Sydney Girls High School

2013 Assessment Task 3

MATHEMATICS

Extension One

Year 12

Time allowed: 60 minutes (plus 5 minutes reading time)

Topics: Polynomials, Integration by Substitution, Circle Geometry, Parametric Parabola, Inverse Functions and Inverse Trigonometric Functions

Instructions:

- Attempt all 5 questions
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Write on one side of the paper only
- A table of standard integrals is supplied

Name:__ ___ Teachers Name______________

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

Note: $\ln x = \log_e x, \quad x > 0$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Question One (12 Marks)

a) The polynomial $x^3+x^2-2x+1=0$ has roots α, β and γ. Fnd the value of:

i) $\alpha+\beta+\gamma$ [1]

ii) $\alpha\beta\gamma$ [1]

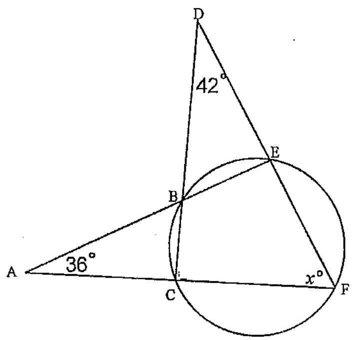
iii) $\alpha\beta+\alpha\gamma+\beta\gamma$ [1]

iv) $\alpha^2+\beta^2+\gamma^2$ [2]

v) $(\alpha+1)(\beta+1)(\gamma+1)$ [2]

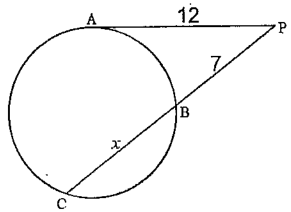
b) Find $\int x\sqrt{x^2+5}\,dx$ using the substitution $u=x^2+5$ [2]

c) In the diagram below $\angle BDE = 42^o, \angle BAC = 36^o$ and $\angle EFC = x^o$.
Find the size of x giving reasons for your answer. [3]



Question Two (12 Marks)

a) In the diagram below AP is a tangent to the circle at A and CP is a secant meeting the circle at B and C. Given that AP=12, BP =7 and CB = x, find the value of x. [2]



b) Evaluate $\displaystyle\int_0^1 \frac{\left(\sin^{-1}x\right)^3}{\sqrt{1-x^2}}\,dx$ using the substitution $u=\sin^{-1}x$ [3]

c) The equation of the chord of contact of two tangents to the parabola $x^2=8y$ is $3x-4y+10=0$. Find the point of intersection of the two tangents. [2]

d) i) Solve the equation $x^3-4x^2-x+4=0$ [2]

ii) Hence sketch $P(x)=x^3-4x^2-x+4$ [1]

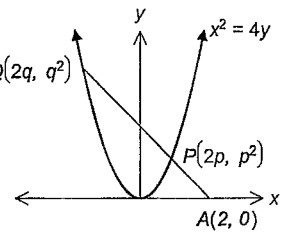
iii) Hence solve $x^3-4x^2-x+4>0$ [2]

Question Three (12 Marks)

a) Find the volume of the solid generated when the region bounded by $y=0$, $x=0$ and $x=\frac{1}{3}$ and $y=\frac{2}{\sqrt{1+9x^2}}$ is rotated about the x-axis. [3]

b) Sketch $y=5\cos^{-1}\left(1-\frac{x}{2}\right)$ clearly showing its domain and range [3]

c) The chord joining the points $P(2p,p^2)$ and $Q(2q,q^2)$ on the curve $x^2=4y$ always passes through the point $A(2,0)$ when produced.



i) Show that $p+q=pq$ [2]

ii) Find the coordinates of M the midpoint of PQ [1]

iii) Find the Cartesian equation of the locus of M [3]

Question Four (12 Marks)

a) Find the equation of the tangent to $y=2\tan^{-1}x$ at the point where $x=\frac{1}{\sqrt{3}}$ [3]

b) Given the function $f(x)=\frac{-1}{x+2}$

i) Find the equation $y=f^{-1}(x)$ [1]

ii) Find the coordinates of any points of intersection of $f(x)$ and $f^{-1}(x)$ [2]

c) i) If $\theta=\tan^{-1}A+\tan^{-1}B$ show that $\tan\theta=\frac{A+B}{1-AB}$ [1]

ii) Hence solve $\frac{\pi}{4}=\tan^{-1}3x+\tan^{-1}2x$ [2]

d) The polynomial $P(x)=(x-a)^3+c$ has a zero at $x=1$ and when divided by x, the remainder is -7. Find the values of a and c. [3]

Question Five (12 Marks)

a) i) Show that the equation of the tangent to the parabola $x^2 = 4ay$ at any point $P(2ap, ap^2)$ is given by $px - y - ap^2 = 0$ [1]

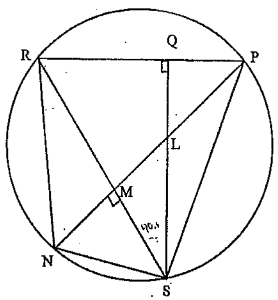
ii) If S is the focus of the parabola and T the point of intersection of the tangent with the y-axis, show that $SP = ST$ [3]

b) i) Differentiate $x\sin^{-1}x + \sqrt{1-x^2}$ [2]

ii) Hence or otherwise find the exact area bounded by $y = \sin x$, the $y-$axis and the line $y = \dfrac{1}{2}$ [1]

c) Solve the equation $\sin^{-1} 2x = \cos^{-1} x$ [2]

d) In the diagram below the altitudes PM and SQ of triangle PSR meet at L. PM produced cuts the circle through P,S and R at N.
Prove that LM=MN [3]



End of Task

Question One - Yr 12 - Ext 1 - Ass (3)

a) $x^3 + x^2 - 2x + 1 = 0$

$a = 1,\ b = 1,\ c = -2,\ d = 1$

i) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= -1$ (1)

ii) $\alpha\beta\gamma = -\frac{d}{a}$

$= -1$ (1)

iii) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= -2$ (1)

12

iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= (-1)^2 - 2(-2)$

$= 1 + 4$

$= 5$ (2)

v) $(\alpha+1)(\beta+1)(\gamma+1) = (\alpha\beta + \alpha + \beta + 1)(\gamma + 1)$

$= (\alpha\beta\gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1)$

$= -1 - 2 - 1 + 1$

$= -3$ (2)

b) $\int x\sqrt{x^2+5}\ dx = \int \frac{1}{2}\sqrt{u}\ du$

let $u = x^2 + 5$

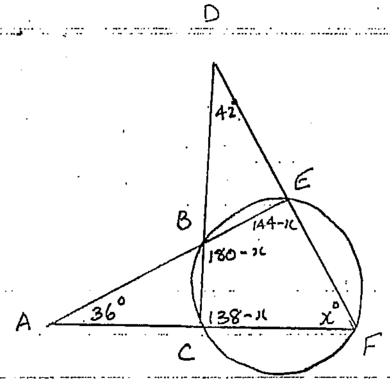
$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int u^{1/2}\ du$

$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$

$= \frac{1}{3}\sqrt{(x^2+5)^3} + C$ (2)



$\angle EBC = 180 - x$ (opp. $\angle$ of cyclic quad BEFC) (1)

$\angle BEF = 180 - (36 + x)$

$= 144 - x$ (angle sum of ΔAEF) (1)

$\angle BCF = 180 - (42 + x)$ (angle sum of ΔDCF)

$= 138 - x$

$\therefore 144 - x + 180 - x + 138 - x + x = 360°$ (angle sum of cyclic quad)

$462 - 2x = 360$

$462 - 360 = 2x$

$102 = 2x$

$\therefore x = 51°$ (1)

Ext 1 2013 Task 3

Q2

a) $12^2 = 7(7+x)$

$144 = 49 + 7x$

$7x = 95$

$x = 13\frac{4}{7}$ ✓✓

b) $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$du = \frac{dx}{\sqrt{1-x^2}}$

$x = 0 \longrightarrow u = 0$

$x = 1 \longrightarrow u = \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} u^3 \, du$ ✓

$\left[\frac{u^4}{4}\right]_0^{\frac{\pi}{2}}$ ✓

$\frac{\left(\frac{\pi}{2}\right)^4}{4}$ ✓

$= \frac{\frac{\pi^4}{16}}{4}$

$= \frac{\pi^4}{64}$

c) $4a = 8$

$a = 2$

$xx_1 = 2a(y + y_1)$

$xx_1 = 4(y + y_1)$

$xx_1 = 4y + 4y_1$

$3x - 4y + 10 = 0$

$xx_1 - 4y - 4y_1 = 0$

$x_1 = 3$ ✓

$-4y_1 = 10$ ✓

$y = -\frac{5}{2}$

$P\left(3, -\frac{5}{2}\right)$

d) i)

$$\begin{array}{r|l} & x^2 - 3x - 4 \\ x - 1 & x^3 - 4x^2 - x + 4 \\ & x^3 - x^2 \\ \hline & -3x^2 - x \\ & -3x^2 + 3x \\ \hline & -4x + 4 \\ & -4x + 4 \\ \hline & 0 \end{array}$$

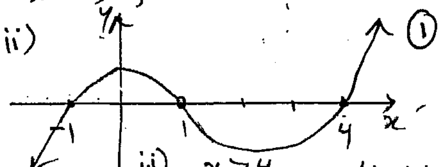
(2)

$(x-1)(x^2 - 3x - 4) = 0$

$(x-1)(x-4)(x+1) = 0$

$x = 1, 4, -1$

ii)



(1)

iii) $x > 4$, $-1 < x < 1$ (1)

Question 3

a)

$$V = \int \pi y^2 dx$$

$$= \pi \int_0^{\frac{1}{3}} \frac{4}{1+9x^2} dx$$

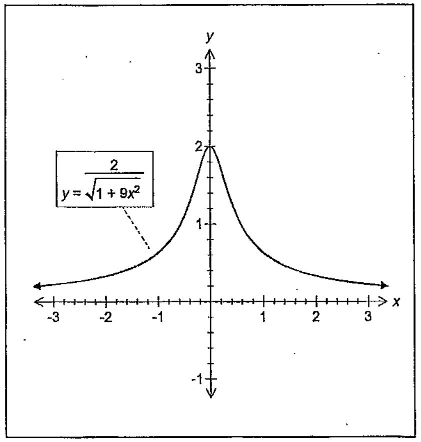
$$= \frac{4\pi}{9} \int_0^{\frac{1}{3}} \frac{1}{\frac{1}{9} + x^2} dx$$

$$= \frac{4\pi}{9} \left[\frac{1}{\frac{1}{3}} \tan^{-1} \frac{x}{\frac{1}{3}} \right]_0^{\frac{1}{3}}$$

$$= \frac{4\pi}{9} \left[3 \tan^{-1} 3x \right]_0^{\frac{1}{3}}$$

$$= \frac{4\pi}{9} \left[\frac{3\pi}{4} - 0 \right]$$

$$= \frac{\pi^2}{3} \text{ units}^3$$



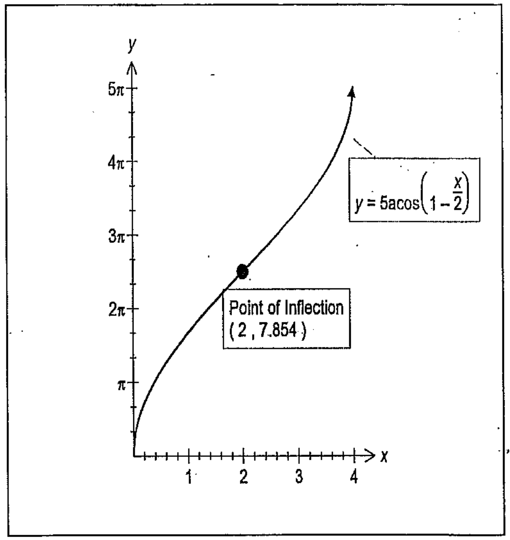
b)

Domain: $-1 \le 1 - \frac{x}{2} \le 1$

$\therefore 0 \le x \le 4$

Range: $0 \le \frac{y}{5} \le \pi$

$\therefore 0 \le y \le 5\pi$



c) Given: $x^2 = 4y$, $P(2p, p^2)$ and $Q(2q, q^2)$.

i) Chord *PQ*:

$$y - p^2 = \frac{p^2 - q^2}{2p - 2q}(x - 2p)$$
$$y - p^2 = \frac{(p-q)(p+q)}{2(p-q)}(x - 2p)$$
$$y - p^2 = \frac{p+q}{2}(x - 2p)$$

Chord passes through the point $A(2,0)$:

$$0 - p^2 = \frac{p+q}{2}(2 - 2p)$$
$$-2p^2 = (p+q)(2-2p)$$
$$-2p^2 = 2p - 2p^2 + 2q - 2pq$$
$$0 = 2(p+q) - 2pq$$
$$2pq = 2(p+q)$$
$$pq = p + q$$

ii) Midpoint $= \left(\frac{2p+2q}{2}, \frac{p^2+q^2}{2}\right)$

$$= \left(p+q, \frac{(p+q)^2 - 2pq}{2}\right)$$

iii) From midpoint: $x = p + q$

and $y = \frac{(p+q)^2 - 2pq}{2}$

that is: $y = \frac{(x)^2 - 2(p+q)}{2}$ (since $pq = p+q$ from i))

$$y = \frac{(x)^2 - 2(x)}{2}$$

$\therefore$ Cartesian equation is given by: $y = \frac{1}{2}x^2 - x$

Question 4.

a) $y = 2\tan^{-1}x$

$y' = \frac{2}{1+x^2}$

When $x = \frac{1}{\sqrt{3}}$

$y = 2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
$= \pi/3$

$y' = 3/2$

$y - \pi/3 = 3/2\left(x - \frac{1}{\sqrt{3}}\right)$

$y = 3/2\, x + \pi/3 - \frac{3}{2\sqrt{3}}$

b) i) $f(x) = \frac{-1}{x+2}$

$x = \frac{-1}{f^{-1}(x) + 2}$

$f^{-1}(x) = -\frac{1}{x} - 2$

ii) $x = \frac{-1}{x+2}$

$x(x+2) = -1$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$

$\therefore y = -1$

$\therefore$ Curves intersect at $(-1,-1)$

c) i) let $\alpha = \tan^{-1} A$ $\quad \beta = \tan^{-1} B$

$A = \tan\alpha$ $\quad B = \tan\beta$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{A+B}{1-AB}$$

$$\therefore \tan\theta = \frac{A+B}{1-AB}$$

ii) $\pi/4 = \tan^{-1}3x + \tan^{-1}2x$

$\tan \pi/4 = \tan(\tan^{-1}3x + \tan^{-1}2x)$

$$1 = \frac{3x+2x}{1-6x^2}$$

$$1 = \frac{5x}{1-6x^2}$$

$6x^2 + 5x - 1 = 0$

$(6x-1)(x+1) = 0$

$\therefore x = 1/6, -1$

but $x > 0$

$\therefore x = 1/6$

d) $P(1) = 0$; $(1-a)^3 + C = 0$ ①

$P(0) = -7$; $(-a)^3 + C = -7$ ②

① − ② :

$(1-a)^3 - (-a)^3 = 7$

$(1-a)^3 + a^3 = 7$

$1 - 3a + 3a^2 - a^3 + a^3 = 7$

$3a^2 - 3a - 6 = 0$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$\therefore a = -1, 2$

When $a = -1$

$2^3 + c = 0$

$c = -8$

When $a = 2$

$(1-2)^3 + C = 0$

$-1 + C = 0$

$C = 1$

5 a) i) $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$0 = px - y - ap^2$ ✓

ii) S is $(0, a)$ T is $(0, -ap^2)$ ✓

$SP^2 = (2ap - 0)^2 + (ap^2 - a)^2$

$= 4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2$

$= a^2p^4 + 2a^2p^2 + a^2$ ✓

$ST^2 = (a - -ap^2)$

$= a^2 + 2a^2p^2 + a^2p^4$ ✓

$= SP^2$

$\therefore ST = SP$

d) $R\hat{Q}S = N\hat{M}S$ (given)

$P\hat{R}Q = P\hat{N}S$ ($\angle$s in same segment)

$R\hat{S}N = Q\hat{S}K$ ($\angle$ sum of Δs MNS and ΔRQS) ✓

In ΔMNS and ΔLMS

$N\hat{M}S = L\hat{M}S$ (given) ✓

$R\hat{S}N = Q\hat{S}M$ (from above)

MS is common

$\therefore \Delta MNS \equiv \Delta LMS$ (AAS) ✓

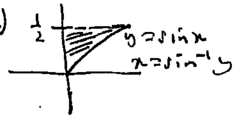
$\therefore LM = MN$ (corresponding sides of congruent Δs)

b) i) $x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$ ✓

$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$ ✓

ii)



$A = \int_0^{\frac{1}{2}} \sin^{-1} y \, dy$

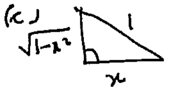
$= \left[y\sin^{-1}y + \sqrt{1-y^2} \right]_0^{\frac{1}{2}}$

$= \frac{1}{2} \times \sin^{-1}\frac{1}{2} + \sqrt{1-(\frac{1}{2})^2} - \sqrt{1}$

$= \frac{1}{2} \times \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1$ ✓

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

(c)



$\sin^{-1} 2x = \sin^{-1}\sqrt{1-x^2}$

$2x = \sqrt{1-x^2}$ ✓

$4x^2 = 1 - x^2$

$5x^2 = 1$

$x^2 = \frac{1}{5}$

$x = \frac{1}{\sqrt{5}}$ ✓