NORTH SYDNEY GIRLS HIGH SCHOOL

2010
TRIAL HSC EXAMINATION

Mathematics

General Instructions
- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120
Attempt Questions 1–10
All questions are of equal value

At the end of the examination, place your solution booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

Student Number:____________________ **Teacher:**____

Student Name:____________________

QUESTION	MARK
1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
8	/12
9	/12
10	/12
TOTAL	/120

Total Marks – 120
Attempt Questions 1–10
All questions are of equal value

Begin each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $e^5 - 3\log_e 5$ correct to two significant figures. **2**

(b) Factorise $64 - x^3$. **2**

(c) Find a primitive of $\frac{1}{4x} + e^{2x}$. **2**

(d) Express $\frac{x}{x-2} - \frac{8}{x^2-4}$ as a single fraction in its simplest terms. **2**

(e) Find the values of x for which $|4+x| \le 7$. **2**

(f) Find the coordinates of the vertex of the parabola $y = x^2 + 4x - 3$. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the equation of the tangent to the curve $y=\log_e(2x-1)$ at the point where $x=1$. **2**

(b) Differentiate $\sqrt{5+\log_e x}$. **1**

(c) Show that if $y=\dfrac{1+\sin x}{\cos x}$ then $\dfrac{dy}{dx}=\dfrac{1}{1-\sin x}$. **3**

(d) Find (i) $\displaystyle\int 6e^{\frac{x}{2}}\,dx$. **1**

(ii) $\displaystyle\int \frac{x}{1-x^2}\,dx$. **1**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}}\left(1-\sec^2 2x\right)dx$. **2**

(f) Solve $4\sin^2\theta-3=0$ for $-\pi\le\theta\le\pi$. **2**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

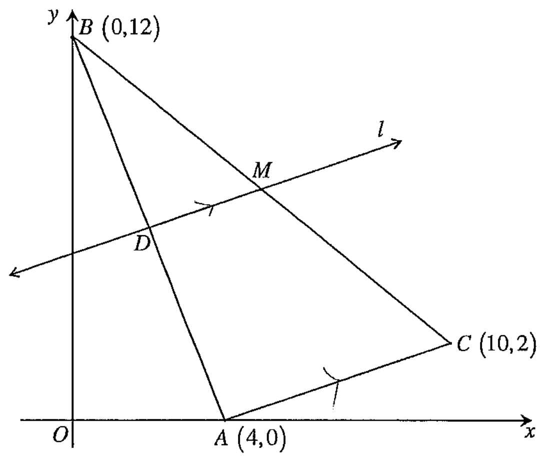
(a) Evaluate $\displaystyle\sum_{n=5}^{11}(2n-5)$. **1**

(b) The triangle PQR has sides $PQ=8$ cm, $PR=7$ cm and $QR=5$ cm.

(i) Show that $\angle PQR=60°$. **2**

(ii) Find the area of the triangle PQR as an exact value. **2**

(c)



In the above diagram, A, B and C are the points $(4,0)$, $(0,12)$ and $(10,2)$ respectively.

(i) Find the gradient of AC. **1**

(ii) Find the coordinates of D, the midpoint of AB. **1**

(iii) The line l is parallel to AC and passes through D. Find the equation of l. **1**

(iv) The line l meets BC at M. Without calculation, explain why M is the midpoint of BC. **1**

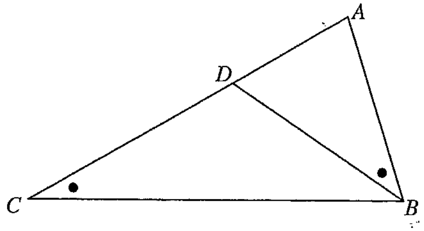
(v) Find the equation of the circle which has BC as diameter. **2**

(vi) Does this circle pass through A? Justify your answer. **1**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of the equation $x^2 - 8x + 5 = 0$ are α and β. **2**
Find the value of $(\alpha - \beta)^2$.

(b) In the diagram below, $\angle ACB = \angle ABD$.



(i) Copy the diagram onto your examination pad.

(ii) Prove that ΔABC is similar to ΔADB. **2**

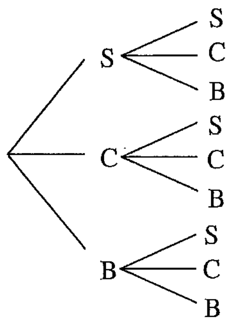
(iii) If $AD = 9$ cm and $AB = 12$ cm, find the length of AC. **2**

(c) A jar contains 27 balls. Twenty of the balls have a star painted on them and ten of the balls have a cross painted on them. Each ball has at least one of these two symbols on it.

(i) If one ball is drawn at random, what is the probability that the ball drawn has two symbols painted on it? **1**

(ii) Two balls are drawn simultaneously from the jar.

(α) Copy and complete the tree diagram below indicating the probabilities on each branch. **1**



Key:
S = star only
C = cross only
B = both

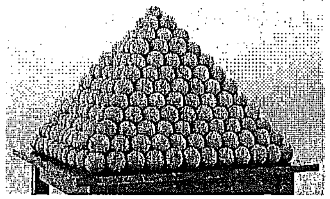
(β) What is the probability that on the two balls drawn

(1) exactly one star appears? **2**

(2) two stars appear? **2**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Con the fruiterer decides to display the oranges in the shape of a square-based pyramid. He has built a frame in the shape of a pyramid and begins by placing a row of oranges around the base.
In subsequent rows, each orange rests only on two oranges in the row below and on the frame. Con completes the pyramid by placing the last orange on top of the four previous oranges.

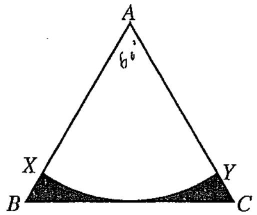


(i) Explain why each row of the pyramid excluding the last orange has 4 fewer oranges than the row on which it rests. **1**

(ii) Con uses 56 oranges in the first row. How many rows of oranges will he have placed *before* he places the last orange? **2**

(iii) How many oranges will Con use in completing his display? **2**

(b) In the diagram, ΔABC is an equilateral triangle with the sides of length 6 cm. An arc with centre A and BC as tangent, cuts AB and AC at X and Y respectively.



(i) Show that the radius of the arc is $3\sqrt{3}$ cm. **2**

(ii) Find the area of the shaded portion in exact form. **2**

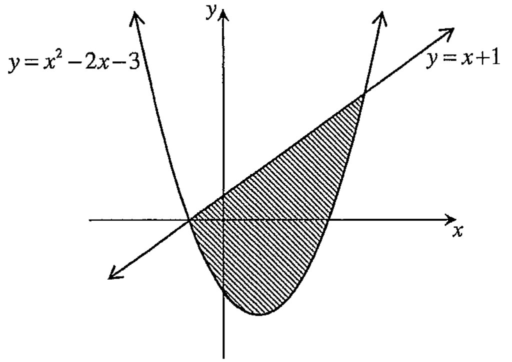
(c) (i) Write down the discriminant of $5x^2 - 2kx + k$. **1**

(ii) For what values of k does $5x^2 - 2kx + k = 0$ have real roots? **2**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $2\ln x = \ln(5+4x)$. **2**

(b) The area enclosed by the graphs of $y = x^2 - 2x - 3$ and $y = x + 1$ is illustrated below. **3**
The graphs intersect at $x = -1$ and $x = 4$.



Find the area enclosed by the curves.

(c) For what value of x is the tangent to $y = e^{3x}$ parallel to the line $y = 6x$? **2**

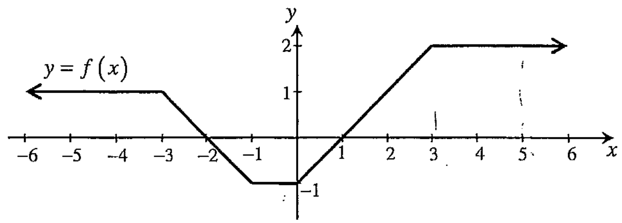
(d) (i) Draw a neat sketch of the curve $y = 3\sin 2x$ for $0 \le x \le 2\pi$. **2**

(ii) On the same diagram, sketch $y = 1 - \cos x$ for $0 \le x \le 2\pi$. **2**

(iii) Hence determine the number of solutions the equation $3\sin 2x + \cos x = 1$ will have in the given domain. **1**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram below illustrates the function $y = f(x)$.



(i) Evaluate $\int_0^5 f(x)\,dx$. **1**

(ii) Find two values of a such that $\int_a^5 f(x)\,dx = 4$. **2**

(b) (i) Differentiate $\sin^2 x - \cos 4x$. **1**

(ii) Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x \cos x + 2\sin 4x)\,dx$. **2**

(c) (i) When Emily is 3 months old, her parents invest an amount of \$500 in an account that earns interest of 8% p.a., the interest being paid every 3 months.
How much will be in the account when Emily turns 15? **1**

(ii) Instead of just making one payment of \$500 into the account, Emily's parents decide to make regular deposits of \$500, every 3 months, starting with the first one when Emily is 3 months old.

(α) Show that the day after Emily's 1st birthday, the value of the account is given by **1**

$$A = 500\left[1 + 1.02 + 1.02^2 + 1.02^3\right]$$

(β) How much money will be in the account the day after Emily turns 15? **2**

(iii) No more payments are made into the account after Emily turns 15 and no withdrawals are made.
Show that the amount in the account on Emily's 16th birthday is \$61 726.53. **1**

(iv) Emily decides that she will withdraw regular amounts of money from this account each birthday, starting with her 16th birthday.
She cannot decide whether she should withdraw \$4 000 or \$5 000 each birthday. By considering the result of part (iii), comment on what will happen in each case. **1**

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $f(x)=x^3-x^2-5x+1$.

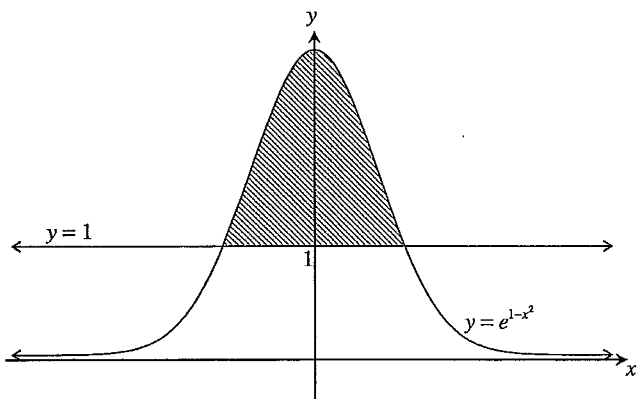
(i) Find the coordinates of the stationary points of the curve $y=f(x)$ and determine their nature. **3**

(ii) Find any points of inflexion. **2**

(iii) Sketch the curve $y=f(x)$ for $-2\le x\le 2$ clearly indicating the endpoints. You need not find the x-intercepts. **2**

(iv) For what values of x is the curve $y=f(x)$ decreasing but concave up? **1**

(b) The diagram shows the region bounded by the curve $y=e^{1-x^2}$, and the line $y=1$.



(i) Show that $x^2=1-\log_e y$. **1**

(ii) The shaded area is rotated about the y-axis. Write down the definite integral equal to the volume formed. **1**

(iii) Evaluate the volume of the solid of revolution using Simpson's rule with three function values. Give your answer correct to two significant figures. **2**

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A golf tournament is organized and the prize money for the first 20 place winners is distributed as follows:

- First Prize = \$1 000 000
- The next 6 placed players each receive 80% of the prize the previous player received.
- The next 13 placed players each receive \$20 000 less than the previous prize winner.
- There are no tied place winners

Find:

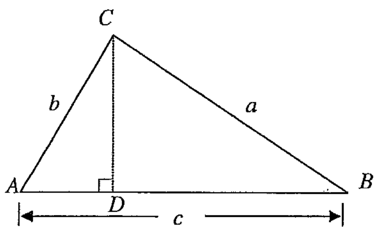
(i) the value of the 7^{th} prize; **1**

(ii) the value of the 20^{th} prize; **1**

(iii) the total amount of prize money available in the tournament. **2**

(b) A product must pass two quality tests before it can be sold. If it passes the first test, it is more likely to pass the second. **3**
The probability that it passes the first test is 85% and the probability that it passes at least one test is 97%.
The probability that it passes one test and not the other is 17·1%.
By drawing a tree diagram or otherwise, find the probability that the product passes the first test and fails the second test.

(c) The triangle ABC has side lengths a, b and c as shown in the diagram.
The point D lies on AB, and CD is perpendicular to AB.



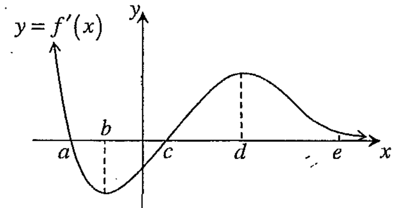
(i) Show that $a\sin B=b\sin A$. **1**

(ii) Show that $c=a\cos B+b\cos A$. **1**

(iii) If $c^2=4ab\cos A\cos B$, show that $a=b$. **3**

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The graph below represents $y=f'(x)$. Specific x-values a, b, c, d and e are as indicated in the diagram.



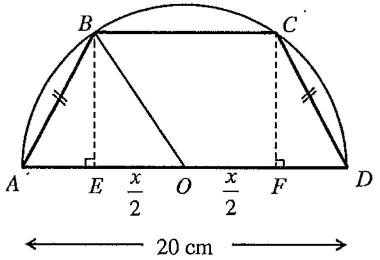
(i) For what value(s) of x will the graph of $y=f(x)$ have a stationary point? **1**

(ii) For what value(s) of x is the graph of $y=f(x)$ increasing? **2**

(iii) When is the graph of $y=f(x)$ concave up? **1**

(iv) Describe what happens to the graph of $y=f(x)$ as $x\to\infty$. **1**

(b) An isosceles trapezium $ABCD$ is drawn with its vertices on a semicircle centre O and diameter 20 cm. Perpendiculars BE and CF are drawn to meet the diameter AD as illustrated in the diagram below.



(i) If $EO=OF=\frac{x}{2}$ show that $BE=\frac{1}{2}\sqrt{400-x^2}$. **2**

(ii) Show that the area of the trapezium $ABCD$ is given by **1**

$$A=\frac{1}{4}(x+20)\sqrt{400-x^2}.$$

(iii) Hence find the length of BC so that the area of the trapezium $ABCD$ is a maximum. **4**

End of paper

Mathematics Trial 2010 Solutions

Question 1

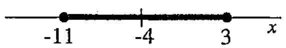
(a) $e^5 - 3\log_e 5 = 143.58...$
$= 140$ (2 sig. fig.)

(b) $64 - x^3 = 4^3 - x^3$
$= (4-x)(16+4x+x^2)$

(c) $\int \left(\frac{1}{4x} + e^{2x}\right) dx = \frac{1}{4}\ln x + \frac{1}{2}e^{2x} + C$

(d) $\frac{x}{x-2} - \frac{8}{x^2-4} = \frac{x}{x-2} - \frac{8}{(x-2)(x+2)}$
$= \frac{x(x+2)-8}{(x-2)(x+2)}$
$= \frac{x^2+2x-8}{(x-2)(x+2)}$
$= \frac{(x-2)(x+4)}{(x-2)(x+2)}$
$= \frac{x+4}{x+2}$

(e) $|4+x| \le 7$
$\therefore |x-(-4)| \le 7$



$\therefore -11 \le x \le 3$

(f) $y = x^2 + 4x - 3$
becomes $y = (x+2)^2 - 7$
$\therefore$ vertex is $(-2,-7)$

Alternatively:
Axis of symmetry is at $x = \frac{-4}{2} = -2$
Then $y = (-2)^2 + 4(-2) - 3 = -7$
$\therefore$ vertex is $(-2,-7)$

Question 2

(a) $y = \log_e(2x-1)$
$y' = \frac{2}{2x-1}$
At $x=1$: $y' = \frac{2}{2(1)-1} = 2$
and $y = \ln(2(1)-1)$
$= \ln 1$
$= 0$
$\therefore$ tangent is $y - 0 = 2(x-1)$
$y = 2x - 2$

(b) $\frac{d}{dx}\sqrt{5+\log_e x} = \frac{d}{dx}(5+\log_e x)^{\frac{1}{2}}$
$= \frac{1}{2}(5+\log_e x)^{-\frac{1}{2}}\left(\frac{1}{x}\right)$
$= \frac{1}{2x\sqrt{5+\log_e x}}$

(c) $y = \frac{1+\sin x}{\cos x}$
$y' = \frac{\cos x(\cos x) - (1+\sin x)(-\sin x)}{\cos^2 x}$
$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$
$= \frac{1+\sin x}{1-\sin^2 x}$
$= \frac{1+\sin x}{(1-\sin x)(1+\sin x)}$
$= \frac{1}{1-\sin x}$

(d) (i) $\int 6e^{\frac{x}{2}}\,dx = 12\int \frac{1}{2}e^{\frac{x}{2}}\,dx$
$= 12e^{\frac{x}{2}} + C$

(ii) $\int \frac{x}{1-x^2}\,dx = -\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx$
$= -\frac{1}{2}\ln(1-x^2) + C$

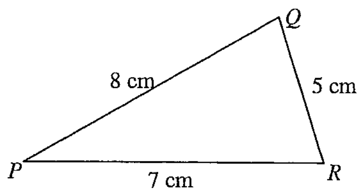
(e) $\int_0^{\frac{\pi}{6}} (1-\sec^2 2x)\,dx = \left[x - \frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{6}}$
$= \frac{\pi}{6} - \frac{\tan\frac{\pi}{3}}{2} - \left(0 - \frac{\tan 0}{2}\right)$
$= \frac{\pi}{6} - \frac{\sqrt{3}}{2}$

(f) $4\sin^2\theta - 3 = 0$ for $-\pi \le \theta \le \pi$
$\sin^2\theta = \frac{3}{4}$
$\sin\theta = \pm\frac{\sqrt{3}}{2}$
$\theta = \frac{\pi}{3}, \pi - \frac{\pi}{3}, -\frac{\pi}{3}, -\pi + \frac{\pi}{3}$
$= \frac{\pi}{3}, \frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{2\pi}{3}$

Question 3

(a) $\sum_{n=5}^{11}(2n-5) = 5+7+9+11+13+15+17$
$= 77$

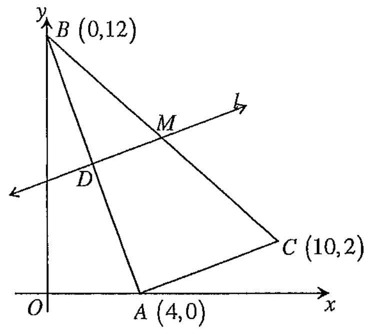
(b)



(i) $\cos\angle PQR = \frac{8^2+5^2-7^2}{2(8)(5)}$ (cos rule)
$= \frac{40}{80}$
$= \frac{1}{2}$
$\therefore \angle PQR = 60°$

(ii) $A = \frac{1}{2}(8)(5)\sin 60°$
$= 20 \times \frac{\sqrt{3}}{2}$
$= 10\sqrt{3}$
$\therefore$ the area of ΔPQR is $10\sqrt{3}$ cm^2

(c)



(i) $m_{AC} = \frac{2-0}{10-4}$
$= \frac{1}{3}$

(ii) $D = (2,6)$

(iii) l: $y - 6 = \frac{1}{3}(x-2)$
$3y - 18 = x - 2$
$x - 3y + 16 = 0$

(iv) $AC \parallel DM$ (same gradient)
$\therefore \frac{AD}{DB} = \frac{CM}{MB}$ (parallel lines preserve ratios)
$1 = \frac{CM}{MB}$
$CM = MB$
$\therefore M$ is the midpoint of CB

(v) Centre is $(5,7)$
$d_{MB} = \sqrt{5^2+5^2}$
$= \sqrt{50}$
$= 5\sqrt{2}$
$\therefore$ the circle is $(x-5)^2 + (y-7)^2 = 50$

(vi) Substitute $(4,0)$ into
$(x-5)^2 + (y-7)^2 = 50$
$(4-5)^2 + (0-7)^2 = 1 + 49$
$= 50$
The point A does lie on the circle

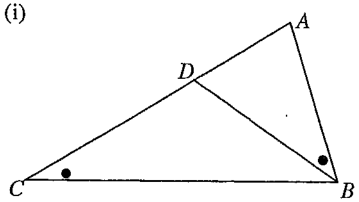
Question 4

(a) $x^2-8x+5=0$

$\therefore\ \alpha+\beta=8$ and $\alpha\beta=5$

$$(\alpha-\beta)^2=\alpha^2-2\alpha\beta+\beta^2$$
$$=\alpha^2+\beta^2-2\alpha\beta$$
$$=(\alpha+\beta)^2-2\alpha\beta-2\alpha\beta$$
$$=(\alpha+\beta)^2-4\alpha\beta$$
$$=8^2-4(5)$$
$$=44$$

(b) (i)



(ii) In ΔABC and ΔADB

1. $\angle A$ is common
2. $\angle ACB=\angle ABD$ (given)

$\therefore \Delta ABC \,|||\, \Delta ADB$ (equiangular)

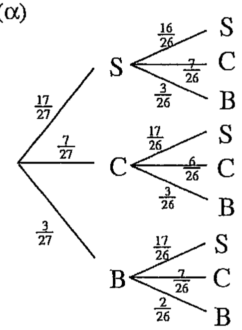
(iii) $\dfrac{AB}{AD}=\dfrac{BC}{DB}=\dfrac{AC}{AB}$

(matching sides of similar Δ's)

$$\frac{12}{9}=\frac{AC}{12}$$
$$AC=\frac{12\times12}{9}$$
$$=16$$

(c) (i) $P(2\text{ symbols})=\dfrac{3}{27}=\dfrac{1}{9}$

(ii) (α)



(β)(1) P(exactly 1 star)

$$=P(SC)+P(CS)+P(CB)+P(BC)$$
$$=\frac{17}{27}\times\frac{7}{26}+\frac{7}{27}\times\frac{17}{26}+\frac{7}{27}\times\frac{3}{26}+\frac{3}{27}\times\frac{7}{26}$$
$$=\frac{140}{351}$$

(2) P(2stars)

$$=P(SS)+P(SB)+P(BS)+P(BB)$$
$$=\frac{17}{27}\times\frac{16}{26}+\frac{17}{27}\times\frac{3}{26}+\frac{3}{27}\times\frac{17}{26}+\frac{3}{27}\times\frac{2}{26}$$
$$=\frac{190}{351}$$

Question 5

(a) (i) Each face has one less orange than in the row below. There are 4 faces. ∴ there are 4 fewer oranges than the row on which it rests.

(ii) $56+52+48+\ldots+4$

$n=?$ $\quad T_n=a+(n-1)d$

$a=56$ $\quad 4=56+(n-1)(-4)$

$T_n=4$

$d=-4$ $\quad 4=56-4n+4$

$$4n=56$$
$$n=14$$

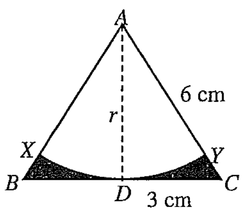
∴ there are 14 rows

(iii) $S_n=\dfrac{n}{2}[a+l]+1$

$$=\frac{14}{2}[56+4]+1$$
$$=421$$

∴ he will use 421 oranges

(b)



(i) $6^2=r^2+3^2$

$$r^2=36-9$$
$$=27$$
$$r=3\sqrt{3}$$

∴ the radius is $3\sqrt{3}$ cm.

(ii) $A=\dfrac{1}{2}bh-\dfrac{1}{2}r^2\theta$ where $\theta=\dfrac{\pi}{3}$

$$A=\frac{1}{2}(6)(3\sqrt{3})-\frac{1}{2}(3\sqrt{3})^2\left(\frac{\pi}{3}\right)$$
$$=9\sqrt{3}-\frac{9\pi}{2}$$

∴ the area is $\left(9\sqrt{3}-\dfrac{9\pi}{2}\right)$ cm^2

(c) (i) For $5x^2-2kx+k$

$$\Delta=(-2k)^2-4(5)(k)$$
$$=4k^2-20k$$

(ii) For real roots: $\Delta\ge0$

$$4k^2-20k\ge0$$
$$4k(k-5)\ge0$$

$k\le0$ or $k\ge5$

Question 6

(a) $2\ln x=\ln(5+4x)$

$$2\ln x=\ln(5+4x)$$
$$\ln x^2=\ln(5+4x)$$
$$x^2=5+4x$$
$$x^2-4x-5=0$$
$$(x-5)(x+1)=0$$
$$x=-1,\ 5$$

But $x>0$ and $4x+5>0$ for the logs to exist

$\therefore\ x=5$ is the only solution

(b) $A=\int_{-1}^{4}\left[x+1-(x^2-2x-3)\right]dx$

$$=\int_{-1}^{4}(3x+4-x^2)\,dx$$
$$=\left[\frac{3x^2}{2}-\frac{x^3}{3}+4x\right]_{-1}^{4}$$
$$=24-\frac{64}{3}+16-\left(\frac{3}{2}+\frac{1}{3}-4\right)$$
$$=20\tfrac{5}{6}$$

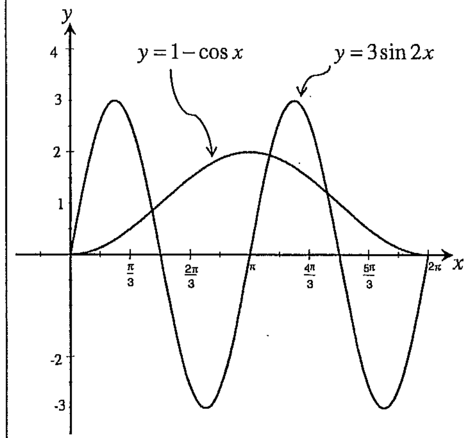
∴ the area is $20\frac{5}{6}$ unit^2

(b) For $y=e^{3x}$, $y'=3e^{3x}$

For $y=6x$, $m=6$

$$\therefore\ 3e^{3x}=6$$
$$e^{3x}=2$$
$$3x=\ln2$$
$$x=\frac{1}{3}\ln2$$

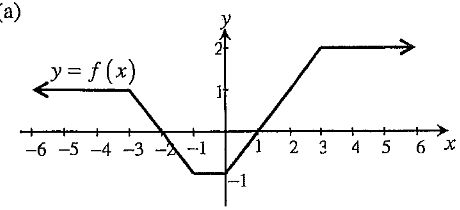
(c) (i) and (ii)



(iii) There are 5 solutions because the curves intersect in 5 different points.

Question 7

(a)



(i) $\int_0^5 f(x)\,dx = -\frac{1}{2}+2+4$

$= 5\frac{1}{2}$

(ii) If $\int_a^5 f(x)\,dx = 4$ we need values of a for which the signed area gives a result of 4. This occurs when $a=3$ or -2

(b) (i) $\frac{d\left(\sin^2 x-\cos 4x\right)}{dx}$

$= 2\sin x\cos x + 4\sin 4x$

(ii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin x\cos x + 2\sin 4x\right)dx$

$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(2\sin x\cos x + 4\sin 4x\right)dx$

$= \frac{1}{2}\left[\sin^2 x - \cos 4x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\sin^2\frac{\pi}{2} - \cos 2\pi - \left(\sin^2\frac{\pi}{4} - \cos\pi\right)\right]$

$= \frac{1}{2}\left[1-1-\left(\frac{1}{2}-(-1)\right)\right]$

$= -\frac{3}{4}$

(c) (i) 8% pa = 2% every ¼ year;
investment periods = 4 × 15 −1= 59
(no interest for the 1st 3 months)

$A = 500(1+0.02)^{59}$

$= 1608.348426$

∴ The amount = \$1608.35

(ii)(α) 1st payment grows to $500(1.02)^3$
2nd payment grows to $500(1.02)^2$
3rd payment grows to $500(1.02)^1$
Last payment remains as 500
∴ Value on the day after 1st birthday is

$A = 500(1.02)^3 + 500(1.02)^2 + 500(1.02)^1 + 500$

$= 500\left[1+1.02+1.02^2+1.02^3\right]$

(β) 1st payment now grows to $500(1.02)^{59}$

$A = 500(1.02)^{59} + 500(1.02)^{58} + ... + 500$

$= 500\left[1+1.02+...+1.02^{58}+1.02^{59}\right]$

$= 500\left[\frac{a\left(r^n-1\right)}{r-1}\right]$ where $a=1$; $r=1.02$; $n=60$

$= 500\left[\frac{1\left((1.02)^{60}-1\right)}{1.02-1}\right]$

$= 57025.769...$

∴ Total in the account on Emily's 16th birthday is \$57025.77.

(iii) During the year, interest is paid 4 times

$A = 57025.76971\times(1.02)^4$

$= 61726.527...$

∴ Amount = \$61726.53 (to nearest cent)

(iv) Interest earned = \$61726.53 − \$57025.77
= \$4700.76

If she withdraws \$4 000, the account will continue to grow.

If she withdraws \$5 000, the money will eventually run out.

Question 8

(a) (i) $f(x) = x^3 - x^2 - 5x + 1$

$f'(x) = 3x^2 - 2x - 5$

$f''(x) = 6x - 2$

Stat points if $f'(x) = 0$

i.e. $3x^2 - 2x - 5 = 0$

$(3x-5)(x+1) = 0$

$\therefore x = -1$ or $\frac{5}{3}$

If $x=-1$: $f''(-1) = 6(-1) - 2$

< 0

$f(-1) = (-1)^3 - (-1)^2 - 5(-1) + 1$

$= 4$

∴ a maximum at $(-1, 4)$

If $x=\frac{5}{3}$: $f''\left(\frac{5}{3}\right) = 6\left(\frac{5}{3}\right) - 2$

> 0

$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 1$

$= -5\frac{13}{27}$

∴ a minimum at $\left(1\frac{2}{3}, -5\frac{13}{27}\right)$

(ii) Points of inflexion when $f''(x) = 0$ and concavity changes

i.e. $6x - 2 = 0$

$x = \frac{1}{3}$

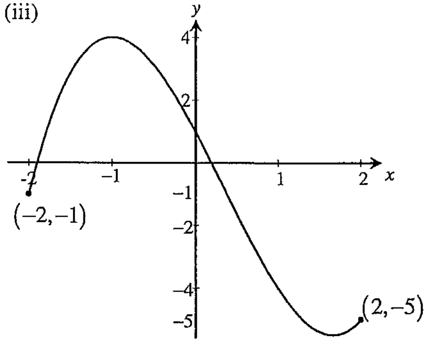
x	$\frac{1}{3}^-$	$\frac{1}{3}$	$\frac{1}{3}^+$
$f''(x)$	−	0	+

If $x=\frac{1}{3}$: $f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 5\left(\frac{1}{3}\right) + 1$

$= -\frac{20}{27}$

∴ an inflection at $\left(\frac{1}{3}, -\frac{20}{27}\right)$

(iii)



(iv) $y=f(x)$ decreasing but concave up when $\frac{1}{3} < x < \frac{5}{3}$

(b) (i) $y = e^{1-x^2}$

$\log_e y = 1 - x^2$

$\therefore\ x^2 = 1 - \log_e y$

(ii) $V = \pi\int_1^e (1-\ln y)\,dy$

(iii)

y	1	$\frac{1+e}{2}$	e
$1-\ln y$	$1-\ln 1$ $=1$	$1-\ln\left(\frac{1+e}{2}\right)$	$1-\ln e$ $=0$

$V = \pi\int_1^e (1-\ln y)\,dy$

$\doteqdot \pi\frac{(e-1)}{6}\left[1+4\left(1-\ln\left(\frac{1+e}{2}\right)\right)+0\right]$

$= 2.2668...$

∴ the volume is 2.3 unit^3 (2 sig. fig.)

Question 9

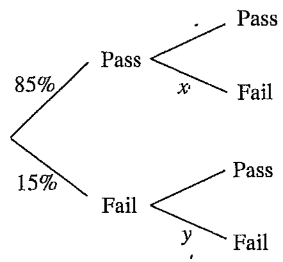
(a) (i) $1000000 + 1000000(0.8) + 1000000(0.8)^2 + \ldots$ to 7 prizes

$$7^{\text{th}} \text{ prize} = \$1000000(0.8)^6$$
$$= \$262144$$

(ii)
$$20^{\text{th}} \text{ prize} = \$262144 - 13 \times \$20000$$
$$= \$2144$$

(iii)
$$\text{Total} = \underbrace{\frac{a(1-r^n)}{1-r}}_{\text{for the first 7 terms}} + \underbrace{\frac{N}{2}[A+L]}_{\text{For the next 13 terms}}$$
$$= \frac{\$1000000\left[1-(0.8)^7\right]}{1-0.8} + \frac{13}{2}[\$242144 + \$2144]$$
$$= \$5539296$$

(b) Let $P(F \text{ second test when passed first test}) = x$
and $P(F \text{ second test when failed first test}) = y$



$$P(\text{at least one } P) = P(PP) + P(PF) + P(FP)$$
$$= 1 - P(FF)$$
$$= 97\%$$

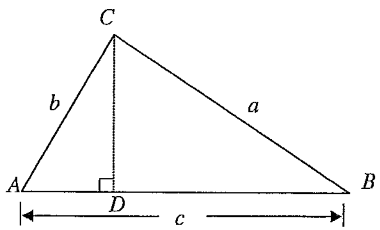
$\therefore\ P(FF) = 3\%$
$\therefore 15\% \times y = 3\%$
$$y = \frac{3\%}{15\%}$$
$$= 20\%$$

$$P(P \text{ only one test}) = P(PF) + P(FP)$$
$$= 85\% \times x + 15\% \times 20\%$$
$$= 0.85x + 0.03$$

$\therefore 17.1\% = 85\% \times x + 15\% \times 20\%$
$0.171 = 0.85x + 0.03$
$x = 0.06$
$= 6\%$

$$P(\text{passes first and fails second}) = P(PF)$$
$$= 85\% \times 6\%$$
$$= 5.1\%$$

(c)



(i) In $\Delta ADC : CD = b \sin A$
In $\Delta CDB : CD = a \sin B$
$\therefore b \sin A = a \sin B$

(ii) In $\Delta ADC : AD = b \cos A$
In $\Delta CDB : DB = a \cos B$
$c = AB$
$= AD + DB$
$= a \cos B + b \cos A$

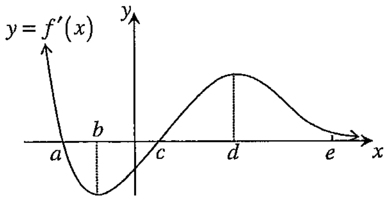
(iii) $c = a\cos B + b\cos A \Rightarrow c^2 = (a\cos B + b\cos A)^2$
But $c^2 = a^2\cos^2 B + 2ab\cos A\cos B + b^2\cos^2 A$
$\therefore a^2\cos^2 B + 2ab\cos A\cos B + b^2\cos^2 A = 4ab\cos A\cos B$
$a^2\cos^2 B + 2ab\cos A\cos B + b^2\cos^2 A - 4ab\cos A\cos B = 0$
$a^2\cos^2 B - 2ab\cos A\cos B + b^2\cos^2 A = 0$
$(a\cos B - b\cos A)^2 = 0$
$\therefore\ a\cos B - b\cos A = 0$
$\therefore a\cos B = b\cos A$ *
$\therefore AD = DB$
$\therefore \Delta ABC$ isosceles [CD perpendicular bisector of AB]
$\therefore a = b$

Alternatively:

$a\cos B = b\cos A$ from *
$a\sin B = b\sin A$ from (i)
$$\therefore\ \frac{a\sin B}{a\cos B} = \frac{b\sin A}{b\cos A}$$
$\therefore\ \tan B = \tan A$
but both angles are acute as they are in ΔABC
$\therefore\ A = B$
$\therefore\ a = b$ (opposite equal angles in ΔABC)

Question 10

(a) The graph below represents $y=f'(x)$. Specific x-values a, b, c, d and e are as indicated in the diagram.



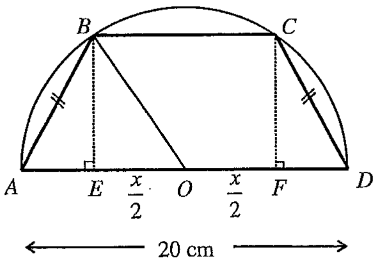
(i) The graph of $y=f(x)$ have a stationary point when $x=a$ or c.

(ii) The graph of $y=f(x)$ is increasing when $x<a$ or $x>c$.

(iii) The graph of $y=f(x)$ is concave up when $f''(x)>0$
i.e. when the gradient graph is increasing.
This is when $b<x<d$.

(iv) As $x\to\infty$ the graph of $y=f(x)$ approaches a horizontal tangent.

(b)



(i) In $\Delta OBE: OB^2=BE^2+OE^2$ (Pythagoras)

$$\therefore 10^2=BE^2+\left(\frac{x}{2}\right)^2$$

$$BE^2=100-\frac{x^2}{4}$$

$$=\frac{1}{4}\left(400-x^2\right)$$

$$\therefore\quad BE=\frac{1}{2}\sqrt{400-x^2}\qquad\text{(length positive)}$$

(ii) $A=\frac{1}{2}h[a+b]$

$$A=\frac{1}{2}(BE)[BC+AD]$$

$$=\frac{1}{2}\cdot\frac{1}{2}\sqrt{400-x^2}\,[x+20]$$

$$=\frac{1}{4}(x+20)\sqrt{400-x^2}$$

(iii) $A=\frac{1}{4}(x+20)\sqrt{400-x^2}$

$$A=\frac{1}{4}(x+20)\left(400-x^2\right)^{\frac{1}{2}}$$

$$A'=\frac{1}{4}(x+20)\cdot\frac{1}{2}\left(400-x^2\right)^{-\frac{1}{2}}(-2x)+\left(400-x^2\right)^{\frac{1}{2}}\cdot\frac{1}{4}$$

$$=-\frac{x}{4}(x+20)\left(400-x^2\right)^{-\frac{1}{2}}+\frac{1}{4}\left(400-x^2\right)^{\frac{1}{2}}$$

Max/min occurs when $A'=0$

i.e. $-\frac{x}{4}(x+20)\left(400-x^2\right)^{-\frac{1}{2}}+\frac{1}{4}\left(400-x^2\right)^{\frac{1}{2}}=0$

$$x(x+20)\left(400-x^2\right)^{-\frac{1}{2}}-\left(400-x^2\right)^{\frac{1}{2}}=0$$

$$\frac{x(x+20)}{\left(400-x^2\right)^{\frac{1}{2}}}-\left(400-x^2\right)^{\frac{1}{2}}=0$$

$$x(x+20)-\left(400-x^2\right)=0$$

$$x^2+20x-400+x^2=0$$

$$2x^2+20x-400=0$$

$$x^2+10x-200=0$$

$$(x-10)(x+20)=0$$

$$x=10,-20$$

But $x>0$ $\quad\therefore x=10$

x	10^-	10	10^+
A'	+	0	−

$\therefore$ the maximum occurs when $x=10$
i.e. when $BC=10$ cm

Alternatively:

$$A' = -\frac{x}{4}(x+20)\left(400-x^2\right)^{-\frac{1}{2}} + \frac{1}{4}\left(400-x^2\right)^{\frac{1}{2}}$$

$$= -\frac{1}{4}\left(400-x^2\right)^{-\frac{1}{2}}\left[x^2+20x+400-x^2\right]$$

$$= -\frac{1}{4}\left(400-x^2\right)^{-\frac{1}{2}}[20x+400]$$

$$= -\left(400-x^2\right)^{-\frac{1}{2}}(5x+100)$$

$$A'' = -\left(400-x^2\right)^{-\frac{1}{2}}[5] + (5x+100)\left[\frac{1}{2}\left(400-x^2\right)^{-\frac{3}{2}}(-2x)\right]$$

$$= -5\left(400-x^2\right)^{-\frac{1}{2}} - x(5x+100)\left(400-x^2\right)^{-\frac{3}{2}}$$

If $x = 10$:

$$A'' = -5\left(400-10^2\right)^{-\frac{1}{2}} - 10(50+100)\left(400-10^2\right)^{-\frac{3}{2}}$$

$$< 0$$

$\therefore$ the maximum occurs when $x = 10$

i.e. when $BC = 10$ cm

End of solutions