SCEGGS Darlinghurst

2010

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks)

(a) Find $\int \cos^3 x \sin x \, dx$ **2**

(b) Find $\int \dfrac{1}{1+e^{-x}} \, dx$ **2**

(c) Evaluate $\int_0^1 \dfrac{x^2}{\sqrt{1-x^2}} \, dx$ **3**

(d) Use the substitution $x = \sec\theta$ to evaluate $\int_1^{\sqrt{2}} \dfrac{1}{x\sqrt{x^2-1}} \, dx$ **3**

(e) (i) Express $\dfrac{3}{(x+1)(x^2+2)}$ in the form $\dfrac{a}{x+1} + \dfrac{bx+c}{x^2+2}$, where a, b and c are constants. **3**

(ii) Hence find $\int \dfrac{3}{(x+1)(x^2+2)} \, dx$. **2**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 2 + 3i$ and $w = \bar{z}$. Find, in the form $x + iy$, where x and y are real:

(i) wz **1**

(ii) $\dfrac{z}{w}$. **1**

(b) (i) Find the square roots of $-3 + 4i$ in the form $a + ib$ where a and b are real. **3**

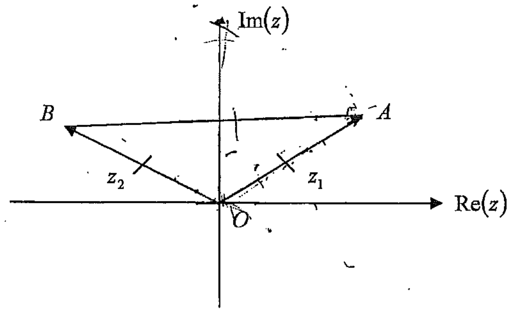
(ii) Hence, solve the equation $(1 + i)z^2 - z - i = 0$. **3**

(c) Sketch the region in the complex plane where the inequalities $z + \bar{z} < 8$, $|z| \geq 4$ and $|\arg z| < \dfrac{\pi}{3}$ hold simultaneously. **3**

Question 2 continues on page 4

Question 2 (continued) **Marks**

(d) In the Argand diagram, vectors $\vec{OA}$ and $\vec{OB}$ represent the complex numbers z_1 and z_2 respectively.



Given that ΔAOB is isosceles and $\angle BOA = \dfrac{2\pi}{3}$:

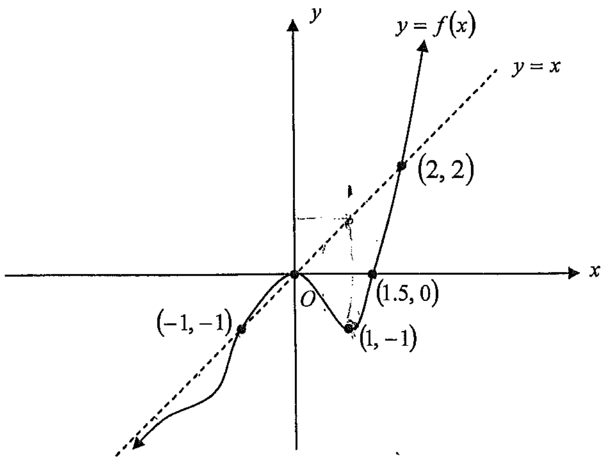
(i) find an expression for z_2 in terms of z_1 **1**

(ii) show that $(z_1 + z_2)^2 = z_1 z_2$. **3**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows $y = f(x)$. The line $y = x$ is an asymptote.



Draw separate one-third page sketches of the graphs of the following. Clearly label important features.

(i) $y = f(-x)$ **1**

(ii) $y = f(x+2)$ **1**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = x.f(x)$ **2**

Question 3 continues on page 6

Question 3 (continued)

(b) The equation $x^3 + 4x^2 + 2x - 1 = 0$ has roots α, β and γ.

(i) Evaluate $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) Evaluate $\alpha^3 + \beta^3 + \gamma^3$ **2**

(iii) Find a cubic polynomial with integer coefficients whose roots are α^2, β^2 and γ^2. **2**

(c) Find the equation of the tangent to the curve defined by $x^2 - xy + y^2 = 5$ at the point $(2, -1)$. **3**

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 - 3x^2 - 9x + k = 0$ has a double root. **2**

Find the possible values of k.

(b) Use the substitution $t = \tan\frac{x}{2}$ to find $\displaystyle\int \frac{1}{5 + 4\cos x + 3\sin x}\,dx$ **3**

(c) (i) By completing the square, show that $4x^2 + 9y^2 + 24x - 36y + 36 = 0$ represents an ellipse in the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ **1**

(ii) Find the eccentricity, e. **1**

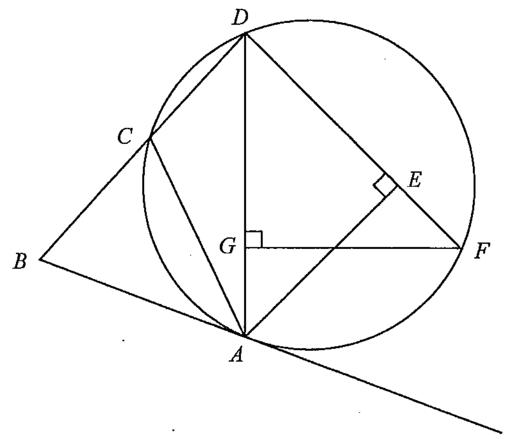
(iii) Sketch the ellipse showing the centre, the foci and the directrices. **3**

Question 4 continues on page 8

Marks

Question 4 (continued)

(d)



In the diagram given, BA is a tangent to the circle at A and the secant BD cuts the circle at C.

DA and DF are two chords such that FG and AE are perpendicular to DA and DF respectively.

Copy the diagram.

(i) Prove that $\angle ACB = \angle BAD$. **2**

(ii) Explain why $AGEF$ is a cyclic quadrilateral with diameter AF. **1**

(iii) Prove that $\angle AGE = \angle ACD$. **2**

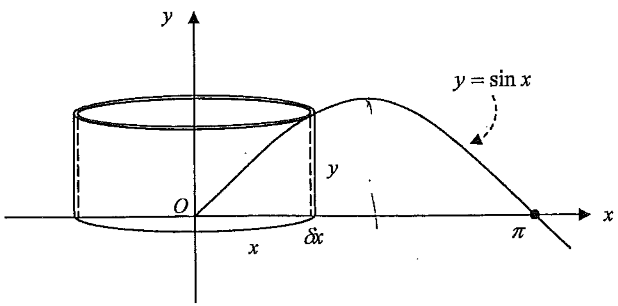
End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) Sebastian, a mathematically-minded sculptor, decided to make a series of pieces modelled on volumes formed by mathematical curves.

His first piece entitled "Give me a Sine" was formed by taking the area under the curve $y = \sin x$ between $x = 0$ to $x = \pi$ and rotating it about the y-axis.



(i) Using the method of cylindrical shells, show that the volume, V, of the resulting solid of revolution is given by **2**

$$V = 2\pi \int_0^{\pi} x \sin x \ dx$$

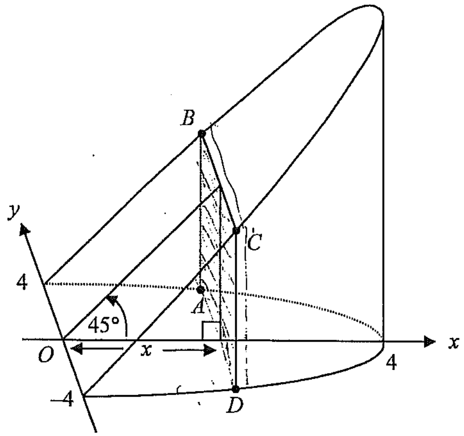
(ii) Use integration by parts to find the exact volume. **3**

Question 5 continues on page 10

Marks

Question 5 (continued)

(b) Sebastian's second sculpture, "The Wedge" was obtained by cutting a right cylinder of radius 4 units at 45° through a diameter of its base.



A rectangular slice $ABCD$, of thickness δx, is taken perpendicular to the base of the wedge at a distance x from the y-axis.

(i) Show that the area of $ABCD$ is given by $2x\sqrt{16 - x^2}$. **2**

(ii) Find the exact volume of the wedge. **3**

(c) (i) Prove that $\cos(A - B)x - \cos(A + B)x = 2\sin Ax \sin Bx$. **1**

(ii) Using the above result, show that the equation $\sin 3x \sin x = 2\cos 2x + 1$ can be written as a quadratic equation in terms of $\cos 2x$. **2**

(iii) Hence find the general solution of $\sin 3x \sin x = 2\cos 2x + 1$. **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) In preparation for a school formal, a committee of three is to be chosen from four Year 12 Prefects and n Year 12 non-Prefects $(n \geq 2)$.

(i) Show that the number of possible committees containing exactly one Prefect is $2n(n-1)$. **1**

(ii) Find the number of possible committees containing exactly two Prefects. **1**

(iii) Deduce that the probability P of the committee containing either one or two Prefects is **1**

$$P = \frac{12n}{(n+4)(n+3)}$$

(b) For each integer $n \geq 0$, let $I_n = \int_1^2 x(\ln x)^n \, dx$.

(i) Show that for $n \geq 1$, **2**

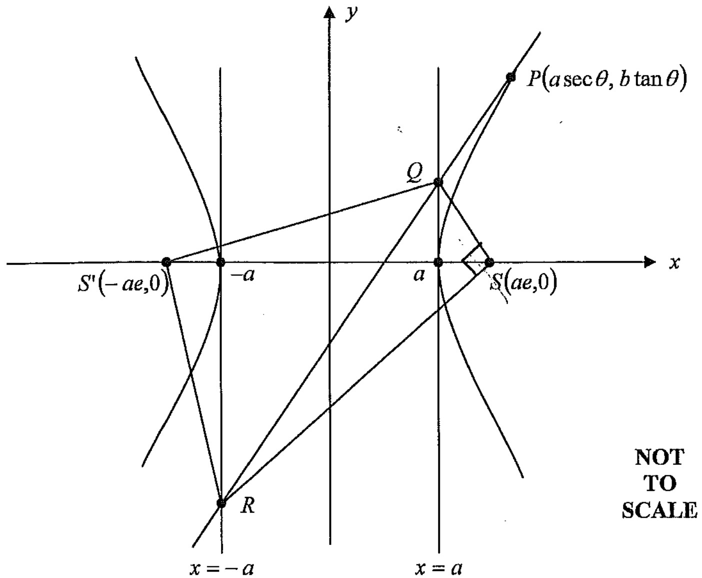
$$I_n = 2(\ln 2)^n - \frac{n}{2} I_{n-1}$$

(ii) Hence evaluate I_3. **3**
(Leave your answer in exact form.)

Question 6 continues on page 12

Question 6 (continued) **Marks**

(c)



$P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
The tangent at P meets the lines $x = a$ and $x = -a$ at Q and R respectively.

(i) Show that the equation of the tangent is given by $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. **2**

(ii) Find the coordinates of Q and R. **1**

(iii) Show that QR subtends a right angle at the focus $S(ae,0)$. **2**

(iv) Deduce that Q, S, R, S' are concyclic. **2**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A family of six are sitting at a round table. In how many ways can they be arranged so that Mum and Dad sit together and the youngest daughter, Chelsea, does not sit opposite Dad? **2**

(b) (i) Factorise the cubic polynomial $z^3 - 64$:

(A) over the real numbers. **1**

(B) over the complex numbers. **1**

(ii) Let ω be one of the complex roots of the equation $z^3 - 64 = 0$.

(A) Show that $\omega^2 = -4(\omega + 4)$. **1**

(B) Hence evaluate $(4\omega + 16)^3$. **1**

(c) A sequence of numbers T_1, T_2, T_3, ... is defined by $T_1 = 1$, $T_2 = 5$ and $T_k = 5T_{k-1} - 6T_{k-2}$.

(i) Show that the statement $T_n = 3^n - 2^n$ is true for $n = 1, 2, 3$. **2**

(ii) Prove by induction that $T_n = 3^n - 2^n$ for all integers $n \geq 1$. **2**

(d) (i) Using the substitution $u = a - x$, show that **1**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(ii) Hence evaluate $\int_0^\pi x\sin^2 x\,dx$. **4**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(x) = x - \frac{1}{2}\tan x$.

(i) Show that $f(x)$ is an odd function. **1**

(ii) Find the value of any stationary points in the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and determine their nature. **3**

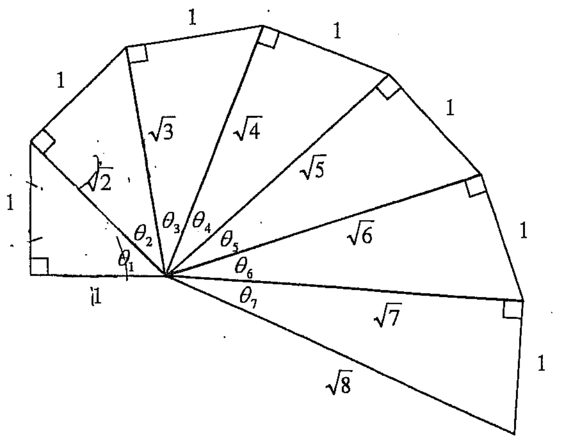
(iii) Sketch the curve $y = f(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. (You do not need to find the values of all x-intercepts.) **2**

(iv) Hence, or otherwise, show that $x \geq \frac{1}{2}\tan x$ for $0 \leq x \leq \frac{\pi}{4}$ and state when equality holds in the given domain. **2**

Question 8 continues on page 16

Question 8 (continued) **Marks**

(b) A spiral is created by constructing a right-angled triangle on the hypotenuse of the previous triangle as shown in the diagram.



Each triangle has an altitude of 1 unit and the hypotenuse lengths form a sequence $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots$

Let the angle, θ, in each triangle be $\theta_1, \theta_2, \theta_3, \ldots$ as shown in the diagram. The angle in the nth triangle is given by θ_n.

(i) Write down expressions for $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_n$. **1**

(ii) Within what range of values does θ lie? **1**

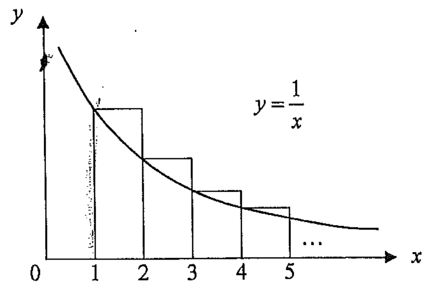
(iii) Using the result from part (a) (iv), show that $\sum_{n=1}^{k}\theta_n \geq \frac{1}{2}\sum_{n=1}^{k}\frac{1}{n}$. **3**

Question 8 continues on page 17

Question 8 (continued) **Marks**

(b) (continued)

(iv) The curve $y = \frac{1}{x}$ is drawn in the first quadrant and upper rectangles are drawn as shown in the diagram. **1**



Show that $\sum_{n=1}^{k}\frac{1}{n} > \int_1^{k+1}\frac{1}{n}\,dn$.

(v) Hence, deduce that $\sum_{n=1}^{k}\theta_n > \ln\sqrt{k+1}$. **1**

End of paper

Ext 2 Trial HSC Examination 2010. (1)

QUESTION 1

a) $\int \cos^3 x \sin x\,dx$

$= -\frac{\cos^4 x}{4} + C$ ✓ ✓

Using Reverse Chain rule.

$\int (f(x))^n f'(x)\,dx = \frac{f(x)^{n+1}}{n+1} + C$

There are other answers possible but this is the easiest.

b) $\int \frac{1}{1+e^{-x}}\,dx$

(multiply top & bottom by e^x)

$= \int \frac{e^x}{e^x+1}\,dx$ ✓

$= \ln(e^x+1) + c$ ✓

using $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$

Note this technique if you weren't sure.

Substitution of $u = e^{-x}$ is possible but it's a much longer solution for Q1b!

c) $\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$

method 1

Rewrite the top.

$\int_0^1 \frac{1-(1-x^2)+1}{\sqrt{1-x^2}}\,dx$

now split up

$= \int_0^1 \frac{-(1-x^2)\,dx}{\sqrt{1-x^2}} + \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

$= -\int_0^1 \sqrt{1-x^2}\,dx + \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ ✓

($\frac{1}{4}$ of a circle)

$= -\frac{1}{4}\pi \times 1^2 + \left[\sin^{-1} x\right]_0^1$ ✓

$= -\frac{\pi}{4} + (\sin^{-1} 1 - \sin^{-1} 0)$

$= -\frac{\pi}{4} + \frac{\pi}{2}$

$= \frac{\pi}{4}$ ✓

method 2

using a trig substitution.

$x = \sin\theta$

$dx = \cos\theta\,d\theta$.

$\theta = \sin^{-1} x$

Limits

$x = 1\ \theta = \frac{\pi}{2}$

$x = 0\ \theta = 0$

$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cdot \cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}}$

$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cancel{\cos\theta}}{\sqrt{\cancel{\cos^2\theta}}}\,d\theta$

$= \int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= \frac{1}{2}\left\{\frac{\pi}{2} - \frac{1}{2}\sin\pi - (0)\right\}$

$= \frac{1}{2} \times \frac{\pi}{2}$

Use either method shown here.

method 1 has some tricks that are worth noticing.

Method 2 is a standard approach. You should recognise that you can use a trig substitution without it being specifically given.

Q1 continued (2)

d) $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

(See standard integrals)

$\sec\theta = x$

$\frac{1}{\cos\theta} = x$

$\cos\theta = \frac{1}{x}$

$\theta = \cos^{-1}\left(\frac{1}{x}\right)$

Change limits

$x = \sqrt{2}\quad \theta = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$x = 1\quad \theta = \cos^{-1} 1 = 0$

$\int_1^{\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}} \times \sec\theta\tan\theta\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\cancel{\sec\theta}\cancel{\sqrt{\tan^2\theta}}}\cancel{\sec\theta}\cancel{\tan\theta}\,d\theta$ ✓

$= \int_0^{\pi/4} 1\,d\theta$ ✓

$= \left[\theta\right]_0^{\pi/4}$

$= \frac{\pi}{4} - 0$ ✓

$= \frac{\pi}{4}$

(*) the substitution and 'integration steps were well done but it really isn't hard to find the new limits.

You should be able to do

$\sec\theta = \sqrt{2}$

$\frac{1}{\cos\theta} = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

e) $\frac{3}{(x+1)(x^2+2)} = \frac{a}{x+1} + \frac{bx+c}{x^2+2}$

$3 = a(x^2+2) + (bx+c)(x+1)$

Substitute $x = -1$

$3 = 3a + 0$

$a = 1$ ✓

Substitute $x = 0, a = 1$

$3 = 1 \times 2 + c \times 1$

$3 = 2 + c$

$c = 1$ ✓

Subst. $x = 1, a = 1, c = 1$

$3 = 1(1+2) + (b+1)(1+1)$

$3 = 3 + 2(b+1)$

$0 = 2(b+1)$

$b = -1$. ✓

$\therefore \frac{3}{\ldots} = \frac{1}{\ldots} + \frac{-x+1}{\ldots}$

this part was very well done by everyone ☺

Q1 continued.

e) ii) $\int \frac{3}{(x+1)(x^2+2)}\,dx$

$= \int \left(\frac{1}{x+1} + \frac{-x+1}{x^2+2}\right) dx$

$= \int \frac{1}{x+1}dx - \frac{1}{2}\int \frac{2x}{x^2+2}dx + \int \frac{1}{x^2+\sqrt{2}^2}dx$

$= \ln(x+1) - \frac{1}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$

✓ ✓

This part was very well done, just a few careless errors with signs.

QUESTION 2.

a) $z = 2+3i$
$w = \bar{z}$
$= 2-3i$

i) $wz = (2+3i)(2-3i)$
$= 4+9$
$= 13$ ✓

ii) $\frac{z}{w} = \frac{2+3i}{2-3i}$
realise the denominator

$= \frac{2+3i}{2-3i} \times \frac{2+3i}{2+3i}$

$= \frac{4+6i+6i-9}{4+9}$

$= \frac{-5+12i}{13}$

$= -\frac{5}{13} + \frac{12}{13}i$ ✓

This part was very easy. Well done!

Q2 cont'.

b) i) $a+ib = \pm\sqrt{-3+4i}$

$(a+ib)^2 = -3+4i$

$a^2-b^2+2abi = -3+4i$

Equate real and imaginary parts

$a^2-b^2 = -3$ ①
$2ab = 4$ ② $\Rightarrow a = \frac{2}{b}$ ③ ✓

Substitute ③ into ①

$\left(\frac{2}{b}\right)^2 - b^2 = -3$

$4 - b^4 = -3b^2$

$b^4 - 3b^2 - 4 = 0$

$(b^2-4)(b^2+1) = 0$

$b^2-4=0$ $\quad$ $b^2+1=0$
$b^2=4$ $\quad$ No real soln.
$b=\pm 2$ ✓

Subst. into ③

$b=2 \quad a=\frac{2}{2}=1$ ✓
$b=-2 \quad a=\frac{2}{-2}=-1$

$\therefore \pm\sqrt{-3+4i} = \pm(1+2i)$

This is a standard question. You should be able to solve simultaneously in cases like this.

ii) $(1+i)z^2 - z - i = 0$

$z = \frac{-(-1)\pm\sqrt{(-1)^2-4.(1+i).-i}}{2(1+i)}$

$= \frac{1\pm\sqrt{1+4i-4}}{2(1+i)}$

$= \frac{1\pm\sqrt{-3+4i}}{2(1+i)}$ ✓

using part (i)

$= \frac{1\pm(1+2i)}{2+2i}$

$z = \frac{1+1+2i}{2+2i}$
$= 1$ ✓

$z = \frac{1-1-2i}{2+2i}$
$= \frac{-2i}{2+2i}$
$= \frac{-i}{1+i} \times \frac{(1-i)}{(1-i)}$
$= \frac{-1-i}{2}$ ✓

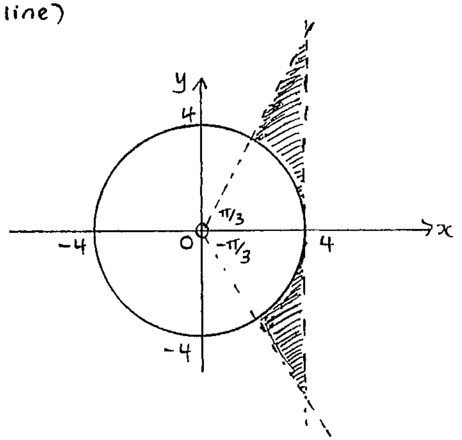
Well done. Be careful with signs!

Question 2 (continued)

c) $z+\bar{z}<8$
$x+iy+x-iy<8$
$2x<8$
$x<4$
(vertical line)

$|z|\geq 4$
circle
centre $(0,0)$
$r=4$

$|\arg z|<\frac{\pi}{3}$
$-\frac{\pi}{3}<\arg z<\frac{\pi}{3}$



Comm 3

Circle
vertical line } ✓

arg z ✓

region shaded ✓

This part was surprisingly poorly done. It is a standard question, You should recognise each of these regions and how to find cartesian form.

d) i) $\overrightarrow{OB} = \text{cis}\,\frac{2\pi}{3}\,\overrightarrow{OA}$ (rotation of OA in an anticlockwise direction)

$z_2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)z_1$

$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z_1$ ✓

Reas 4

ii) $(z_1+z_2)^2 = \left(z_1 + \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)z_1\right)^2$

(Easier to do in cis form)

$= \left(z_1 + \text{cis}\frac{2\pi}{3}z_1\right)^2$

$= z_1^2 + 2\,\text{cis}\frac{2\pi}{3}z_1^2 + \left(\text{cis}\frac{2\pi}{3}z_1\right)^2$ ✓

De moivres.

$= z_1^2 + 2\,\text{cis}\frac{2\pi}{3}z_1^2 + \text{cis}\frac{4\pi}{3}z_1^2$

(quad 3)

$= z_1^2 + 2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)z_1^2 + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)z_1^2$ ✓

$= z_1^2\left(1 - 1 + \sqrt{3}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= z_1^2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

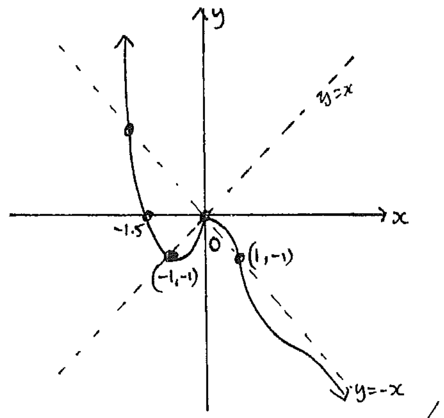
$= z_1\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)z_1$ ✓

$= z_1 z_2$.

There is more than one way to do this. Here's one method using Demoivres and careful expansion of brackets & using Cartesian form.

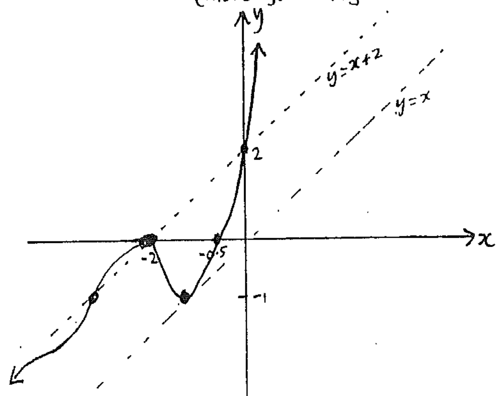
Question 3.

a) i) $y=f(-x)$ ⇒ reflect original in the y axis.



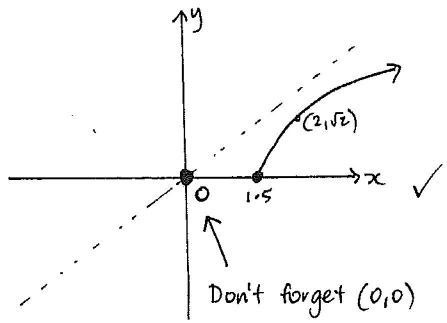
✓

ii) $y=f(x+2)$ ⇒ shift original left 2 units. (move y axis right 2 units)



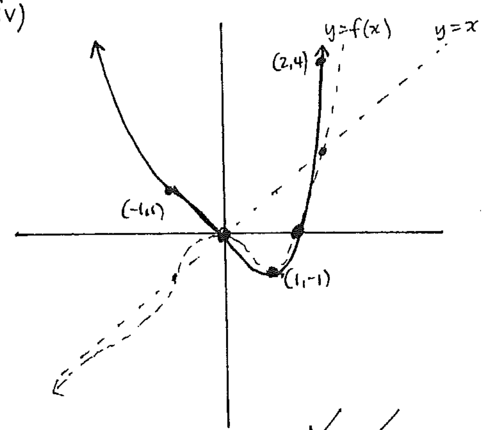
✓

iii) $y=\sqrt{f(x)}$



✓ ✓

iv)



✓ ✓

Some things to make your graphs stand out.

- ⊛ Highlight the answer clearly if you've drawn the original as well.
- ⊛ Use some points for more difficult concepts/limits
- ⊛ Drawing asymptotes will help as well.

Comm 6

b) $x^3 + 4x^2 + 2x - 1 = 0$

i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= \left(-\frac{b}{a}\right)^2 \quad - 2\left(\frac{c}{a}\right)$ ✓

$= \left(-\frac{4}{1}\right)^2 \quad -2\left(\frac{2}{1}\right)$

$= 16 - 4$ ✓

$= 12$

This part was easy.

ii) the roots α, β, γ satisfy $x^3+4x^2+2x-1=0$

$\therefore \alpha^3 + 4\alpha^2 + 2\alpha - 1 = 0$

$\beta^3 + 4\beta^2 + 2\beta - 1 = 0$

$\gamma^3 + 4\gamma^2 + 2\gamma - 1 = 0$

Adding these equations

$\alpha^3+\beta^3+\gamma^3 + 4(\alpha^2+\beta^2+\gamma^2) + 2(\alpha+\beta+\gamma) - 3 = 0$ ✓

$\therefore \alpha^3+\beta^3+\gamma^3 = 3 - 2(\alpha+\beta+\gamma) - 4(\alpha^2+\beta^2+\gamma^2)$

$= 3 - 2(-4) \quad - 4(12)$

$= 3 + 8 \quad - 48$ ✓

$= -37$

practise this method if you couldn't do it.

iii) Roots are α^2, β^2 and γ^2

Let $y = x^2$

$x = \sqrt{y}$

Substitute into original.

$(\sqrt{y})^3 + 4(\sqrt{y})^2 + 2\sqrt{y} - 1 = 0$ ✓

$y\sqrt{y} + 4y + 2\sqrt{y} - 1 = 0$

$y\sqrt{y} + 2\sqrt{y} = 1 - 4y$

$\sqrt{y}(y+2) = 1 - 4y$

Square both sides.

$y(y+2)^2 = (1-4y)^2$

$y(y^2+4y+4) = 1 - 8y + 16y^2$

$y^3 + 4y^2 + 4y - 1 + 8y - 16y^2 = 0$

$y^3 - 12y^2 + 12y - 1 = 0$

$\therefore$ New equation is $\quad x^3 - 12x^2 + 12x - 1 = 0$ ✓

Reas 2

this technique can take different forms but this is an exam favourite.

there are other ways to do this question.

c) $x^2 - xy + y^2 = 5$

Differentiate w.r.t x. (implicit differentiation)

$2x - \left(x\frac{dy}{dx} + y\cdot 1\right) + 2y\frac{dy}{dx} = 0$ ✓

(make $\frac{dy}{dx}$ the subject)

$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y - x) = y - 2x$

$\frac{dy}{dx} = \frac{y-2x}{2y-x}$ ✓

<u>At $(2,-1)$ gradient.</u> $\quad m_T = \frac{-1-4}{-2-2}$

$= \frac{-5}{-4}$

$= \frac{5}{4}$ ✓

<u>Eqn tangent</u>

$y + 1 = \frac{5}{4}(x-2)$

$4y + 4 = 5x - 10$

$5x - 4y - 14 = 0$

Too many mistake with the negative sign combined with the product rule. Don't do steps in your head. Set your work out clearly includi brackets and sign errors are less likel to occur.

Standard techniques using implicit differentiation that you must be able to do are

- basic $\frac{d}{dx}(x^n)$
- chain rule $\frac{d}{dx}(y^n)$
- product rule $\frac{d}{dx}(xy)$
- quotient rule $\frac{d}{dx}\left(\frac{x}{y}\right)$

Question 4.

a) $x^3-3x^2-9x+k=0$

$P(x)=x^3-3x^2-9x+k$

$P'(x)=3x^2-6x-9$

Since $P(x)$ has a double root, $P'(x)=0$ gives the possible values of the root.

$3x^2-6x-9=0$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3 \qquad x=-1$

these are the possible roots.

when $x=3$, $P(3)=0$

$27-3\times9-9\times3+k=0$

$k=27$

when $x=-1$, $P(-1)=0$

$-1-3+9+k=0$

$k=-5$

Calc 2

[See next page for another (longer solution) to Q4 a)]

If you didn't know the Multiple Root Theorem, Look it up and practise it.

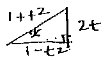
b) $t=\tan\frac{x}{2}$

$\frac{x}{2}=\tan^{-1}t$

$x=2\tan^{-1}t$

$dx=\frac{2}{1+t^2}\,dt$

$\cos x=\frac{1-t^2}{1+t^2}$



$\sin x=\frac{2t}{1+t^2}$

$$\int\frac{1}{5+4\cos x+3\sin x}\,dx$$

$$=\int\frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)+3\left(\frac{2t}{1+t^2}\right)}\times\left(\frac{2}{1+t^2}\right)dt$$

$$=\int\frac{2}{5(1+t^2)+4(1-t^2)+3(2t)}\,dt$$

$$=\int\frac{2}{5+5t^2+4-4t^2+6t}\,dt$$

$$=\int\frac{2}{t^2+6t+9}\,dt$$

$$=\int\frac{2}{(t+3)^2}\,dt$$

$$=\int 2(t+3)^{-2}\,dt$$

$$=\frac{-2}{t+3}+c$$

$$=\frac{-2}{\tan\frac{x}{2}+3}+c$$

Calc 3

(*) You must know these results.

This part was well done except for algebraic mistakes. Set your working out clearly and watch your signs are correct.

Don't forget to write the answer back in terms of original variable.

Q4 a) Another method but this takes longer and is open to algebraic mistakes if you're not careful

$x^3-3x^2-9x+k=0$

Double root α,α,β

Sum

$\alpha+\beta+\gamma=-\frac{b}{a}$

$2\alpha+\beta=3 \qquad (1)$

Sum Two

$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$

$\alpha^2+\alpha\beta+\alpha\beta=-9$

$\alpha^2+2\alpha\beta=-9 \qquad (2)$

Product

$\alpha\beta\gamma=-\frac{d}{a}$

$\alpha^2\beta=-k \qquad (3)$

Solve (1) & (2) simultaneously.

from (1) $\beta=3-2\alpha$

sub. into (2)

$\alpha^2+2\alpha(3-2\alpha)=-9$

$\alpha^2+6\alpha-4\alpha^2=-9$

$-3\alpha^2+6\alpha+9=0$

$3\alpha^2-6\alpha-9=0$

$\alpha^2-2\alpha-3=0$

$(\alpha-3)(\alpha+1)=0$

$\alpha=3$ or $\alpha=-1$

Substitute into (1) to find $\beta=3-2\alpha$

$\alpha=3$

$\beta=3-2\times3$

$=-3$

$\alpha=-1$

$\beta=3-2\times-1$

$=3+2$

$=5$

Now find k from (3)

$k=-\alpha^2\beta$

$\alpha=3,\ \beta=-3$

$k=-3^2\times-3$

$=-9\times-3$

$=27$

$\alpha=-1,\ \beta=5$

$k=-(-1)^2\times5$

$=-1\times5$

$=-5$

$\therefore k=27$

or $k=-5$

c) i) $4x^2 + 9y^2 + 24x - 36y + 36 = 0$

$4x^2 + 24x \quad + 9y^2 - 36y \quad = -36$

$4(x^2 + 6x \underline{+9}) + 9(y^2 - 4y \underline{+4}) = -36 + 36 + 36$

$4(x+3)^2 + 9(y-2)^2 = 36$

$\frac{(x+3)^2}{9} + \frac{(y-2)^2}{4} = 1$ ✓

* Completing the square is not as easy when the coefficient of x^2 is not one. Do some practice of these if you couldn't do it.

ii) $a = 3$, $b = 2$

$b^2 = a^2(1-e^2)$
$4 = 9(1-e^2)$
$1 - e^2 = \frac{4}{9}$
$e^2 = \frac{5}{9}$
$e = \frac{\sqrt{5}}{3}$ ✓

A bit harder because the ellipse is shifted.

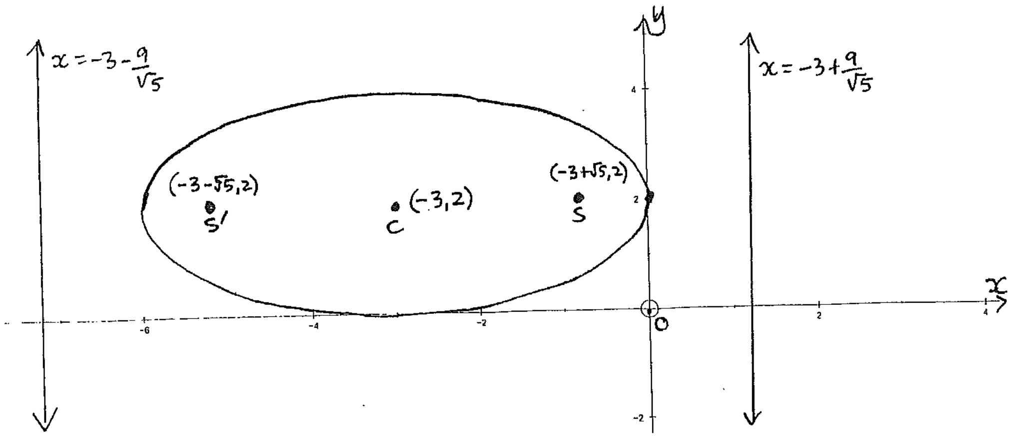
iii) Centre $(-3, 2)$ ✓centre

Comm 3

Foci $(-3 \pm ae, 2) = (-3 \pm \sqrt{5}, 2)$ ✓foci directrices

Directrices $x = -3 \pm \frac{a}{e}$

$x = -3 \pm \frac{9}{\sqrt{5}}$ ✓(x, y axes) position



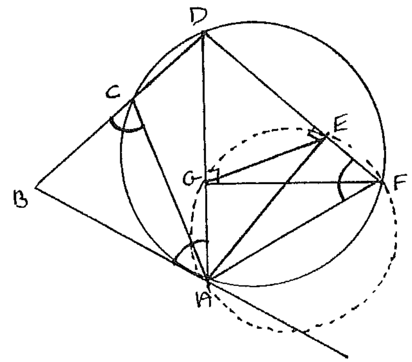
Reas 5

There is more than one way to do this question.

The instruction says Copy the Diagram so you must do it. The marker must be able to clearly see any labels that you have added so that they can follow your reasoning.

Be kind!

d)



i) Construct AF

$\angle BAD = \angle DFA$ (Angle between a tangent and a chord equals the angle in the alternate segment). ✓

$\angle DFA = \angle ACB$ (AFDC is a cyclic quad.) (the exterior angle in a cyclic quadrilateral is equal to the opposite interior angle.) ✓

$\therefore \angle ACB = \angle BAD$

---

ii) $\angle AGF = 90°$ (given)

$\therefore$ A, G, F are concyclic points since the angle in a semicircle is $90°$.

$\therefore$ AGF lie on a circle with diameter AF

Similarly

$\angle AEF = 90°$ (given) ✓

$\therefore$ AEF lie on a circle with diameter AF

$\therefore$ AGEF is a cyclic quadrilateral with diameter AF.

This part was well done by those who did it. Drawing the new circle through AGEF with help with the logic and reasoning.

---

iii) Let $\angle AFE = x$

Since AGEF is a cyclic quad. the opposite angles are supplementary.

$\therefore \angle AGE = 180 - x$ ✓

$\angle ACB = \angle AFD = x$ from part (i) ✓

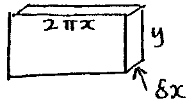
$\therefore \angle ACD = 180 - x$ (Angles on a straight line add to $180°$).

$\therefore \angle AGE = \angle ACD$.

This part wasn't answered very well and even those who did parts (i) & (ii) really well, got mixed up. Have another go. It's not too bad.

Question 5. (12)

a) i)



Each cylinder $\begin{cases}\text{radius } x \\ \text{height } y \\ \text{thickness } \delta x\end{cases}$

has volume

$\delta V = 2\pi x y \ \delta x$ ✓

$= 2\pi x \sin x \ \delta x$

Total volume

$V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} 2\pi x \sin x \ \delta x$ ✓

$= 2\pi \int_0^{\pi} x \sin x \ dx$

Calc 5

Drawing the cylindrical slice as a rectangle is the easiest method.

ii) $V = 2\pi \int_0^{\pi} x \sin x \ dx$

I.B.P.
$u = x \quad v' = \sin x$
$u' = 1 \quad v = -\cos x$
$\int uv' = uv - \int vu'$

$= 2\pi\left(\left[-x\cos x\right]_0^{\pi} + \int_0^{\pi} \cos x \ dx\right)$ ✓

$= 2\pi\left(\{-\pi\cos\pi - 0\} + \left[\sin x\right]_0^{\pi}\right)$ ✓

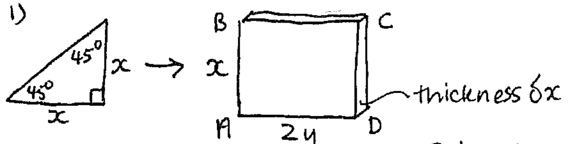
$= 2\pi\left\{-\pi \times -1 + (\sin\pi - \sin 0)\right)$

$= 2\pi \times \pi$

$= 2\pi^2$ units^3 ✓

This part was well done. It is an easy integration by parts.

b) i)



Circle base
$x^2 + y^2 = 16$
$y^2 = 16 - x^2$
$y = \sqrt{16 - x^2}$
(quad 1)

ABCD Area $= 2xy$ ✓

$= 2x\sqrt{16 - x^2}$ ✓

Calc 5

Students who had greatest success in this part drew a clear diagram of the crosssection ABCD and showed the area clearly.

Q5 (cont') (13)

b) ii) Each volume

$\delta V = A \ \delta x$

$= 2x\sqrt{16 - x^2} \ \delta x$

Total volume

$V = \lim_{\delta x \to 0} \sum_{x=0}^{4} 2x\sqrt{16 - x^2} \ \delta x$ ✓

$= \int_0^4 2x\sqrt{16 - x^2} \ dx$

$= -\int_0^4 -2x(16 - x^2)^{\frac{1}{2}} \ dx$ ✓

$= -\left[\frac{2}{3}(16 - x^2)^{3/2}\right]_0^4$

$= -\left\{\frac{2}{3}\left(0 - 16^{3/2}\right)\right\}$

$= -\frac{2}{3} \times -64$

$= \frac{128}{3}$

$= 42\frac{2}{3}$ units^3 ✓

This part was well done.

Look for short cuts.

use reverse chain rule

$\int f'(x) f(x)^n dx = \frac{f(x)^{n+1}}{n+1}$

OR use a trig. substitution.

c) i) $\cos(A - B)x - \cos(A + B)x$

$= \cos Ax \cos Bx + \sin Ax \sin Bx$

$- (\cos Ax \cos Bx - \sin Ax \sin Bx)$ ✓

$= 2\sin Ax \sin Bx$.

Easy but don't cramp the setting out.

ii) $\sin 3x \sin x = 2\cos 2x + 1$

LHS $= \sin 3x \sin x$ (using part (i))

$= \frac{1}{2}(\cos(3-1)x - \cos(3+1)x)$

$= \frac{1}{2}(\cos 2x - \cos 4x)$

$\therefore \frac{1}{2}(\cos 2x - \cos 4x) = 2\cos 2x + 1$ ✓

$\cos 2x - \cos 4x = 4\cos 2x + 2$

$\cos 4x + 3\cos 2x + 2 = 0$

Lots of algebraic mistakes in this part which was a shame because it's a straight forward trig. substitution question.

Q5 (cont')

c) ii) (continued)

$\cos 4x + 3\cos 2x + 2 = 0$

using the substitution.

$\cos 4x = 2\cos^2 2x - 1$

$\therefore 2\cos^2 2x - 1 + 3\cos 2x + 2 = 0$

$2\cos^2 2x + 3\cos 2x + 1 = 0$ ✓

This is a quadratic equation in terms of $\cos 2x$.

(*) If you're going to use a substitution make sure you have the signs correct.

iii) Solving

$2\cos^2 2x + 3\cos 2x + 1 = 0$

$(2\cos 2x + 1)(\cos 2x + 1) = 0$

$\cos 2x = -\frac{1}{2}$	$\cos 2x = -1$
general solutions.	
$2x = \pm\cos^{-1}(-\frac{1}{2}) + 2n\pi$	$2x = \pm\cos^{-1}(-1) + 2n\pi$
$x = \pm\frac{1}{2}\cos^{-1}(-\frac{1}{2}) + n\pi$	$x = \pm\frac{1}{2}\cos^{-1}(-1) + n\pi$
$= \pm\frac{1}{2} \times \frac{2\pi}{3} + 2\pi$	$x = \pm\frac{1}{2}\pi + n\pi$
$= \pm\frac{\pi}{3} + n\pi$	$x = \pm\frac{\pi}{2} + n\pi$

(where n is an integer)

✓ ✓

Regardless of the quadratic equation used, The general solutions were poorly done. Learn how to do these properly and understand the logic of what you're doing will help as well!

Question 6

Reas 3

This question really isn't as hard as some students thought.

Well done to those who got it all done.

a) i) Committees with one prefect

$^4C_1 \times {}^nC_2$

$= \frac{4!}{1!\,3!} \times \frac{n!}{2!\,(n-2)!}$

$= \frac{4n(n-1)}{2}$

$= 2n(n-1)$ ✓

ii) Committees with two prefects

$^4C_2 \times {}^nC_1$

$= \frac{4!}{2!\,2!} \times \frac{n!}{1!\,(n-1)!}$

$= \frac{4 \times 3 \times n}{2}$

$= 6n$ ✓

iii) Total number of committees with no restrictions.

$^{n+4}C_3 = \frac{(n+4)!}{3!\,(n+1)!}$

$= \frac{(n+4)(n+3)(n+2)}{6}$

It's easier to simplify this first and then use it to find the probability.

<u>Probability of one or two prefects</u>

$= \dfrac{2n(n-1) + 6n}{\frac{(n+4)(n+3)(n+2)}{6}}$

$= \frac{6(2n^2 + 4n)}{(n+4)(n+3)(n+2)}$

$= \frac{12n(n+2)}{(n+4)(n+3)(n+2)}$

$= \frac{12n}{(n+4)(n+3)}$ ✓

b) $I_n = \int_1^2 x(\ln x)^n \, dx$

using I.B.P. $u = (\ln x)^n$ $\quad v' = x$
$u' = n(\ln x)^{n-1} \cdot \frac{1}{x}$ $\quad v = \frac{x^2}{2}$
$\int uv' = uv - \int vu'$

$$I_n = \left[\frac{x^2}{2}(\ln x)^n\right]_1^2 - \int_1^2 n(\ln x)^{n-1}\frac{1}{x}\cdot\frac{x^2}{2}\,dx$$

$$= \left(\frac{4}{2}(\ln 2)^n - \frac{1}{2}(\ln 1)^n\right) - \int_1^2 n(\ln x)^{n-1}\frac{x}{2}\,dx$$

$$= 2(\ln 2)^n - \frac{n}{2}\int_1^2 x(\ln x)^{n-1}\,dx$$

$$= 2(\ln 2)^n - \frac{n}{2} I_{n-1}$$

iii) $I_3 = 2(\ln 2)^3 - \frac{3}{2} I_2$

$$= 2(\ln 2)^3 - \frac{3}{2}\left[2(\ln 2)^2 - \frac{2}{2} I_1\right]$$

$$= 2(\ln 2)^3 - 3(\ln 2)^2 + \frac{3}{2} I_1$$

$$= 2(\ln 2)^3 - 3(\ln 2)^2 + \frac{3}{2}\left(2\ln 2 - \frac{1}{2} I_0\right)$$

Evaluate $I_0 = \int_1^2 x(\ln x)^0\,dx$
$= \int_1^2 x\,dx$
$= \left[\frac{x^2}{2}\right]_1^2$
$= \frac{4}{2} - \frac{1}{2}$
$= \frac{3}{2}$

$$\therefore \quad I_3 = 2(\ln 2)^3 - 3(\ln 2)^2 + 3\ln 2 - \frac{3}{4} \times \frac{1}{2} \times \frac{3}{2}$$

$$= 2(\ln 2)^3 - 3(\ln 2)^2 + 3\ln 2 - \frac{9}{8}$$

Calc 5

Excellent work in this question, well done!

I_0 is easier to find than I_1 but you can do I_1 if you want.

c) i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiate wrt x

$$\frac{2x}{a^2} - \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\frac{2x}{a^2}}{\frac{2y}{b^2}}$$

$$\frac{dy}{dx} = \frac{2x}{a^2} \times \frac{b^2}{2y}$$

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

At $P(a\sec\theta, b\tan\theta)$, gradient

$$m_T = \frac{b^2 \cdot a\sec\theta}{a^2 \cdot b\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

Equation of tangent at P.

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$a\tan\theta y - ab\tan^2\theta = b\sec\theta x - ab\sec^2\theta$$

$$b\sec\theta x - a\tan\theta y = ab(\sec^2\theta - \tan^2\theta)$$

$$b\sec\theta x - a\tan\theta y = ab$$

$(\div ab)$ $\quad \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

ii) Tangent cuts Q when $x = a$

$$\frac{a\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

$$\frac{y\tan\theta}{b} = \sec\theta - 1$$

$$y = \frac{b}{\tan\theta}(\sec\theta - 1)$$

$\therefore Q\left(a, \frac{b}{\tan\theta}(\sec\theta - 1)\right)$

Calc 2

This question part (i) is standard bookwork.

Everybody should be able to find equations of tangents & normals to hyperbolas & ellipse without a problem.

(continued)

c) ii) Tangent cuts $\mathcal{R}$ at $x=-a$

$$-\frac{a\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

$$\frac{y\tan\theta}{b} = -\sec\theta - 1$$

$$y = -\frac{b}{\tan\theta}(\sec\theta+1)$$

$\therefore R\left(-a,\ -\frac{b}{\tan\theta}(\sec\theta+1)\right)$

✓ Q and R

Only one mark for both points Q and R.

Be careful with signs!

$Q\left(a, \frac{b}{\tan\theta}(\sec\theta-1)\right)$ $\quad S(ae,0)$

iii) $$m_{QS} = \frac{\frac{b}{\tan\theta}(\sec\theta-1) - 0}{a-ae}$$

$$= \frac{b(\sec\theta-1)}{\tan\theta \cdot a(1-e)}$$

Again, be careful with negative signs and no fudging please.

$R\left(-a, -\frac{b}{\tan\theta}(\sec\theta+1)\right)$ $\quad S(ae,0)$

$$m_{RS} = \frac{-\frac{b}{\tan\theta}(\sec\theta+1) - 0}{-a-ae}$$

$$= \frac{-b(\sec\theta+1)}{-\tan\theta \cdot a(1+e)}$$

$$m_{QS} \times m_{RS} = \frac{b(\sec\theta-1)}{a\tan\theta(1-e)} \times \frac{b(\sec\theta+1)}{a\tan\theta(1+e)} \quad ✓$$

$$= \frac{b^2(\sec^2\theta-1)}{\tan^2\theta \quad a^2(1-e^2)}$$

for hyperbola $b^2=a^2(e^2-1)$ so change signs.

$$= \frac{b^2}{-a^2(e^2-1)} \frac{(\cancel{\sec^2\theta-1})}{(\cancel{\tan^2\theta})} \quad ✓$$

$$= \frac{b^2}{-b^2}$$

$$= -1 \qquad \therefore QS \perp RS \quad \therefore \angle QSR = 90^\circ$$

iv) Similarly using $S'(-ae,0)$

$$m_{QS'} = \frac{\frac{b}{\tan\theta}(\sec\theta-1)}{a+ae}$$

$$m_{RS'} = \frac{-\frac{b}{\tan\theta}(\sec\theta+1)}{-a+ae}$$

$$m_{QS'} \times m_{RS'} = \frac{-b^2(\cancel{\sec^2\theta-1})}{\cancel{\tan^2\theta} \quad -a^2(1-e^2)}$$

change signs.

$$= \frac{-b^2}{a^2(e^2-1)}$$

$$= \frac{-b^2}{b^2}$$

$$= -1$$

$\therefore QS' \perp RS'$

$\angle QS'R = 90^\circ$ ✓

Since $\angle QSR = 90^\circ$, Q, S and R lie on a circle with diameter QR

$\angle QS'R = 90^\circ$ $\therefore$ Q, S'R lie on a circle with diameter QR

Since $\angle QSR = 90^\circ$ and $\angle QS'R = 90^\circ$ the opposite angles are supplementary

$\therefore$ QSRS' are concyclic with diameter QR. ✓

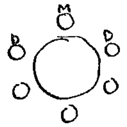
Reas 2

This part was well done by those that got this far.

It's not very difficult to do this proof.

Question 7.

a)



seat Mum		seat Dad		seat Chelsea		Rest
1	x	2	x	3	x	3!

$= 36$ ✓✓

Reas 2

This is an Ext 1 question.

b) i) $z^3 - 64$

(A) $= (z-4)(z^2+4z+16)$

factorised over $\mathbb{R}$ ✓

(B) $= (z-4)\left((z+2)^2 + 12\right)$ by completing the square

$= (z-4)\left((z+2)^2 - (\sqrt{12}\, i)^2\right)$

difference of two squares

$= (z-4)(z+2+\sqrt{12}\, i)(z+2-\sqrt{12}\, i)$

factorised over $\mathbb{C}$ ✓

BASIC!!!

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

Noone should get this part wrong.

there are other ways to do part (B) but this one's pretty quick.

ii) (A) Since ω is a complex root of $z^3 - 64 = 0$ then $\omega \neq 4$ so it satisfies $\omega^2 + 4\omega + 16 = 0$

$\therefore \omega^2 = -4\omega - 16$

$\omega^2 = -4(\omega + 4)$ ✓

(B) $(4\omega+16)^3 = (-\omega^2)^3$

$= -\omega^6$

$= -(\omega^3)^2$

$= -(64)^2$

$= -4096$ ✓

using $\omega^2 = -(4\omega+16)$ and $\omega^3 - 64 = 0$, $\omega^3 = 64$

If you didn't recognise this type of question, look up the complex roots of a number using ω (omega).

c) i) $T_1 = 1, \quad T_2 = 5$

$T_k = 5T_{k-1} - 6T_{k-2}$

i) Show that $T_n = 3^n - 2^n$ is true for $n = 1, 2, 3$

for $n=1$ $\quad T_1 = 1$ (given)

$T_1 = 3^1 - 2^1$

$= 1$

$\therefore$ True for $n = 1$

for $n=2$ $\quad T_2 = 5$ (given)

$T_2 = 3^2 - 2^2$

$= 9 - 4$

$= 5$

$\therefore$ True for $n = 2$ ✓

for $n=3$ $\quad T_3 = 5T_2 - 6T_1$

$= 5 \times 5 - 6 \times 1$

$= 25 - 6$

$= 19$

$T_3 = 3^3 - 2^3$

$= 27 - 8$

$= 19$ ✓

$\therefore$ True for $n = 3$.

this induction question was very well done.

make sure you set it out clearly and show how you've matched from the given answer to the one you found.

ii) Assume true for $n = k$

$T_k = 3^k - 2^k$ where $T_k = 5T_{k-1} - 6T_{k-2}$.

Prove true for $n = k+1$

$T_{k+1} = 3^{k+1} - 2^{k+1}$ where

$T_{k+1} = 5T_k - 6T_{k-1}$

Question 7 (continued)

c) ii) (continued)

$$T_{k+1} = 5T_k - 6T_{k-1}$$
$$= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$$
(using the assumption)
$$= 5.3^k - 5.2^k - 6.3^k.3^{-1} + 6.2^k.2^{-1}$$
$$= 5.3^k - 5.2^k - 2.3^k + 3.2^k$$
$$= 3.3^k - 2.2^k$$
$$= 3^{k+1} - 2^{k+1}$$

$\therefore$ If the statement is true for $n=k$ it is true for $n=k+1$.
Since it is true for $n=1,2,3$, it is true for all integers $n \geq 1$ by the principle of mathematical induction.

Comm 2

Very well done.
This was an easy question.

Question 7 (continued)

d) i) $\int_0^a f(x)\,dx$

	change limits
$u = a - x$	$x=0 \quad u=a$
$x = a - u$	$x=a \quad u=a-a=0$
$dx = -du$	

$$= \int_a^0 f(a-u).-du$$
$$= -\int_a^0 f(a-u)\,du$$
$$= \int_0^a f(a-u)\,du$$
(variable doesn't matter)
$$= \int_0^a f(a-x)\,dx$$

Calc 1

This part is standard bookwork.
You should be able to do this technique.

ii) $\int_0^\pi x\sin^2 x\,dx$

$$= \int_0^\pi (\pi - x)\sin^2(\pi - x)\,dx$$
[note that (Quad 2) $\sin(\pi - x) = \sin x$]
$$= \int_0^\pi (\pi - x)\sin^2 x\,dx$$
$$= \int_0^\pi \pi\sin^2 x\,dx - \int_0^\pi x\sin^2 x\,dx$$

$$\therefore 2\int_0^\pi x\sin^2 x\,dx = \int_0^\pi \pi\sin^2 x\,dx$$

$$\int_0^\pi x\sin^2 x\,dx = \frac{\pi}{2}\int_0^\pi \tfrac{1}{2}(1-\cos 2x)\,dx$$
$$= \frac{\pi}{4}\left[x - \tfrac{1}{2}\sin 2x\right]_0^\pi$$
$$= \frac{\pi}{4}\left\{\pi - \tfrac{1}{2}\sin 2\pi - 0\right\}$$
$$= \frac{\pi^2}{4}$$

Many students missed the point of part (i).
By doing the substitution, the question becomes much easier.
You do not have to use integration by parts at all.

Some careless mistakes with signs and algebraic steps.
Set your work out clearly & carefully from line to line.

Calc 4

Question 8.

a) i) $f(x) = x - \frac{1}{2}\tan x$

$f(-x) = -x - \frac{1}{2}\tan(-x)$

$= -x + \frac{1}{2}\tan x$ (Since $\tan(-x) = -\tan x$)

$= -(x - \frac{1}{2}\tan x)$

$= -f(x)$

$\therefore f(x)$ is an odd function.

This question part a) is not difficult and the students who did it were very successful.

ii) $f(x) = x - \frac{1}{2}\tan x$

$f'(x) = 1 - \frac{1}{2}\sec^2 x$

$= 1 - \frac{1}{2}(\cos x)^{-2}$

$f''(x) = \frac{2}{2}(\cos x)^{-3} \times -\sin x$

$= -\frac{\sin x}{\cos x} \times \frac{1}{\cos^2 x}$

$= -\tan x \, . \sec^2 x$

Calc 3

Easy work here!

Stationary points $f'(x) = 0$

$1 - \frac{1}{2}\sec^2 x = 0$

$2 - \sec^2 x = 0$

$\sec^2 x = 2$

$\left(\frac{1}{\cos x}\right)^2 = 2$

$(\cos x)^2 = \frac{1}{2}$

$\cos x = \pm\sqrt{\frac{1}{2}}$

$\cos x = \frac{1}{\sqrt{2}}$ (positive case only since $-\frac{\pi}{2} < x < \frac{\pi}{2}$)

$x = \pm\frac{\pi}{4}$

Question 8 (cont')

when $x = \frac{\pi}{4}$ $\quad y = \frac{\pi}{4} - \frac{1}{2}\tan\frac{\pi}{4}$

$= \frac{\pi}{4} - \frac{1}{2}$

$\left(\frac{\pi}{4}, \frac{\pi}{4} - \frac{1}{2}\right)$

$\doteqdot (0.8, 0.3)$

when $x = -\frac{\pi}{4}$ $\quad y = -\frac{\pi}{4} - \frac{1}{2}\tan(-\frac{\pi}{4})$

$= -\frac{\pi}{4} + \frac{1}{2}\tan\frac{\pi}{4}$

$= -\frac{\pi}{4} + \frac{1}{2}$

$\left(-\frac{\pi}{4}, -\frac{\pi}{4} + \frac{1}{2}\right)$

$\doteqdot (-0.8, -0.3)$

Test nature

when $x = \frac{\pi}{4}$

$f''(\frac{\pi}{4}) = -\tan\frac{\pi}{4} \sec^2\frac{\pi}{4}$

$= -1 \times 0.5$

$= -0.5$

< 0

$\therefore$ Concave down

$\therefore$ maximum turning point $\left(\frac{\pi}{4}, \frac{\pi}{4} - \frac{1}{2}\right)$

when $x = -\frac{\pi}{4}$

$f''(-\frac{\pi}{4}) = -\tan(-\frac{\pi}{4}) \sec^2(-\frac{\pi}{4})$

$= \tan\frac{\pi}{4} \sec^2(-\frac{\pi}{4})$

$= 1 \times 0.5$

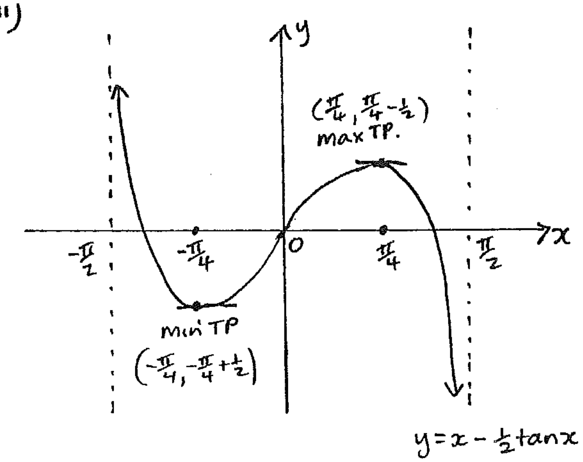
$= 0.5$

> 0

$\therefore$ Concave up

$\therefore$ minimum turning point $\left(-\frac{\pi}{4}, -\frac{\pi}{4} + \frac{1}{2}\right)$

a) iii)



✓ shape of graph. ✓ asymptotes

Comm 2

Don't forget to draw in the vertical asymptotes.

iv) From the graph it can be seen that for $0 \le x \le \frac{\pi}{4}$ the curve $f(x) = x - \frac{1}{2}\tan x$ lies on/above the x-axis

$$f(x) \geq 0$$

$$\therefore \quad x - \frac{1}{2}\tan x \geq 0 \quad ✓$$

$$x \geq \frac{1}{2}\tan x$$

Equality holds when $x = 0$. ✓

Reas 2

Explain this using clear language or symbols in logical steps.

b) i) $\tan\theta_1 = \frac{1}{1}$

$\tan\theta_1 = 1$

$\tan\theta_2 = \frac{1}{\sqrt{2}}$

$\tan\theta_3 = \frac{1}{\sqrt{3}}$ ✓

$\vdots$

$\tan\theta_n = \frac{1}{\sqrt{n}}$

An easy mark here!

ii) The largest angle occurs in the first triangle where $\tan\theta_1 = 1 \quad \therefore \theta_1 = \frac{\pi}{4}$

The angles reduce in size.

$\therefore \quad 0 < \theta \le \frac{\pi}{4}$ ✓

iii) $\sum_{n=1}^{k} \theta_n$

$= \theta_1 + \theta_2 + \theta_3 + \ldots + \theta_k$

(using a) iv)) ✓

$\geq \frac{1}{2}\tan\theta_1 + \frac{1}{2}\tan\theta_2 + \frac{1}{2}\tan\theta_3 + \ldots + \frac{1}{2}\tan\theta_k$

$= \frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{2}\times\frac{1}{\sqrt{3}} + \ldots + \frac{1}{2}\times\frac{1}{\sqrt{k}}$ ✓

$\geq \frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{3} + \ldots + \frac{1}{2}\times\frac{1}{k}$

$= \frac{1}{2}\sum_{n=1}^{k}\frac{1}{n}$ ✓

(Since $\frac{1}{x} \geq \frac{1}{\sqrt{x}}$ for $0 < x \le 1$

For $0 < \theta \le \frac{\pi}{4}$

$0 < \tan\theta \le 1$

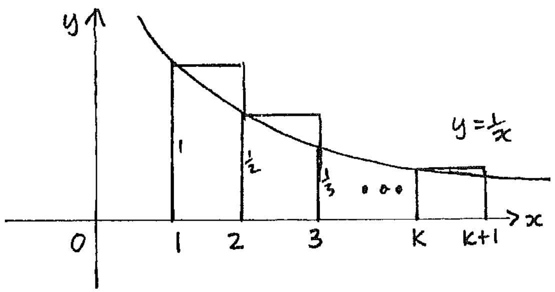
$0 < x \le 1$)

direct use of a) iv) as instructed. You might not get the instructions in HSC.

Excellent work those who got this step of logic. You should include the reason for this step as well.

Reas 3

b) iv) Sum of the areas of the rectangles is greater than the exact area under the curve $y=\frac{1}{x}$ from $x=1$ to $x=k+1$.



Area of rectangles > Exact area

$$(1\times1)+(1\times\tfrac{1}{2})+(1\times\tfrac{1}{3})+\dots+(1\times\tfrac{1}{k}) > \int_1^{k+1}\frac{1}{x}\,dx$$

$$\sum_{n=1}^{k}\frac{1}{n} > \int_1^{k+1}\frac{1}{x}\,dx$$

(variable doesn't matter.)

$$\sum_{n=1}^{k}\frac{1}{n} > \int_1^{k+1}\frac{1}{n}\,dn$$

✓

Reas 2 this page

Although these were the last 2 parts they were actually easy marks to get in a Question 8.

Well done if you got the marks for these parts.

v) $$\sum_{n=1}^{k}\theta_n \geq \frac{1}{2}\sum_{n=1}^{k}\frac{1}{n}$$

(from part b) iii))

$$> \frac{1}{2}\int_1^{k+1}\frac{1}{n}\,dn$$

$$= \frac{1}{2}\left[\ln n\right]_1^{k+1}$$

$$= \frac{1}{2}\ln(k+1)$$

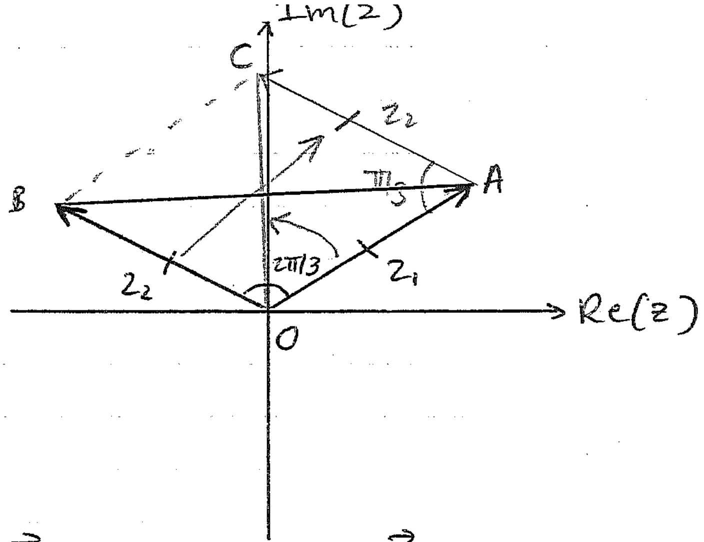
$$= \ln(k+1)^{\frac{1}{2}}$$

$$= \ln\sqrt{k+1}$$

$\therefore \sum_{n=1}^{k}\theta_n > \ln\sqrt{k+1}$ as required.

✓

✱ Show the step that $\frac{1}{2}\ln(k+1)=\ln(k+1)^{\frac{1}{2}}$.



move $\vec{OB}$ to position $\vec{AC}$

$\angle OAC = \pi/3$, co-interior angles on parallel lines are supplementary

$AC = OA$ since $OB = AC$

$\therefore \Delta AOC$ is an equilateral triangle

$\angle ACO = \angle AOC$ ($\angle$s opp = sides in isos. Δ are =)

$2\angle ACO = 2\pi/3$ ($\angle$ sum in Δ is $180°$)

$\angle ACO = \pi/3 = \angle AOC$

$\vec{OC} = z_1 + z_2$

$\therefore LHS = (\vec{OC})^2$

$= (z_1 \text{ cis } \pi/3)^2$

$= \left(|z_1| \text{ cis}(\arg(z_1)) \cdot 1 \text{ cis } \pi/3 \right)^2$

$= \left(|z_1| \text{ cis}(\arg(z_1) + \pi/3) \right)^2$

$= |z_1||z_1| \text{ cis}\left(2\arg(z_1) + \frac{2\pi}{3}\right)$

$RHS = z_1 z_2$

$= z_1 \cdot z_1 \text{ cis } 2\pi/3$

$= |z_1| \text{ cis}(\arg(z_1)) \cdot |z_1| \text{ cis}(\arg(z_1)) \cdot 1 \cdot \text{cis } 2\pi/3$

$= |z_1||z_1| \text{ cis}(\arg(z_1) + \arg(z_2) + 2\pi/3)$

$= |z_1||z_1| \text{ cis}(2\arg(z_1) + 2\pi/3)$

$= LHS$

nb. can cut to