1

	Syllabus outcomes and marking guide
(a) $P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ $m:n=3:-2$ $=P\left(\frac{3(2)+(-2)(-2)}{3+(-2)}, \frac{3(1)+(-2)(4)}{3+(-2)}\right)$ $=P(10,-5)$	PE2 • Correctly uses formula. AND • Gives correct answer 2
(b) $\frac{2}{x+1} \geq 3$, $x - 1$ $(x+1)^2 \times \frac{2}{x+1} \geq 3(x+1)^2$ $2(x+1) \geq 3(x+1)^2$ $3(x+1)^2 - 2(x+1) \leq 0$ $(x+1)[3(x+1)-2] \leq 0$ $(x+1)(3x+1) \leq 0$ $-1 < x \leq -\frac{1}{3}$	PE3 • Shows correct factorisation. AND • Gives correct solution .... 2 • Shows correct factorisation. OR • Gives correct solution 1
(c) $\int_0^4 \frac{dx}{x^2+16} = \frac{1}{4}\left[\tan^{-1}\left(\frac{x}{4}\right)\right]_0^4$ using standard integral $=\frac{1}{4}[\tan^{-1}1 - \tan^{-1}0]$ $=\frac{1}{4}\left[\frac{\pi}{4} - 0\right]$ $=\frac{\pi}{16}$	HE4, HE6 • Gives correct standard integral. AND • Shows correct substitution and simplification .... 2 • Gives correct standard integral. OR • Shows correct substitution and simplification. 1

Question 1 **(Continued)**

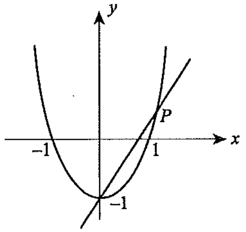
Syllabus outcomes and marking guide

(d) $y = x^2 - 1$ $\quad y = 2x - 1$

$y' = 2x$ $\quad y' = 2$

Point of intersection is

$$x^2 - 1 = 2x - 1$$
$$x^2 - 2x = 0$$
$$x(x - 2) = 0$$
$$x = 0, 2$$
$$\therefore P(2, 3)$$



$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$
$$= \left|\frac{2 - 4}{1 + 2 \times 4}\right|$$
$$= \frac{2}{9}$$
$$\theta = \tan^{-1}\left(\frac{2}{9}\right)$$
$$\theta = 12.53° = 13° \text{ (nearest degree)}$$

PE3
- Correctly completes:
  - point of intersection
  - solving m_1 and m_2
  - formula and solution 3
- Correctly completes two of the above . . . 2
- Correctly completes one of the above . . . 1

(e) Put $M = x + \frac{1}{x}$

$$\therefore M^2 - 5M + 6 = 0$$
$$(M - 2)(M - 3) = 0$$
$$M = 2 \text{ or } 3$$
$$\therefore x + \frac{1}{x} = 2 \quad \text{or} \quad x + \frac{1}{x} = 3$$
$$x^2 - 2x + 1 = 0 \quad \text{or} \quad x^2 - 3x + 1 = 0$$
$$(x - 1)^2 = 0 \quad \text{or} \quad x = \frac{3 \pm \sqrt{5}}{2}$$
$$\therefore x = 1 \quad \text{or} \quad x = \frac{3 \pm \sqrt{5}}{2}$$

PE6
- Correctly completes:
  - factorisation
  - solving $x + \frac{1}{x} = 2$
  - solving $x + \frac{1}{x} = 3$ 3
- Correctly completes two of the above . . . 2
- Correctly completes one of the above . . . 1

Question 2

Syllabus outcomes and marking guide

(a) $\lim\limits_{\theta \to 0} \frac{\tan 2\theta}{8\theta} = \frac{1}{4} \cdot \lim\limits_{\theta \to 0} \frac{\tan 2\theta}{2\theta}$

$$= \frac{1}{4}$$

PE2
- Gives correct answer 1

(b) Given $v = e^{-x}$

$$\frac{dv}{dx} = -e^{-x}$$
$$v\frac{dv}{dx} = e^{-x} \times -e^{-x}$$
$$a = -e^{-2x}$$

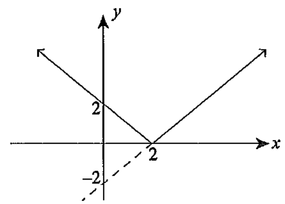
Alternatively, $v = e^{-x}$

$$v^2 = e^{-2x}$$
$$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \times -2e^{-2x}$$
$$a = -e^{-2x}$$

HE5
- Correctly gives $\frac{dv}{dx}$ or $\frac{1}{2}v^2$.

AND
- Gives the expression for acceleration . . . 2
- Only gives $\frac{dv}{dx}$ or $\frac{1}{2}v^2$ 1

(c) (i)



PE4
- Correctly sketches $y = |x - 2|$ 1

Question 2 (Continued)

(ii) $|x-2| < \frac{1}{2}x$

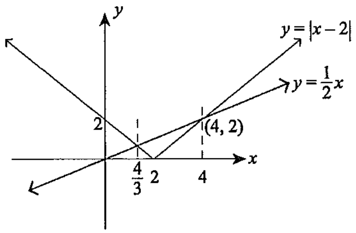
$x - 2 < \frac{1}{2}x$ and $x - 2 > -\frac{1}{2}x$

$2x - 4 < x$ $\quad$ $2x - 4 > -x$

$x < 4$ $\quad$ $3x > 4$

$\therefore$ Solution is: $\frac{4}{3} < x < 4$

Alternatively (using graph in (c)(i),



$x - 2 = \frac{1}{2}x$ and $-x + 2 = \frac{1}{2}x$

$\frac{1}{2}x = 2$ $\quad$ $\frac{3}{2}x = 2$

$x = 4$ $\quad$ $x = \frac{4}{3}$

$\therefore \frac{4}{3} < x < 4$

Syllabus outcomes and marking guide

PE4

- Correctly finds $x < 4$ and $x > \frac{4}{3}$ 2
- Correctly finds $x < 4$ or $x > \frac{4}{3}$ 1

(d) $\int_0^{\frac{\pi}{4}} \cos^2 3x \, dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}[1 + \cos 6x]dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 6x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{6}\sin\frac{3\pi}{2}\right] - \frac{1}{2}[0 + 0]$

$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{6}\right]$ or $\frac{3\pi - 2}{12}$

HE6, HE7

- Shows correct integration, substitution and correctly gives the answer 3
- Any two of the above................ 2
- Any one of the above................ 1

Question 2 (Continued)

(e) $y = \tan^{-1}\sqrt{x}$

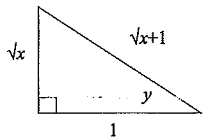
$\tan y = \sqrt{x}$

$x = (\tan y)^2$

$\frac{dx}{dy} = 2(\tan y)^1 \times \sec^2 y$

$= \frac{2\tan y}{\cos^2 y}$

$\therefore \frac{dy}{dx} = \frac{\cos^2 y}{2\tan y}$



Or, $\frac{dx}{dy} = \frac{2\tan y}{\cos^2 y}$

$= \tan y \sec^2 y$

$= 2\tan y(1 + \tan^2 y)$

$= 2\sqrt{x}(1 + x)$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}(1 + x)}$

Since $\tan y = \sqrt{x}$

$\frac{dy}{dx} = \frac{\left(\frac{1}{\sqrt{x+1}}\right)^2}{2\sqrt{x}}$

$= \frac{1}{2\sqrt{x}(x+1)}$, as required.

Syllabus outcomes and marking guide

HE4

- Correctly completes:
  - writing $x = f(y)$
  - correct derivative
  - showing result 3
- Correctly completes two of the above . . . 2
- Correctly completes one of the above . . . 1

Question 3

	Sample answer	Syllabus outcomes and marking guide

(a)	$\dfrac{\cos A + \sin A}{\cos A - \sin A} \times \dfrac{\cos A + \sin A}{\cos A + \sin A}$
	$= \dfrac{\cos^2 A + 2\cos A \cdot \sin A + \sin^2 A}{\cos^2 A - \sin^2 A}$
	$= \dfrac{1 + \sin 2A}{\cos 2A}$
	$= \dfrac{1}{\cos 2A} + \dfrac{\sin 2A}{\cos 2A}$
	$= \sec 2A + \tan 2A$, as required
		PE4, H5
		• Correctly completes:
		  • simplification process
		  • using double angle results
		  • showing the required expression . . 3
		• Correctly completes two of the above . . . 2
		• Correctly completes one of the above . . . 1

(b)	$y = 2x \cdot \ln\sqrt{\cos x}$
	$= 2x \cdot \frac{1}{2}\ln(\cos x)$
	$\therefore y = x\ln(\cos x)$
	Using product rule,
	$\dfrac{dy}{dx} = 1 \cdot \ln(\cos x) + x \cdot \dfrac{-\sin x}{\cos x}$
	$= \ln(\cos x) - x\tan(x)$
		H5
		• Correctly completes:
		  • simplifying $y = f(x)$
		  • derivative
		  • simplification. 3
		• Correctly completes two of the above . . . 2
		• Correctly completes one of the above . . . 1

(c)	Committee of 6 boys and 7 girls.
	Total number of different committees:
	${}^7C_4 \times {}^6C_3$
	$= \dfrac{7!}{4!3!} \times \dfrac{6!}{3!3!}$
	$= 700$
	Committees **without** Jill and **with** Jack:
	${}^6C_4 \times {}^5C_2$
	$= \dfrac{6!}{4!2!} \times \dfrac{5!}{2!3!}$
	$= 150$
	$\therefore$ Probability that the sub-committee includes Jack and excludes Jill $= \dfrac{150}{700}$
	$= \dfrac{3}{14}$
		PE3, HE3
		• Correctly:
		  • determines total number of different committees
		  • determines number of committees satisfying conditions
		  • finds probability. 3
		• Correctly completes two of the above . . . 2
		• Correctly completes one of the above . . . 1

Question 3 (Continued)

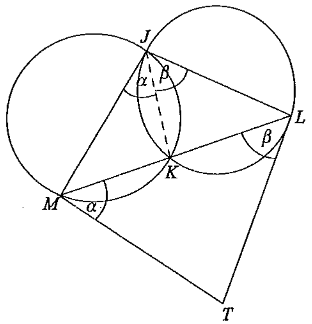
	Sample answer	Syllabus outcomes and marking guide

(d)	$\displaystyle\int_2^3 \frac{x\,dx}{(x-1)^2}$, $u = x - 1$
	$\therefore du = dx$
	$x = 2, \quad u = 1$
	$x = 3, \quad u = 2$
	$\therefore \displaystyle\int_2^3 \frac{x\,dx}{(x-1)^2} = \int_1^2 \frac{u+1}{u^2} \cdot du$
	$= \displaystyle\int_1^2 \frac{1}{u} + u^{-2} \cdot du$
	$= \left[\ln u - \frac{1}{u}\right]_1^2$
	$= \left(\ln 2 - \frac{1}{2}\right) - (\ln 1 - 1)$
	$= \ln 2 + \frac{1}{2}$
		HE6
		• Correctly completes:
		  • changing variable
		  • integration
		  • answer 3
		• Correctly completes two of the above . . . 2
		• Correctly completes one of the above . . . 1

Question 4

Sample answer | **Syllabus outcomes and marking guide**

(a)



Let $\angle LMT = \alpha$

Let $\angle LMT = \beta$

$\therefore \angle LTM = 180 - (\alpha + \beta)$

(angle sum of a triangle)

Construct JK.

$\angle TMK = \angle MJK = \alpha$

(alternate segment theorem)

$\angle KLT = \angle KJL = \beta$

(alternate segment theorem)

$\therefore \angle MJL = \alpha + \beta$

$\therefore TMJL$ is a cyclic quadrilateral

(opposite angles are supplementary $\angle LTM + \angle MJL = 180°$)

PE2, PE3

- Correctly:
  - uses angle sum of a triable
  - uses alternate segment theorem
  - shows $TMJL$ is a cyclicquadrilateral 3
- Correctly completes two of the above . . . 2
- Correctly completes one of the above . . . 1

(b) $\left(x^2 - \frac{2}{x^4}\right)^6$

$\binom{6}{k}(x^2)^{6-k}\left(-\frac{2}{x^4}\right)^k$

$= \binom{6}{k}x^{12-2k}(-2x^{-4})^k$

To be independent of x:

$12 - 2k - 4k = 0$

$k = 2$

$\therefore T_3 = \binom{6}{2}(x^2)^4\left(-\frac{2}{x^4}\right)^2$

$= \frac{6!}{2!4!} \times (-2)^2$

$= 60$

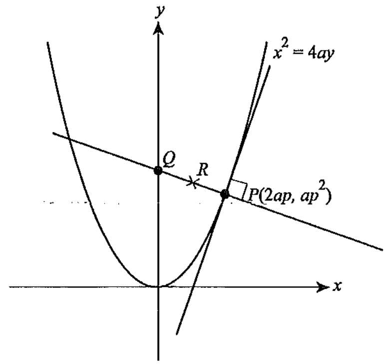
HE3

- Correctly:
  - uses the general term
  - finds the term that is independent of x ($k = 2$)
  - determines the coefficient 3
- Correctly completes two of the above . . . 2
- Correctly completes one of the above . . . 1

Question 4 **(Continued)**

Sample answer | **Syllabus outcomes and marking guide**

(c) (i)



$x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\therefore \frac{dy}{dx} = \frac{x}{2a}$

at P;

$M_T = \frac{2ap}{2a} = p$

$M_N = -\frac{1}{p}$

Equation of normal:

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

i.e. $x + py = 2ap + ap^3$ as required

PE4

- Gives gradient of tangent/normal at P.

AND

- Gives equation of normal as required . . . 2

- Gives gradient of tangent/normal at P.

OR

- Gives equation of normal as required . . . 1

(ii) coordinates of Q; put $x = O$

$O + py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore Q(O, a(2 + p^2))$

PE4

- Correctly finds coordinates for Q 1

(iii) R is the midpoint of PQ:

$x_R = \frac{0 + 2ap}{2} = ap$

$y_R = \frac{a(2 + p^2) + ap^2}{2} = \frac{2a + 2ap^2}{2}$

$= a(1 + p^2)$

$\therefore R(ap, a(1 + p^2))$

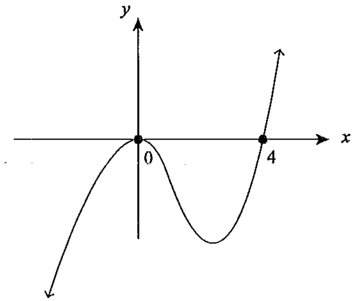
PE4

- Correctly identifies midpoint R 1

Question 4 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iv) Since $x = ap \Rightarrow p = \frac{x}{a}$ and $y = a(1+p^2)$ $\therefore y = a\left(1+\frac{x^2}{a^2}\right)$ $\therefore y = a + \frac{x^2}{a}$ i.e. $x^2 = a(y-a)$ This is a parabola with $V(0, a)$, focal length $\frac{a}{4}$ and $S\left(0, \frac{5a}{4}\right)$.	PE4 • Finds equation of locus of R. AND • Describes locus, stating the parabola and one other fact .... 2 • Finds equation of locus of R. OR • Describes locus, stating the parabola and one other fact 1

Question 5

Sample answer	Syllabus outcomes and marking guide
(a) (i) $P(x) = x^3 - 4x^2 = x^2(x-4)^1$ 	PE3, HE3 • Correctly graphs $P(x)$ 1
(ii) $\ln(x^3-4x^2)$ only exists for: $x^3 - 4x^2 > 0$ i.e. $x > 4$ Domain of $\ln(x^3-4x^2)$ is $\{x : x > 4\}$	PE3, HE3 • Gives the correct domain 1
(b) Newton's method: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ Now, $f(x) = x - \cos x$ $f'(x) = 1 + \sin x$ Given $x_0 = 0.5$ $f\left(\frac{1}{2}\right) = \frac{1}{2} - \cos\frac{1}{2}$ $f'\left(\frac{1}{2}\right) = 1 + \sin\frac{1}{2}$ $\therefore x_1 = \frac{1}{2} - \left[\frac{\frac{1}{2} - \cos\frac{1}{2}}{1 + \sin\frac{1}{2}}\right]$ $x_1 = 0.755222...$ $x_1 = 0.76$ (to 2 decimal places)	HE3, HE7 • States and applies Newton's method. AND • Gives correct answer .... 2 • States and applies Newton's method. OR • Gives correct answer 1
(c) (i) $\frac{dT}{dt} = 0 + B \cdot ke^{kt}$ $\frac{dT}{dt} = k \times Be^{kt}$ $\frac{dT}{dt} = k(T-A)$, as required.	HE3 • Gives the correct answer 1

Question 5 (Continued)

(ii) Finding the value of constant B:

$95 = 15 + Be^{k\times 0}$

$B = 80$

Finding the value of k:

$40 = 15 + 80 \times e^{5k}$

$e^{5k} = \frac{25}{80} = \frac{5}{16}$

$5k = \ln\left(\frac{5}{16}\right)$

$k = \frac{1}{5}\ln\left(\frac{5}{16}\right)$, or -0.232630162

Finding the time to reach 25°C:

$25 = 15 + 80 \times e^{kt}$

$\frac{1}{8} = e^{kt}$

$t = \frac{1}{k}\ln\left(\frac{1}{8}\right)$

$t = \dfrac{\ln\left(\frac{1}{8}\right)}{\frac{1}{5}\ln\left(\frac{5}{16}\right)} = 8.9388$ minutes ≈ 9 minutes

Syllabus outcomes and marking guide

HE3
- Correctly finds:
  - the value of B
  - the value of k or CNE
  - the time to reach 25°C. 3
- Correctly gives any two of the above. . . . 2
- Correctly gives any one of the above. . . . 1

(d) Required to show that for $n \geq 1$:

$n(n+3) = 2P$ i.e. even

For $n = 1$;

$LHS = 1(1+3)$

$= 4$ This is a multiple of 2, $\therefore$ true.

Assume true for $n = k$; $\therefore k(k+3) = 2P$

Now for $n = k+1$;

$(k+1)((k+1)+3) = (k+1)(k+4)$

$= k^2 + 5k + 4$

$= k^2 + 3k + 2k + 4$

$= k(k+3) + 2(k+2)$

$= 2P + 2(k+2)$

$= 2[P + k + 2]$

$= 2Q$, where Q is a positive integer

$\therefore$ If true for $n = k$, then true for $n = k+1$.

Since true for $n = 1$, then true for $n = 1+1 = 2$, $n = 2+1 = 3$ and so on for all integer values $n \geq 1$.

Syllabus outcomes and marking guide

HE2
- Correctly:
  - establishes truth for $n = 1$
  - writes statement for $n = k+1$ assuming true for $n = k$
  - shows true for $n = k+1$ if true for $n = k+1$
  - deduces truth of statement from previous steps 4
- Correctly completes three of the above . . 3
- Correctly completes two of the above . . . 2
- Correctly completes one of the above . . . 1

Question 6

Sample answer

(a) $x^3 - 4x^2 - 11x + 30 = 0$

$\alpha + \beta + \gamma = 4$

$\alpha\beta + \alpha\gamma + \beta\gamma = -11$

$\alpha\beta\gamma = -30$

but $\alpha = \beta + \gamma$

$\alpha + (\beta + \gamma) = 4$

$\alpha + \alpha = 4$

$2\alpha = 4$

$\alpha = 2$

$\therefore (x-2)$ is a factor.

$$
\begin{array}{r|l}
 & x^2 - 2x - 15 \\
\hline
(x-2) & x^3 - 4x^2 - 11x + 30 \\
 & \underline{x^3 - 2x^2} \\
 & -2x^2 - 11x + 30 \\
 & \underline{-2x^2 + 4x} \\
 & -15x + 30 \\
 & \underline{-15x + 30} \\
 & 0
\end{array}
$$

$\therefore P(x) = (x-2)(x^2 - 2x - 15)$

$= (x-2)(x+3)(x-5)$

For $P(x) = 0$

$(x-2)(x+3)(x-5) = 0$

$\therefore x = -3, 2, 5$

Syllabus outcomes and marking guide

PE3
- Correctly:
  - finds $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\Sigma\alpha\beta\gamma$
  - establishes $\alpha = 2$
  - completely factorises $P(x)$ or solves $P(x) = 0$. 3
- Correctly completes two of the above . . . 2
- Correctly completes one of the above . . . 1

ALTERNATIVELY

$\alpha\beta\gamma = -30$

$\therefore \beta\gamma = -15$ (1)

$\beta + \gamma = 2$ (2)

$\therefore \beta - \frac{15}{\beta} = 2$

$\beta^2 - 2\beta - 15 = 0$

$(\beta - 5)(\beta + 3) = 0$

$\beta = 5$ or -3

If $\beta = 5$, $\gamma = -3$ and if $\beta = -3$, $\gamma = 5$

$\therefore$ Roots of $P(x) = 0$ are 2, −3 and 5.

- Correctly:
  - establishes $\alpha = 2$
  - establishes quadratic expression in β or γ
  - solves $P(x) = 0$. 3
- Correctly completes two of the above . . . 2
- Correctly completes one of the above . . . 1

(b) (i) $x = \frac{3}{2}\sin 2t + 2\cos 2t$

$\dot{x} = 3\cos 2t - 4\sin 2t$

$\ddot{x} = -6\sin 2t - 8\cos 2t$

$\ddot{x} = -4\left[\frac{3}{2}\sin 2t + 2\cos 2t\right]$

$\ddot{x} = -4x$

$\therefore$ The motion in SHM with the particle oscillating about the origin with a period of π seconds.

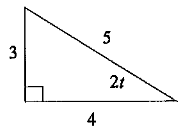
HE3
- Shows $\ddot{x} = -4x$.

AND
- Describes the motion of the particle 2
- Shows $\ddot{x} = -4x$.

OR
- Describes the motion of the particle 1

Question 6 **(Continued)**

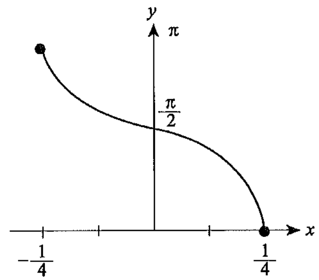
Sample answer	Syllabus outcomes and marking guide
(ii) Maximum displacement is when $\dot{x}=0$. i.e. $3\cos 2t - 4\sin 2t = 0$ $\therefore 4\sin 2t = 3\cos 2t$ $\frac{\sin 2t}{\cos 2t} = \frac{3}{4}$ $\therefore \tan 2t = \frac{3}{4}$	HE3 • Determines t for maximum displacement. AND • Finds $x_{max} = 2.5$ m .... 2 • Determines t for maximum displacement. OR • Finds $x_{max} = 2.5$ m 1



$\therefore$ Maximum displacement:

$$x_{max} = \frac{3}{2}\sin 2t + 2\cos 2t$$
$$= \frac{3}{2}\cdot\left(\frac{3}{5}\right) + 2\left(\frac{4}{5}\right)$$
$$= \frac{9}{10} + \frac{8}{5}$$
$$x_{max} = 2.5 \text{ m}$$

Sample answer	Syllabus outcomes and marking guide
(c) (i) $y = \cos^{-1} 4x$ $-1 \le 4x \le 1$ $-\frac{1}{4} \le x \le \frac{1}{4}$ $\therefore$Domain: $-\frac{1}{4} \le x \le \frac{1}{4}$ Range: $0 \le y \le \pi$	HE4 • Correctly states the domain and range... 1
(ii) See graph below	HE4 • Correctly sketches $y = \cos^{-1} 4x$ 1



 TENME1_SS_10.FM

Question 6 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(iii) $y = \cos^{-1} 4x$ $\frac{dy}{dx} = -\frac{4}{\sqrt{1-16x^2}}$ $x = -\frac{1}{8}$, $\frac{dy}{dx} = -\frac{4}{\sqrt{1-\frac{1}{4}}}$ $= -\frac{8}{\sqrt{3}}$ $x = -\frac{1}{8}$, $y = \cos^{-1}\left(-\frac{1}{2}\right)$ $= \frac{2\pi}{3}$ $\therefore$Equation of the tangent: $y - \frac{2\pi}{3} = -\frac{8}{\sqrt{3}}\left(x + \frac{1}{8}\right)$ $y = \frac{-8}{\sqrt{3}}x - \frac{1}{\sqrt{3}} + \frac{2\pi}{3}$	HE4 • Correctly defines $\frac{dy}{dx}$. • Correctly finds the gradient of the tangent. • Correctly finds the equation of the tangent .... 3 • Any two of the above .... 2 • Any one of the above 1

 TENME1_SS_10.FM

Question 7

Sample answer	Syllabus outcomes and marking guide
(a) $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dt}=-20$ mm^3 per second $\frac{dV}{dr}=4\pi r^2$ $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$ $-20=4\pi r^2\times\frac{dr}{dt}$ $\frac{dr}{dt}=\frac{-20}{4\pi r^2}=\frac{-5}{\pi r^2}$ $SA=4\pi r^2$, $\frac{dSA}{dr}=8\pi r$ $\frac{dSA}{dt}=\frac{dSA}{dr}\times\frac{dr}{dt}$ $\frac{dSA}{dt}=8\pi r\times\frac{-5}{\pi r^2}=\frac{-40}{r}$ when $r=10$: $\frac{dSA}{dt}=\frac{-40}{10}=-4$ mm^2 per second	HE5, HE7 • Correctly: • shows $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$ • finds $\frac{dr}{dt}$ • shows $\frac{dSA}{dt}=\frac{dSA}{dr}\times\frac{dr}{dt}$ • shows $\frac{dSA}{dt}=\frac{40}{r}$ • finds $\frac{dSA}{dt}=4$ mm^2 per second .... 5 • Correctly shows $\frac{dSA}{dt}=\frac{40}{r}$ .... 4 • Correctly reaches $\frac{dSA}{dt}=\frac{dSA}{dr}\times\frac{dr}{dt}$..... 3 • Correctly finds $\frac{dr}{dt}$ .... 2 • Only shows $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$ 1
(b) (i) $x=Vt\cos\alpha$ and $y=-\frac{1}{2}gt^2+Vt\sin\alpha$ $\therefore y=-\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2+V\left(\frac{x}{V\cos\alpha}\right)$ i.e. $\therefore y=x\tan\alpha-\frac{gx^2\sec^2\alpha}{2V^2}$, as required	HE3 • Correctly gives x and y in terms of t. AND • Correctly gives the equation of trajectory of the missile .... 2 • Only gives x and y in terms of t........ 1
(ii) $X=OR\times\cos\theta$ and $Y=OR\times\sin\theta$ substituting into y above (part i): $OR\times\sin\theta=OR\times\cos\theta\tan\alpha-\frac{g\times OR^2\cos^2\theta\sec^2\alpha}{2V^2}$ $\sin\theta=\cos\theta\tan\alpha-\frac{g\times OR\cos^2\theta\sec^2\alpha}{2V^2}$ $OR\times\frac{g\cos^2\theta}{2V^2\cos^2\alpha}=\tan\alpha\times\cos\theta-\sin\theta$ $OR=\frac{(\tan\alpha\times\cos\theta-\sin\theta)\times 2V^2\cos^2\alpha}{g\cos^2\theta}$ $OR=\frac{2V^2\cos\alpha}{g\cos^2\theta}(\sin\alpha\times\cos\theta-\cos\alpha\times\sin\theta)$ $OR=\frac{2V^2\cos\alpha}{g\cos^2\theta}\times\sin(\alpha-\theta)$, as required.	HE3 • Correctly expresses X and Y in terms of OR and θ. AND • Correctly expresses OR in terms of α and θ............................ 2 • Expresses X and Y in terms of OR and θ. OR • Expresses OR in terms of α and θ 1

Question 7 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iii) $\frac{dOR}{d\alpha}=\frac{2V^2}{g\cos^2\theta}[\cos\alpha\cdot\cos(\alpha-\theta)-\sin\alpha\cdot\sin(\alpha-\theta)]$ $\frac{dOR}{d\alpha}=\frac{2V^2}{g\cos^2\theta}[\cos(\alpha+(\alpha-\theta))]$ $\frac{dOR}{d\alpha}=\frac{2V^2}{g\cos^2\theta}[\cos(2\alpha-\theta)]$ and $\frac{d^2OR}{d\alpha^2}=\frac{-4V^2}{g\cos^2\theta}[\sin(2\alpha-\theta)]$ Therefore, $\frac{dOR}{d\alpha}=0$, when $\cos(2\alpha-\theta)=0$ Hence, $2\alpha-\theta=\frac{\pi}{2}$, i.e. $\alpha=\frac{1}{2}\theta+\frac{\pi}{4}$ At $\alpha=\frac{1}{2}\theta+\frac{\pi}{4}$, $\frac{d^2OR}{d\alpha^2}=\frac{-4V^2}{g\cos^2\theta}<0$ $\therefore$ OR is a maximum when $\alpha=\frac{1}{2}\theta+\frac{\pi}{4}$, as required.	HE3 • Correctly shows $\alpha=\frac{1}{2}\theta+\frac{\pi}{4}$. AND • Correctly shows a maximum.......... 2 • Shows $\alpha=\frac{1}{2}\theta+\frac{\pi}{4}$.................. 1
(iv) When $\alpha=\frac{1}{2}\theta+\frac{\pi}{4}$, $OR_{max}=\frac{2V^2}{g}\left(\frac{\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\times\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\cos^2\theta}\right)$ $OR_{max}=\frac{2V^2}{g}\left(\frac{\frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}{1-\sin^2\theta}\right)$ $OR_{max}=\frac{V^2}{g}\left(\frac{\left(1-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{(1-\sin\theta)(1+\sin\theta)}\right)$ $OR_{max}=\frac{V^2}{g}\left(\frac{(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}\right)$ $OR_{max}=\frac{V^2}{g}\left(\frac{1}{1+\sin\theta}\right)$, as required.	HE3 • Correctly substitutes and simplifies..... 1