NSW INDEPENDENT

NSW INDEPENDENT SCHOOLS

2010
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used.
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

Student name / number ....

Marks

Question 1 **Begin a new booklet**

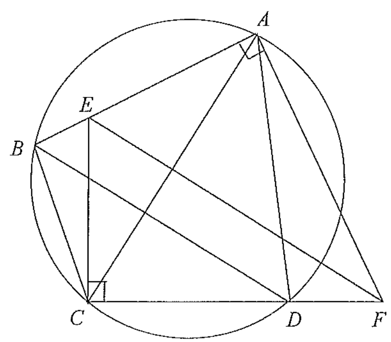
(a) Find the value of k such that $(x-2)$ is a factor of $P(x)=x^3+2x+k$. **2**

(b) Find the acute angle between the lines $y=3x-2$ and $y=2-x$, giving your answer correct to the nearest degree. **2**

(c) Find the number of ways in which 3 boys and 3 girls can be arranged in a straight line so that the tallest boy and the tallest girl occupy the two middle positions. **2**

(d) Find $\frac{d}{dx}\left(e^x\tan^{-1}x\right)$. **2**

(e)



$ABCD$ is a cyclic quadrilateral. The perpendicular to CD drawn from C meets AB at E. The perpendicular to AB drawn from A meets CD produced at F.

(i) Give a reason why $AECF$ is a cyclic quadrilateral. **1**

(ii) Hence show that $BD \parallel EF$. **3**

Student name / number ....

Question 2 **Begin a new booklet** **Marks**

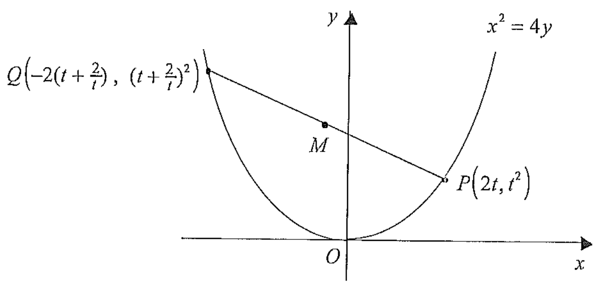
(a) Find the value of $\lim_{x \to 0} \frac{\sin 3x}{2x}$. **2**

(b) Solve the inequality $\frac{x+1}{x} > 0$. **2**

(c) Find the coordinates of the point P that divides the interval joining the points $A(-2, 5)$ and $B(7, -1)$ internally in the ratio $2:1$. **2**

(d) Find $\int \cos^2 2x \, dx$. **2**

(e)



$P(2t, t^2)$ and $Q\left(-2(t+\frac{2}{t}), (t+\frac{2}{t})^2\right)$ are two points on the parabola $x^2 = 4y$.
M is the midpoint of PQ.

(i) By considering the gradient of QP, show that QP is normal to the parabola at P. **2**

(ii) Find the equation of the locus of M as P moves on the parabola. **2**

Student name / number ....

Question 3 **Begin a new booklet** **Marks**

(a) Consider the function $f(x) = \frac{3x}{x^2 - 1}$.

(i) Show that the function is odd. **1**

(ii) Show that the function is decreasing for all values of x in its domain. **1**

(iii) Sketch the graph of the function showing clearly the equations of any asymptotes. **2**

(b)(i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\operatorname{cosec} x + \cot x = \cot\frac{x}{2}$. **2**

(ii) Hence evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x + \cot x)\, dx$, giving the answer in simplest exact form. **3**

(c) At time t hours after an oil spill occurs, a circular oil slick has radius r km, where $r = \sqrt{t+1} - 1$. Find the rate at which the area of the slick is increasing when its radius is 1 km, giving your answer correct to 2 decimal places. **3**

Student name / number ....

Marks

Question 4 **Begin a new booklet**

(a) At time t years after observation begins, the number N of birds in a colony is given by $N = 100 + 400\,e^{-0.1t}$.

(i) Sketch the graph of N as a function of t showing clearly the initial population size and the limiting population size. **2**

(ii) Find the time taken for the population size to fall to half its initial value, giving the answer correct to the nearest year. **2**

(b) Use the substitution $u = x + 1$ to find the value of $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$. **4**

(c) Use Mathematical Induction to show that $3^n - 2n - 1$ is divisible by 4 for all positive integers $n \geq 2$. **4**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a) Consider the function $f(x) = 2\cos^{-1}(x-1)$.

(i) Find the domain and range of the function. **2**

(ii) Sketch the graph of the function. **1**

(iii) Find the equation of the inverse function. **1**

(b) Four people visit a town with four restaurants A, B, C and D. Each person chooses a restaurant at random.

(i) Find the probability that they all choose different restaurants. **2**

(ii) Find the probability that exactly two of them choose restaurant A. **2**

(c) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} and acceleration a ms^{-2} given by $a = x + \frac{3}{2}$. Initially the particle is 5m to the right of O and moving towards O with a speed of 6 ms^{-1}.

(i) Explain whether the particle is initially speeding up or slowing down. **1**

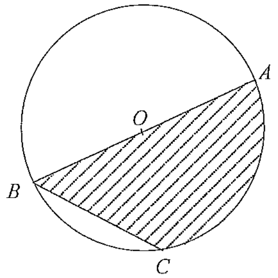
(ii) Show that $v^2 = x^2 + 3x - 4$. **2**

(iii) Find where the particle changes direction. **1**

Student name / number ....

Question 6 **Begin a new booklet** **Marks**

(a)



AB is a diameter of a circle with centre O and radius 1 cm. C is a point on the circle such that $\angle ABC = \theta$ radians and the perimeter of the shaded region is 5 cm.

(i) Show that $\theta + \cos\theta - 1\cdot 5 = 0$. **2**

(ii) Show that $0\cdot 8 < \theta < 0\cdot 9$. **2**

(iii) Use one application of Newton's method with an initial value $\theta_0 = 0\cdot 85$ to find the next approximation to the value of θ, giving your answer correct to 2 decimal places. **2**

(b) A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point O on the line, where $x = 4\cos^2 t - 1$.

(i) Show that $\ddot{x} = -4(x-1)$. **2**

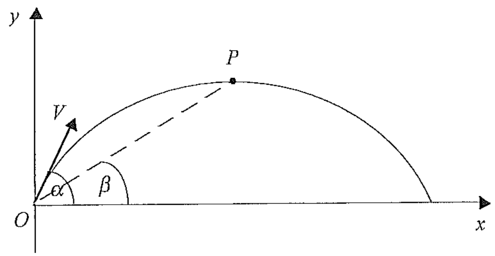
(ii) Sketch the graph of x as a function of t for $0 \le t \le \pi$, showing clearly the times when the particle passes through O. **2**

(iii) For $0 \le t \le \pi$, find the time when the velocity of the particle is increasing most rapidly, and find this maximum rate of increase in the velocity. **2**

Student name / number ....

Question 7 **Begin a new booklet** **Marks**

(a)



A particle is projected from a point O with speed V ms^{-1} at an angle α above the horizontal, where $0 < \alpha < \frac{\pi}{2}$. It moves in a vertical plane subject to gravity where the acceleration due to gravity is 10 ms^{-2}. At time t seconds it has horizontal and vertical displacements x metres and y metres respectively from O. At point P where it attains its greatest height the angle of elevation of the particle from O is β radians.

(i) Use integration to show that $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - 5t^2$. **2**

(ii) Show that $\tan\beta = \frac{1}{2}\tan\alpha$. **3**

(iii) If the particle has greatest height 80 m above O at a horizontal distance 120 m from O, find the exact values of α and V. **3**

(b)(i) Write down the binomial expansion of $(1+x)^{2n+1}$. **1**

(ii) Hence show that $\sum_{r=0}^{n} \binom{2n+1}{r} = 4^n$. **3**

Question 1

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• uses the factor theorem to write an equation for k	1
• solves for k	1

Answer

$(x-2)$ is a factor $\therefore P(2)=0$. Hence $8+4+k=0$. $\therefore k=-12$

b. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• writes a numerical expression for $\tan\theta$	1
• finds the angle to the nearest degree.	1

Answer

The lines have gradients 3 and -1. $\therefore \tan\theta=\left|\dfrac{3-(-1)}{1+3\times(-1)}\right|=2$ $\therefore \theta \approx 63°$

c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• realises that the tallest boy and girl can be positioned in two ways	1
• includes the factor 4! to arrange the remaining students	1

Answer

• • * * • • Position the tallest boy and girl (*'s) in 2 ways, then the others (•'s) in 4! ways.
Hence number of arrangements is $2\times 4!=48$.

d. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
• quotes the derivative of the inverse tan function	1
• applies the product rule	1

Answer

$$\frac{d}{dx}\left(e^x\tan^{-1}x\right)=e^x\tan^{-1}x+\frac{e^x}{1+x^2}$$

1e. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • explains why $AECF$ is cyclic	1
ii • quotes property of cyclic quad $ABCD$ to deduce $\angle BDC=\angle BAC$	1
• quotes property of cyclic quad $AECF$ to deduce $\angle EAC=\angle EFC$	1
• identifies equal corresponding angles to deduce $BD \parallel EF$	1

Answer

i. In quadrilateral $AECF$, opposite interior angles at A and C are supplementary. Hence $AECF$ is cyclic.

ii. $\angle BDC=\angle BAC$ (*∠'s subtended at the circumference of circle ABCD by same arc BC are equal*)
$\angle EAC=\angle EFC$ (*∠'s subtended at the circumference of circle AECF by same arc EC are equal*)
$\therefore \angle BDC=\angle EFC$ (*∠BAC, ∠EAC same angle, since B, E, A collinear*)
$\therefore BD \parallel EF$ (*equal corresponding ∠'s on transversal CF*)

Question 2

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• rearranges subject of limit into appropriate form	1
• applies known limit	1

Answer

$$\lim_{x\to 0}\frac{\sin 3x}{2x}=\frac{3}{2}\lim_{x\to 0}\frac{\sin 3x}{3x}=\tfrac{3}{2}\times 1=\tfrac{3}{2}$$

b. Outcomes assessed : P4, PE3

Marking Guidelines

Criteria	Marks
• finds one inequality for x	1
• correctly combines this with a second inequality to quote all the solutions	1

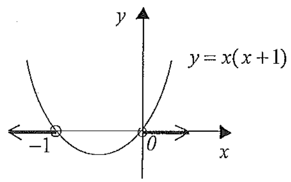
Answer

Multiplying both sides by x^2,

$\dfrac{x+1}{x}>0 \Rightarrow x(x+1)>0$ and $x\neq 0$

$\therefore x<-1$ *or* $x>0$

(by inspection of the graph)



2c. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• finds the x coordinate	1
• finds the y coordinate	1

Answer

$A(-2\,,5)\quad B(7\,,-1)$

$2 \quad : \quad 1$

$$P\left(\frac{1\times(-2)+2\times 7}{2+1}\,,\ \frac{1\times 5+2\times(-1)}{2+1}\right)$$

$\therefore P(4\,,1)$

d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• uses an appropriate trigonometric identity	1
• finds the primitive function	1

Answer

$$\int \cos^2 2x\ dx = \tfrac{1}{2}\int \left(1+\cos 4x\right)\, dx = \tfrac{1}{2}x + \tfrac{1}{8}\sin 4x + c$$

e. Outcomes assessed : PE2, PE4

Marking Guidelines

Criteria	Marks
i • finds gradient of normal at P	1
• finds gradient of QP	1
ii • finds and simplifies x and y coordinates of M in terms of t	1
• eliminates t to find Cartesian equation of locus of M	1

Answer

i. At the general point $(2t\,,t^2)$

$y=t^2 \Rightarrow \dfrac{dy}{dt}=2t$

$x=2t \Rightarrow \dfrac{dx}{dt}=2$

$\therefore \dfrac{dy}{dx}=\dfrac{2t}{2}=t$

Hence gradient of normal at P is $\dfrac{-1}{t}$.

Join of $P\left(2t\,,t^2\right)$ and $Q\left(-2(t+\frac{2}{t})\,,(t+\frac{2}{t})^2\right)$ has gradient

$$\frac{(t+\frac{2}{t})^2-t^2}{-2(t+\frac{2}{t})-2t} = \frac{4+4(\frac{1}{t})^2}{-4t-4(\frac{1}{t})} = \frac{4(t^2+1)}{-4t(t^2+1)} = \frac{-1}{t}$$

$\therefore QP$ is normal to the parabola at P.

2e ii. At M,

$$x=\tfrac{1}{2}\left\{2t-2\left(t+\tfrac{2}{t}\right)\right\}=-\tfrac{2}{t}$$

$$y=\tfrac{1}{2}\left\{t^2+\left(t+\tfrac{2}{t}\right)^2\right\}=t^2+2+\tfrac{2}{t^2}$$

$\therefore$ locus of M has equation $y=\frac{4}{x^2}+2+\frac{1}{2}x^2$

Question 3

a. Outcomes assessed : P5, H6

Marking Guidelines

Criteria	Marks
i • formally proves f is odd	1
ii • finds $f'(x)$ and considers sign to deduce f is decreasing	1
iii • shows branches of curve for $\lvert x\rvert>1$ with asymptotes $x=1,\ x=-1,\ y=0$	1
• shows branch of curve for $\lvert x\rvert<1$	1

Answer

i. $f(-x)=\dfrac{3(-x)}{(-x)^2-1} = -\dfrac{3x}{x^2-1}$

$\therefore f(-x)=-f(x)$ for all $x\neq\pm1$

$\therefore f(x)$ is an odd function.

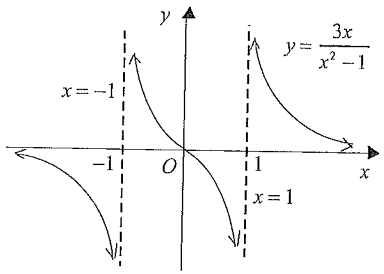
ii.

$$f'(x)=3\ \frac{1.(x^2-1)-x\,.\,2x}{(x^2-1)^2} = \frac{-3\,(x^2+1)}{(x^2-1)^2}$$

$\therefore f'(x)<0$ for $x\neq\pm1$

$\therefore f(x)$ is decreasing throughout its domain.

iii.



b. Outcomes assessed : H3, H5

Marking Guidelines

Criteria	Marks
i • writes one of cosec x or $\cot x$ in terms of t	1
• writes the second reciprocal trig. ratio in terms of t, then simplifies sum to obtain result	1
ii • finds primitive function	1
• evaluates, giving surd values for trigonometric ratios	1
• uses log laws to obtain simplest exact answer.	1

Answer

i. $t=\tan\frac{x}{2} \Rightarrow \quad \operatorname{cosec}x+\cot x = \dfrac{1+t^2}{2t}+\dfrac{1-t^2}{2t} = \dfrac{1+t^2+1-t^2}{2t}=\dfrac{1}{t}$

$\therefore \operatorname{cosec}x+\cot x=\cot\frac{x}{2}$

3b ii. Let $I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(\operatorname{cosec}x+\cot x\right)dx$

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cot\tfrac{x}{2}\,dx$$

$$=2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\frac{1}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dx$$

$$=2\left[\ln\left(\sin\tfrac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$\therefore I=2\left\{\ln\left(\sin\tfrac{\pi}{4}\right)-\ln\left(\sin\tfrac{\pi}{6}\right)\right\}$$

$$=2\left(\ln\tfrac{1}{\sqrt{2}}-\ln\tfrac{1}{2}\right)$$

$$=2\ln\left(2^{-\frac{1}{2}}\div 2^{-1}\right)$$

$$=2\ln 2^{\frac{1}{2}}$$

$$=\ln 2$$

c. Outcomes assessed : HE5, HE7

Marking Guidelines

Criteria	Marks
• writes the area A as a function of t and differentiates	1
• finds either t or $\sqrt{t+1}$ when $r=1$	1
• substitutes for t to find the required rate to stated accuracy, giving units	1

Answer

$$A=\pi\left\{\sqrt{t+1}-1\right\}^2$$

$$\frac{dA}{dt}=2\pi\left\{\sqrt{t+1}-1\right\}.\tfrac{1}{2}(t+1)^{-\frac{1}{2}}$$

$$=\pi\,\frac{\sqrt{t+1}-1}{\sqrt{t+1}}$$

$$r=1\ \Rightarrow\ \sqrt{t+1}-1=1$$

$$\sqrt{t+1}=2$$

$$\therefore\ \frac{dA}{dt}=\pi\,\frac{2-1}{2}=\frac{\pi}{2}$$

Ans. 1·57 km/h

Question 4

a. Outcomes assessed : H3, HE3

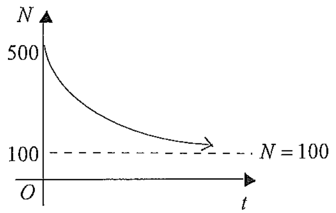
Marking Guidelines

Criteria	Marks
i • shows vertical intercept 500 with decreasing function	1
• shows curve approaching asymptote $N=100$ as $t\to\infty$	1
ii • finds value of $e^{-0.1t}$	1
• solves equation for t	1

Answer

i. $N=100+400\,e^{-0.1t}$

$t=0\ \Rightarrow N=500$ and $t\to\infty\ \Rightarrow\ N\to 100$



ii. $N=250\ \Rightarrow\ 400e^{-0.1t}=150$

$$e^{-0.1t}=\tfrac{3}{8}$$

$$-\tfrac{1}{10}t=\ln\tfrac{3}{8}$$

$$\therefore t=-10\ln\tfrac{3}{8}\approx 9\cdot 8$$

Ans. 10 years (to the nearest year)

4b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes integral in terms of u	1
• rearranges integrand into sum of index expressions in u	1
• finds primitive function	1
• evaluates	1

Answer

$u=x+1$

$du=dx$

$x=0\ \Rightarrow\ u=1$

$x=3\ \Rightarrow\ u=4$

$$\int_0^3\frac{x}{\sqrt{x+1}}\,dx=\int_1^4\frac{u-1}{\sqrt{u}}\,du$$

$$=\int_1^4\left(u^{\frac{1}{2}}-u^{-\frac{1}{2}}\right)du$$

$$=\left[\tfrac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]_1^4$$

$$=\tfrac{2}{3}(4^{\frac{3}{2}}-1)-2(4^{\frac{1}{2}}-1)$$

$$=2\tfrac{2}{3}$$

c. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• defines an appropriate sequence of statements to be tested by Mathematical Induction	1
• verifies that the first statement is true	1
• shows that the truth of the k^{th} statement implies the truth of the next statement in the sequence	1
• completes the process of Mathematical Induction	1

Answer

Let $S(n)$, $n=2,3,4,\ldots$ be the sequence of statements defined by $S(n)$: $3^n-2n-1=4I$ for some integer I.

Consider $S(2)$: $3^2-2\times 2-1=4\times 1$ Hence $S(2)$ is true.

If $S(k)$ is true, $k\ge 2$, $3^k-2k-1=4I$ for some integer I. *

Consider $S(k+1)$: $3^{k+1}-2(k+1)-1=3\left\{3^k-2k-1\right\}+4k$

$=3\times 4I+4k$ if $S(k)$ is true, using *

$=4\,(3I+k)$ where $(3I+k)$ is an integer.

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(2)$ is true, hence $S(3)$ is true and then $S(4)$ is true and so on.

$\therefore S(n)$ is true, ie. 3^n-2n-1 is divisible by 4, for all positive integers $n\ge 2$.

Question 5

a. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i • finds domain	1
• finds range	1
ii • sketches curve with correct shape, position and intercepts on the coordinate axes	1
iii • finds the equation of the inverse function	1

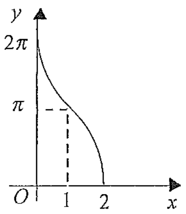
Answer

i. $f(x) = 2\cos^{-1}(x-1)$

$-1 \le x-1 \le 1$ and $0 \le \cos^{-1}(x-1) \le \pi$

$\therefore$ Domain: $\{x : 0 \le x \le 2\}$ Range : $\{y : 0 \le y \le 2\pi\}$

ii.



iii. $0 \le y \le 2\pi$ and

$$y = 2\cos^{-1}(x-1)$$
$$\tfrac{1}{2}y = \cos^{-1}(x-1)$$
$$\cos(\tfrac{1}{2}y) = x-1$$
$$x = 1+\cos(\tfrac{1}{2}y)$$

Hence, interchanging x and y,

$$f^{-1}(x) = 1+\cos(\tfrac{1}{2}x), \quad 0 \le x \le 2\pi$$

b. Outcomes assessed : H5, HE3

Marking Guidelines

Criteria	Marks
i • recognises that there are 4! ways for 4 people to choose different restaurants	1
• divides by 4^4 to find the probability	1
ii • recognises the binomial distribution and writes an expression for the probability	1
• calculates the probability	1

Answer

i. $\dfrac{4!}{4^4} = \dfrac{4\times3\times2\times1}{4\times4\times4\times4} = \dfrac{3}{32}$

ii. Probability that a person chooses restaurant A is $\frac{1}{4}$

$P(\textit{exactly 2 choose A}) = {}^4C_2(\tfrac{1}{4})^2(\tfrac{3}{4})^2 = \tfrac{27}{128}$

c. Outcomes assessed : HE5

Marking Guidelines

Criteria	Marks
i • notes that initially v, a have opposite signs to deduce that particle is slowing down	1
ii • writes a as a derivative with respect to x and integrates to find expression for v^2	1
• uses initial conditions to evaluate the constant of integration	1
iii • considers $v=0$ to find that particle changes direction when it reaches $x=1$	1

5c. Answer

i. $a = x+\frac{3}{2}$. Initially $x=5$ and $v=-6$

$\therefore a = \frac{13}{2} > 0$ and $v<0$. Hence particle is slowing down.

ii.
$$\frac{d(\frac{1}{2}v^2)}{dx} = x+\tfrac{3}{2}$$
$$\tfrac{1}{2}v^2 = \tfrac{1}{2}x^2+\tfrac{3}{2}x+c_1$$
$$v^2 = x^2+3x+c$$
$$\left.\begin{matrix} x=5 \\ v=-6 \end{matrix}\right\} \Rightarrow \quad 36 = 25+15+c$$
$$\therefore c=-4$$
$$\therefore v^2 = x^2+3x-4$$

iii. $v=0 \Rightarrow (x+4)(x-1)=0$

Particle starts at $x=5$ and moves left towards O.
When particle reaches $x=1$, $v=0$ and $a=\frac{5}{2}>0$.
Hence particle is instantaneously at rest at $x=1$, then moves off to the right and subsequently continues moving right while speeding up ($v>0$ and $a>0$)
Hence only change of direction is at a point 1 m to the right of O.

Question 6

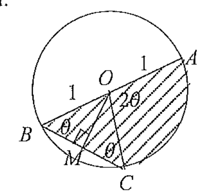
a. Outcomes assessed : H5, PE3, HE1

Marking Guidelines

Criteria	Marks
i • finds BC in terms of θ	1
• uses perimeter in terms of θ to write equation	1
ii • shows that $f(0\cdot8)$, $f(0\cdot9)$ have opposite signs	1
• notes that f is continuous and makes required deduction about θ	1
iii • finds $f'(\theta)$ and substitutes into Newton's formula	1
• calculates next approximation.	1

Answer

i.



Let M be the foot of the perpendicular form O to BC.
Then $BC = 2BM = 2\cos\theta$ (*radius ⊥ chord bisects chord*)

In ΔOBC, $\angle OCB = \angle OBC = \theta$ (*equal ∠'s opp. equal sides*)
$\therefore \angle AOC = 2\theta$ (*ext. ∠ is sum of int. opp. ∠'s*)
Hence *Arc* $AC = 2\theta$

Perimeter is 5 cm $\therefore 2\theta + 2\cos\theta + 2 = 5$
$\theta + \cos\theta - 1\cdot5 = 0$

ii. Let $f(\theta) = \theta+\cos\theta-1\cdot5$.
Then $f'(\theta) = 1-\sin\theta$. We want roots of $f(\theta)=0$.
f is a continuous, increasing function such that $f(0\cdot8) \approx -0\cdot003 < 0$ and $f(0\cdot9) \approx 0\cdot022 > 0$
Hence there is exactly one root θ and $0\cdot8 < \theta < 0\cdot9$.

iii. Applying Newton's method with initial approximation $\theta_0 = 0\cdot85$, next approximation is

$$\theta \approx 0\cdot85 - \frac{0\cdot85+\cos0\cdot85-1\cdot5}{1-\sin0\cdot85} \approx 0\cdot85 - \frac{0\cdot009983}{0\cdot248720} \approx 0\cdot81$$

6b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • differentiates once to find $\dot{x}$	1
• differentiates a second time and rearranges to obtain required result	1
ii • shows curve with correct position, shape and endpoints	1
• shows intercepts on t axis	1
iii • states time when v increases most rapidly	1
• states maximum rate of increase	1

Answer

i. $x = 4\cos^2 t - 1$

$= 2(1+\cos 2t) - 1$

$= 1 + 2\cos 2t$

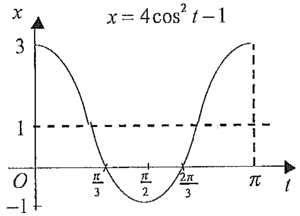
$\dot{x} = -4\sin 2t$

$\ddot{x} = -8\cos 2t$

$= -4(2\cos 2t)$

$= -4(x-1)$

ii.



iii. v is increasing most rapidly when $\ddot{x}$ takes its greatest positive value, ie. when x takes its least value.

This extreme value of $\ddot{x}$ is $-4(-1-1) = 8$ when $t = \frac{\pi}{2}$ and $x = -1$.

Hence v increases most rapidly at time $\frac{\pi}{2}$ seconds and this maximum rate of increase is $8\ \text{ms}^{-2}$.

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • finds expression for x by integration	1
• finds expression for y by integration	1
ii • finds t at greatest height	1
• finds x and y in terms of V, α for this t value	1
• uses the gradient of OP to establish result	1
iii • finds $\tan\alpha$ and hence the exact value of α	1
• finds the exact value of $\sin\alpha$ and writes an equation for V^2	1
• finds the exact value of V	1

Answer

i. $t = 0 \Rightarrow x = y = 0,\quad \dot{x} = V\cos\alpha,\ \dot{y} = V\sin\alpha$

$\ddot{x} = 0 \Rightarrow \dot{x}$ is constant

$\therefore \dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + c_1,\quad c_1$ constant

$t = 0,\ x = 0 \Rightarrow c_1 = 0$

$\therefore x = Vt\cos\alpha$

$\ddot{y} = -10 \Rightarrow \dot{y} = -10t + c_2,\quad c_2$ constant

$t = 0, \dot{y} = V\sin\alpha \Rightarrow c_2 = V\sin\alpha \qquad \therefore \dot{y} = -10t + V\sin\alpha$

$y = -5t^2 + Vt\sin\alpha + c_3,\quad c_3$ constant

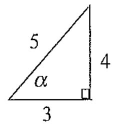
$t = 0,\ y = 0 \Rightarrow c_3 = 0 \qquad \therefore y = Vt\sin\alpha - 5t^2$

7a ii. At greatest height, $\dot{y} = 0 \Rightarrow 10t = V\sin\alpha \qquad \therefore t = \frac{1}{10}V\sin\alpha$

$\therefore$ at point P, $\quad x = \frac{1}{10}V^2\sin\alpha\cos\alpha \quad$ and $\quad y = (\frac{1}{10} - \frac{5}{100})V^2\sin^2\alpha = \frac{1}{20}V^2\sin^2\alpha$

Then $\tan\beta = \textit{gradient OP} = \dfrac{\frac{1}{20}V^2\sin^2\alpha}{\frac{1}{10}V^2\sin\alpha\cos\alpha} = \frac{1}{2}\tan\alpha$

iii. $P(120, 80) \Rightarrow \tan\beta = \frac{80}{120} \Rightarrow \frac{1}{2}\tan\alpha = \frac{2}{3} \qquad \therefore \tan\alpha = \frac{4}{3}$



$80 = \frac{1}{20}V^2\sin^2\alpha = \frac{1}{20}V^2 \times \frac{16}{25} \qquad \therefore V^2 = 2500$

$\therefore \alpha = \tan^{-1}\frac{4}{3},\quad V = 50$

b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • writes expansion, with or without using sigma notation	1
ii • substitutes $x = 1$	1
• rewrites sum using only binomial coefficients $\binom{2n+1}{r}$, $0 \le r \le n$	1
• deduces required result	1

Answer

i. $(1+x)^{2n+1} = \sum_{r=0}^{2n+1}\binom{2n+1}{r}x^r = \binom{2n+1}{0} + \binom{2n+1}{1}x + \ldots + \binom{2n+1}{n}x^n + \binom{2n+1}{n+1}x^{n+1} + \ldots + \binom{2n+1}{2n+1}x^{2n+1}$

ii. Substituting $x = 1$, $\quad 2^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1} + \ldots + \binom{2n+1}{n} + \binom{2n+1}{n+1} + \ldots + \binom{2n+1}{2n} + \binom{2n+1}{2n+1}$

$2^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1} + \ldots + \binom{2n+1}{n} + \binom{2n+1}{n} + \ldots + \binom{2n+1}{1} + \binom{2n+1}{0}$

using $\binom{2n+1}{k} = \binom{2n+1}{2n+1-k}$

$2^{2n+1} = 2\sum_{r=0}^{n}\binom{2n+1}{r}$

$\therefore \sum_{r=0}^{n}\binom{2n+1}{r} = 2^{2n} = 4^n$