Sydney Girls High School

2013

HSC Assessment Task 3

Mathematics Extension 2

Topics Assessed: Polynomials, Integration.

General Instructions:

- Reading time – 5 minutes.
- Working time – 60 minutes.
- There are 3 questions which are of equal value.
- Write using black or blue pen.
- Board-approved calculators may be used.
- All necessary working should be shown in every question.
- Write your student number clearly at the top of each question and clearly number each question.

- **Total Marks: 60**

Question 1 (20 marks).

(a) Find the following integrals:

(i) $\int \frac{x^2+6}{x^2+4}\,dx$ **2**

(ii) $\int \sin x \sec^2 x\,dx$ **2**

(iii) $\int \frac{1}{\sqrt{x^2-6x+8}}\,dx$ **2**

(iv) $\int \frac{\sin x}{\cos x+2}\,dx$ **1**

(v) $\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$ **3**

(vi) $\int \sqrt{\frac{5-x}{5+x}}\,dx$ **3**

(b) Without evaluating the integral, explain why $\int_{-2}^{2} \sin x e^{-x^2}\,dx = 0$. **1**

(c) Evaluate $\int_0^1 \sqrt{4-x^2}\,dx$. **3**

(d) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$, **1**

(ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\cos^5 x+\sin^5 x}\,dx$. **2**

Question 2 (20 marks).

(a) Evaluate $\int_0^{\pi} e^{2x} \sin x \, dx$. **3**

(b) Use the substitution $t = \tan\frac{x}{2}$ to find $\int \frac{dx}{5+4\cos x+3\sin x}$. **3**

(c) (i) Find real numbers A, B and C such that: **3**

$$\frac{8}{(x+2)(x^2+4)} \equiv \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

(ii) Hence find $\int \frac{8}{(x+2)(x^2+4)} dx$. **3**

(d) A polynomial function is $P(x) = x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$.
Factorise $P(x)$ over the field of:

(i) Real numbers, **3**

(ii) Complex Numbers. **1**

(e) Factorise $x^4 + 4$ over the complex field. **2**

(f) α, β and γ are the roots of $x^3 + ax + b = 0$. Find the cubic polynomial whose roots are $\alpha\beta, \alpha\gamma$ and $\beta\gamma$. **2**

Question 3 (20 marks).

(a) (i) Show that $2+i$ is a root of the equation $2z^3 - 5z^2 - 2z + 15 = 0$. **2**

(ii) Find the other roots. **2**

(b) Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ given that it has a root of multiplicity 4. **3**

(c) Given that α, β, γ are the zeros of $P(x) = 2x^3 - 4x^2 - 3x - 1$, find:

(i) $\sum \alpha^2$ **1**

(ii) $\sum \alpha^3$ **2**

(iii) $\sum \alpha^4$ **2**

(d) The equation $ax^4 + bx^2 + 3 = 0$ has a double root. Show that: **3**

$$b^2 - 12a = 0.$$

(e) (i) Let n be an integer greater than 1 and let $I_n = \int_0^1 \frac{dx}{(x^2+1)^n}$. **3**

Prove that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$.

(ii) Hence, deduce the value of $\int_0^1 \frac{dx}{(x^2+1)^3}$. **2**

1 (a) (i) $\int\left(\frac{x^2+4}{x^2+4}+\frac{2}{x^2+4}\right)dx$

$= \int\left(1+\frac{2}{x^2+4}\right)dx$

$= x + 2\times\frac{1}{2}\tan^{-1}\frac{x}{2} + c$

$= x + \tan^{-1}\frac{x}{2} + c$

(ii) let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-\frac{du}{\sin x} = dx$

$\int \cancel{\sin x}\,.\,u^{-2}\,.\,-\frac{du}{\cancel{\sin x}}$

$= u^{-1} + c$

$= \sec x + c$

(iii) $\int\frac{dx}{\sqrt{(x-3)^2-1}}$

$= \log\left(x-3+\sqrt{x^2-6x+8}\right)+c$

(iv) $-\log(\cos x + 2) + c$

(v) let $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$\frac{du}{\cos\theta} = d\theta$

$\int \cancel{\cos\theta}\,.\,u^{-5}\,\frac{du}{\cancel{\cos\theta}}$

$= \frac{u^{-4}}{-4} + c$

$= -\frac{1}{4\sin^4\theta} + c$

(vi) $\int\sqrt{\frac{5-x}{5+x}\times\frac{5-x}{5-x}}\,dx$

$= \int\frac{5-x}{\sqrt{25-x^2}}\,dx$

$= \int\left(\frac{5}{\sqrt{25-x^2}} - \frac{x}{\sqrt{25-x^2}}\right)dx$

$= 5\sin^{-1}\frac{x}{5} - \frac{1}{2}\int\frac{2x}{\sqrt{25-x^2}}\,dx$

let $u = 25-x^2$

$\frac{du}{dx} = -2x$

$-\frac{du}{2x} = dx$

$= 5\sin^{-1}\frac{x}{5} - \frac{1}{2}\int 2x\,u^{-\frac{1}{2}}\,.\,-\frac{du}{2x}$

$= 5\sin^{-1}\frac{x}{5} + u^{\frac{1}{2}} + c$

$= 5\sin^{-1}\frac{x}{5} + \sqrt{25-x^2} + c$

(b) $\sin x\,.\,e^{-x^2}$ is an odd function

(c) let $x = 2\sin\theta$ when $x=1$, $\theta = \frac{\pi}{6}$

$\frac{dx}{d\theta} = 2\cos\theta$ " $x=0$, $\theta=0$

$\int_0^{\frac{\pi}{6}}\sqrt{4-4\sin^2\theta}\,.\,2\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}}\sqrt{4\cos^2\theta}\,.\,2\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}}4\cos^2\theta\,.\,d\theta$

$= \int_0^{\frac{\pi}{6}}4\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$

$= 2\left(\frac{\pi}{6} + \frac{\sin\frac{\pi}{3}}{2}\right)$

$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

(d)(i) $\int_0^a f(a-x)\,dx$ let $u = a-x$

$\frac{du}{dx} = -1$

$-du = dx$

$= \int_a^0 f(u)\,.\,-du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$ Q.E.D.

(ii) $\int_0^{\frac{\pi}{2}}\frac{\cos^5 x}{\cos^5 x+\sin^5 x}\,dx = \int_0^{\frac{\pi}{2}}\frac{\cos^5(\frac{\pi}{2}-x)}{\cos^5(\frac{\pi}{2}-x)+\sin^5(\frac{\pi}{2}-x)}\,dx$

$= \int_0^{\frac{\pi}{2}}\frac{\sin^5 x}{\sin^5 x+\cos^5 x}\,dx$

$= \int_0^{\frac{\pi}{2}}\left(\frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x} - \frac{\cos^5 x}{\sin^5 x+\cos^5 x}\right)dx$

$2\int_0^{\frac{\pi}{2}}\frac{\cos^5 x}{\cos^5 x+\sin^5 x}\,dx = [x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}}\frac{\cos^5 x}{\cos^5 x+\sin^5 x} = \frac{\pi}{4}$

Question 2

a) $I_0 = \int_0^{2\pi} e^{2x} \sin x \, dx$ $\quad u = e^{2x} \quad dv = \sin x \, dx$

$du = 2e^{2x} dx \quad v = -\cos x \, dx$

$= \left[-e^{2x}\cos x\right]_0^{\pi} + 2\int_0^{\pi} e^{2x}\cos x \, dx$

$= e^{2\pi} + 1 + 2\int_0^{\pi} e^{2x}\cos x \, dx$ $\quad u = e^{2x} \quad dv = \cos x \, dx$

$du = 2e^{2x} \quad v = \sin x$

$= e^{2\pi} + 1 + 2\left[e^{2x}\sin x\right]_0^{\pi} - 2\int_0^{\pi} 2e^{2x}\sin x \, dx$

$= e^{2\pi} + 1 + 2\left[e^{2\pi}\sin\pi - e^{0}\sin 0\right] - 4I_0$

$5I_0 = e^{2\pi} + 1$

$I_0 = \frac{1}{5}(e^{2\pi} + 1)$

b) let $t = \tan\frac{x}{2}$ $\quad dx = \frac{2dt}{1+t^2}$

$\int \frac{dx}{5+4\cos x+3\sin x} = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)+3\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$

$= \int \frac{1+t^2}{5(1+t^2)+4(1-t^2)+6t} \cdot \frac{2dt}{1+t^2}$

$= \int \frac{2\,dt}{9+6t+t^2}$

$= 2\int \frac{dt}{(t+3)^2}$

$= 2\int (t+3)^{-2}$

$= -2(t+3)^{-1} + C$

$= \frac{-2}{t+3} + C$

$= \frac{-2}{\tan\frac{x}{2}+3} + C.$

c) i) $8 = A(x^2+4) + (Bx+C)(x+2)$

Substituting $x = -2$

$8 = 8A + 0$

$A = 1$

Substituting $x = 0$

$8 = 1(0+4) + (0+C)(0+2)$

$8 = 4 + 2C$

$4 = 2C$

$C = 2$

Substituting $x = 1$

$8 = 1(1+4) + (B+2)(1+2)$

$8 = 5 + (B+2)3$

$8 = 5 + 3B + 6$

$-3 = 3B$

$B = -1$

ii) $\int \frac{8}{(x+2)(x^2+4)} dx = \int \frac{1}{x+2} dx + \int \frac{-x+2}{x^2+4} dx$

$= \ln(x+2) - \frac{1}{2}\int \frac{2x}{x^2+4} + 2\int \frac{1}{x^2+4}$

$= \ln(x+2) - \frac{1}{2}\ln(x^2+4) + \tan^{-1}\left(\frac{x}{2}\right) + C$

d) i) $P(x) = x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$

$P(-1) = -1 + 1 - 13 + 13 + 48 - 48$

$= 0$

$\therefore (x+1)$ is a factor

$P(x) = (x+1)(Ax^4 + Bx^3 + Cx^2 + Dx + E)$

By inspection:

$A = 1$

$E = -48$

$P(x) = (x+1)(x^4 + Bx^3 + Cx^2 + Dx - 48)$

Coefficient of x:

$-48 = -48 + D$

$D = 0$

Coefficient of x^2:

$13 = D + C$

$C = 13$

Coefficient of x^3:

$13 = C + B$

$13 = 13 + B$

$B = 0$

$\therefore P(x) = (x+1)(x^4 + 13x^2 - 48)$

$= (x+1)(x^2+16)(x^2-3)$

$= (x+1)(x^2+16)(x-\sqrt{3})(x+\sqrt{3})$

ii) $P(x) = (x+1)(x^2 - (4i)^2)(x-\sqrt{3})(x+\sqrt{3})$

$= (x+1)(x+4i)(x-4i)(x-\sqrt{3})(x+\sqrt{3})$

e) $x^4 + 4 = x^4 + 4x^2 - 4x^2 + 4$

$= (x^2+2)^2 - 4x^2$

$= (x^2+2)^2 - (2x)^2$

$= (x^2+2x+2)(x^2-2x+2)$

$= (x^2+2x+1+1)(x^2-2x+1+1)$

$= [(x+1)^2 - i^2][(x-1)^2 - i^2]$

$= (x+1+i)(x+1-i)(x-1+i)(x-1-i)$

f) $x^3 + ax + b = 0$

$\alpha\beta\gamma = -b$

$\alpha\beta = \frac{-b}{\gamma}$

$x = \alpha, \beta, \gamma$

$y = \frac{-b}{\alpha}, \frac{-b}{\beta}, \frac{-b}{\gamma}$

$y = \frac{-b}{x}$

$x = \frac{-b}{y}$

$\left(\frac{-b}{y}\right)^3 + a\left(\frac{-b}{y}\right) + b = 0$

$\frac{-b^3}{y^3} + \frac{ab}{y} + b = 0$

$-b^3 - aby^2 + by^3 = 0$

$by^3 - aby^2 - b^3 = 0$

$y^3 - ay^2 - b^2 = 0$

Question 3

a) i) $z=2+i$ is a root

That is:

$2(2+i)^3-5(2+i)^2-2(2+i)+15$

$=2[8+12i-6-i]-5[4+4i-1]-4-2i+15$

$=2(2+11i)-5(3+4i)+11-2i$

$=4+22i-15-20i+11-2i$

$=0$

$\therefore z=2+i$ is a root of $P(z)=2z^3-5z^2-2z+15$.

ii) If $z=2+i$ is a root of $P(z)$ where all coeffs are real, then $\bar{z}=2-i$ is also a root, by conjugate root theorem.

To find other roots:

Method 1:

Factors: $(z-2+i)(z-2-i)$

$=(z-2)^2-i^2$

$=z^2-4z+4+1$

$=z^2-4z+5$

$\therefore P(z)=(z^2-4z+5)(az+b)$

That is:

$2z^3-5z^2-2z+15\equiv(z^2-4z+5)(az+b)$

$\equiv az^3+bz^2-4az^2-4bz+5az+5b$

Equating coeffs: $a=2$

$-5=b-4a$

$-5=b-8$

$b=3$

$\therefore P(z)=(z-2+i)(z-2-i)(2z+3)$

$\therefore$ roots of $P(z)$ are: $z=2+i$, $z=2-i$, $z=-3/2$.

Method 2

Let the three roots of $P(z)$ be: $(2-i)$, $(2+i)$ and γ.

$\therefore (2-i)+(2+i)+\gamma=\frac{5}{2}$

$4+\gamma=\frac{5}{2}$

$\gamma=-3/2$

$\therefore$ Other roots of $P(z)$ are $z=2-i$ and $z=-\frac{3}{2}$.

3b) Let $P(x)=x^5+2x^4-2x^3-8x^2-7x-2$

A root of multiplicity 4 means $P'''(x)=0$ has the same root:

$P'(x)=5x^4+8x^3-6x^2-16x-7$

$P''(x)=20x^3+24x^2-12x-16$

$P'''(x)=60x^2+48x-12$

$0=12(5x^2+4x-1)$

$0=(x+1)(5x-1)$

$\therefore x=-1$ or $x=\frac{1}{5}$

$P(-1)=(-1)^5+2(-1)^4-2(-1)^3-8(-1)^2-7(-1)-2$

$=-1+2+2-8+7-2$

$=0$

$\therefore P(x)=(x+1)^4(x+b)$

$b=-2$ by observation of the constant term

$\therefore P(x)=(x+1)^4(x-2)$

$\therefore$ Solution is $x=-1$ or 2.

OR [Sum of roots: $-1\times4+\beta=-2$

$-4+\beta=-2$

$\beta=2$]

Q3 c) $P(x) = \underset{a}{2}x^3 - \underset{b}{4}x^2 - \underset{c}{3}x - \underset{d}{1}$ has roots α, β, γ.

$\Sigma\alpha = \alpha+\beta+\gamma = \frac{-b}{a} = \frac{4}{2} = 2$

$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = -\frac{3}{2}$

$\alpha\beta\gamma = \frac{-d}{a} = \frac{1}{2}$

i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (2)^2 - 2\left(-\frac{3}{2}\right)$

$= 4 + 3$

$= 7$

ii) $P(\alpha)$: $2\alpha^3 - 4\alpha^2 - 3\alpha - 1 = 0$ --- ①

$P(\beta)$: $2\beta^3 - 4\beta^2 - 3\beta - 1 = 0$ --- ②

$P(\gamma)$: $2\gamma^3 - 4\gamma^2 - 3\gamma - 1 = 0$ --- ③

① + ② + ③ :

$2(\alpha^3+\beta^3+\gamma^3) = 4(\alpha^2+\beta^2+\gamma^2) + 3(\alpha+\beta+\gamma) + 3$

$\therefore \Sigma\alpha^3 = \frac{1}{2}[4(7) + 3(2) + 3]$

$= \frac{1}{2} \times 37$

$= 18.5$

iii) $\alpha P(\alpha)$: $2\alpha^4 - 4\alpha^3 - 3\alpha^2 - \alpha = 0$ .... ①

$\beta P(\beta)$: $2\beta^4 - 4\beta^3 - 3\beta^2 - \beta = 0$ .... ②

$\gamma P(\gamma)$: $2\gamma^4 - 4\gamma^3 - 3\gamma^2 - \gamma = 0$ .... ③

① + ② + ③

$2(\alpha^4+\beta^4+\gamma^4) = 4(\Sigma\alpha^3) + 3(\Sigma\alpha^2) + \Sigma\alpha$

$\therefore \alpha^4+\beta^4+\gamma^4 = \frac{1}{2}\left[4\left(\frac{37}{2}\right) + 3(7) + 2\right]$

$\therefore \Sigma\alpha^4 = \frac{1}{2}[74 + 21 + 2]$

$= \frac{97}{2} = 48\frac{1}{2}$

Q3 d) Let $P(x) = ax^4 + bx^2 + 3$

If $P(x)$ has a double root then:

$P(x) = P'(x) = 0$.

$\therefore P'(x) = 4ax^3 + 2bx$

$0 = 2x(2ax^2 + b)$

$\therefore x = 0$ or $2ax^2 = -b$

$\Downarrow$ $\qquad x^2 = \frac{-b}{2a}$

No solution since $P(0) = 3$.

$\therefore P(x^2) = 0$

That is:

$a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + 3 = 0$

$\frac{ab^2}{4a^2} - \frac{b^2}{2a} + 3 = 0$

$b^2 - \frac{4ab^2}{2a} + 3(4a) = 0$

$b^2 - 2b^2 + 12a = 0$

$-b^2 + 12a = 0$

$\therefore b^2 - 12a = 0$.

Q3 e) i) $I_n = \int_0^1 \frac{dx}{(x^2+1)^n}$

Let $u = \frac{1}{(x^2+1)^n}$ $\quad \frac{dv}{dx} = 1$

$du = -n(x^2+1)^{-n-1} \cdot 2x\,dx \quad v = x$

$\therefore du = \frac{-2nx}{(x^2+1)^{n+1}}$

IBP: $\left[uv - \int v\,du\right]$

$I_n = \left[\frac{x}{(x^2+1)^n}\right]_0^1 + \int_0^1 \frac{2nx^2}{(x^2+1)^{n+1}}\,dx$

$I_n = \frac{1}{2^n} + 2n\int \frac{x^2+1-1}{(x^2+1)^{n+1}}\,dx$

$I_n = \frac{1}{2^n} + 2n\left[\int_0^1 \frac{(x^2+1)^1\,dx}{(x^2+1)^{n+1}} - \int_0^1 \frac{1}{(x^2+1)^{n+1}}\,dx\right]$

$I_n = \frac{1}{2^n} + 2n\left[I_n - I_{n+1}\right]$

$I_n = 2^{-n} + 2nI_n - 2nI_{n+1}$

$2nI_{n+1} = 2^{-n} + 2nI_n - I_n$

$2nI_{n+1} = 2^{-n} + (2n-1)I_n$ #.

3e) ii) $\int_0^1 \frac{dx}{(x^2+1)^3} = I_3$.

when $n=2$: $2nI_{n+1} = 2^{-n} + (2n-1)I_n$

$4I_3 = 2^{-2} + 3I_2$

$4I_3 = \frac{1}{4} + 3I_2$

$\therefore I_3 = \frac{1}{16} + \frac{3}{4}I_2$.

when $n=1$: $2nI_{n+1} = 2^{-n} + (2n-1)I_n$

$2I_2 = 2^{-1} + 1 \times I_1$

$\therefore I_2 = \frac{1}{4} + \frac{1}{2}I_1$

$I_1 = \int_0^1 \frac{1}{x^2+1}\,dx$.

$= \left[\tan^{-1}x\right]_0^1$

$= \frac{\pi}{4}$

$\therefore I_3 = \frac{1}{16} + \frac{3}{4}\left[\frac{1}{4} + \frac{1}{2}\left(\frac{\pi}{4}\right)\right]$

$= \frac{1}{16} + \frac{3}{16} + \frac{3}{8} \times \frac{\pi}{4}$

$= \frac{1}{4} + \frac{3\pi}{32}$.