



FORM VI

MATHEMATICS EXTENSION 1

Examination date

Wednesday 13th August 2008

Time allowed

2 hours (plus 5 minutes reading time)

Instructions

- All seven questions may be attempted.
- All seven questions are of equal value.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

- SGS booklets: 7 per boy. A total of 1250 booklets should be sufficient.
- Candidature: 125 boys.

Examiner

DS

QUESTION ONE (12 marks) Use a separate writing booklet.

Marks

- (a) Simplify $\frac{n!}{(n-1)!}$. 1
- (b) Write down the derivative of $y = \cos^{-1} x^2$. 1
- (c) Find $\int \frac{1}{40 + x^2} dx$. 1
- (d) Simplify $\log_e \sqrt{e}$. 1
- (e) Write down a primitive of $2x e^{x^2}$. 1
- (f) Write $\cos 2\theta$ in terms of t , where $t = \tan \theta$. 1
- (g) A is the point $(-6, 2)$ and B is the point $(4, 10)$. Find the coordinates of the point P that divides the interval AB internally in the ratio $7 : 4$. 2
- (h) Sketch the graph of the polynomial function $y = x^3(3 - x)$. (There is no need to find the coordinates of the turning point.) 2
- (i) Use the identity $(1 + x)^n = \sum_{r=0}^n {}^n C_r x^r$ to prove that 2

$${}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n.$$

QUESTION TWO (12 marks) Use a separate writing booklet.

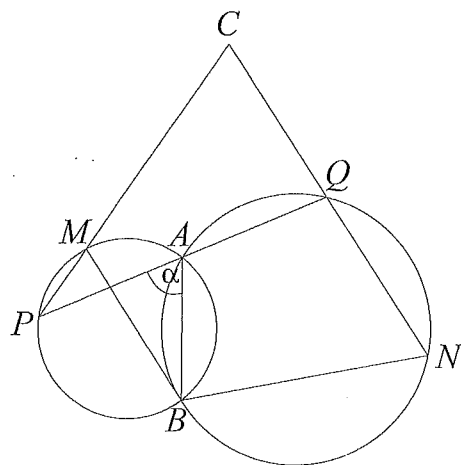
Marks

(a) Use the substitution $x = u - 2$ to find $\int \frac{x}{(x+2)^2} dx$. 3

(b) Solve the inequation $\frac{x}{x+2} > 0$. 3

(c) Show that $\tan(\tan^{-1} 2 - \tan^{-1} \sqrt{2}) = \frac{5\sqrt{2}-6}{7}$. 3

(d)



The diagram above shows two circles intersecting at A and B. The points P, A and Q are collinear, and the chords PM and NQ, when produced, intersect at C. Let $\angle PAB = \alpha$.

- (i) Give a reason why $\angle BNQ = \alpha$. 1
- (ii) Prove that the quadrilateral CMBN is cyclic. 2

QUESTION THREE (12 marks) Use a separate writing booklet.

Marks

(a) An ice-cube is taken out of a freezer and begins to melt. Assume that it remains a cube as it does so. If its edge length is decreasing at the constant rate of 2 mm/min, find the rate at which its volume is decreasing at the instant when the edge length is 15 mm. 4

- (b) It is known that the polynomial equation $6x^3 - 17x^2 - 5x + 6 = 0$ has three real roots, and that two of them have a product of -2 .
- (i) Use the product of the roots to find one of the three roots. 1
 - (ii) Use the sum of the roots, or any other suitable method, to find the other two roots. 3

(c) Find the exact value of $\int_0^{\frac{\pi}{2}} (\cos x - \cos^2 x) dx$. 4

QUESTION FOUR (12 marks) Use a separate writing booklet.

Marks

(a) Prove by mathematical induction that for all positive integer values of n , 4

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \dots + n(n+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5).$$

- (b) Let α be the real root of the equation $\cos x = 2x$.
- (i) On the same diagram, sketch the graphs of the functions $y = \cos x$ and $y = 2x$. 1
 - (ii) Show α on your diagram. 1
 - (iii) Use one application of Newton's method with starting value $\frac{1}{2}$ to estimate α . Write your answer correct to two decimal places. 3

(c) Use the identity $(1+x)^4(1+x)^{96} = (1+x)^{100}$ to prove that 3

$$\binom{96}{4} + \binom{4}{1} \binom{96}{3} + \binom{4}{2} \binom{96}{2} + \binom{4}{3} \binom{96}{1} = \binom{100}{4} - 1.$$

QUESTION FIVE (12 marks) Use a separate writing booklet.

Marks

(a) Find the term independent of x in the expansion of $\left(ax^3 + \frac{b}{x^2}\right)^{5n}$, where n is a positive integer. 4

(b) Newton's law of cooling states that the rate of decrease of the temperature of a heated body is proportional to the excess of the temperature of the body over that of its surroundings. Using t for time (in minutes), H for the temperature of the body (in °C), and S for the constant temperature of the surroundings (also in °C), the law of cooling can be modelled by the differential equation $\frac{dH}{dt} = -k(H - S)$, where k is a positive constant.

(i) Show that the function $H = Ae^{-kt} + S$ satisfies the differential equation, where A is a constant. 1

(ii) Suppose that a body is heated to 80 °C in a room whose temperature is 20 °C, and that after 5 minutes the temperature of the body is 70 °C.

(α) Show that, at any time $t \geq 0$, $H = 20 + 60\left(\frac{5}{6}\right)^{\frac{t}{5}}$. 3

(β) Find, correct to one decimal place, the temperature of the body after one hour. 1

(c) Let $P(a) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$.

(i) Use the factor theorem to show that $a + b$ is a factor of $P(a)$. 2

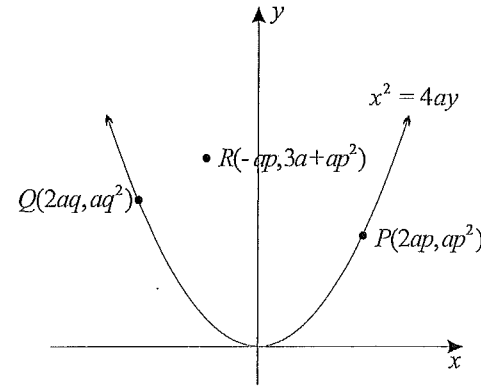
(ii) Hence, or otherwise, factorise $P(a)$. 1

QUESTION SIX (12 marks) Use a separate writing booklet.

Marks

(a) A particle moves along the x -axis. It starts from rest at the point $x = 1$. Its acceleration is given by $\ddot{x} = -4\left(x + \frac{1}{x^3}\right)$. Find its velocity when it is half-way from its starting point to the origin. 4

(b)



In the diagram above, $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are distinct points on the parabola $x^2 = 4ay$, and R is the point $(-ap, 3a + ap^2)$.

(i) Show that the normal to the parabola at P has equation $x + py = 2ap + ap^3$. 2

(ii) Show that the normal at P passes through R . 1

(iii) If the normal at Q also passes through R , show that $q^2 + pq - 1 = 0$. 2

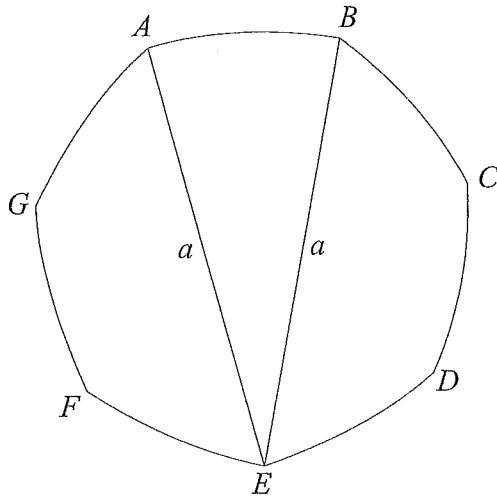
(iv) Show that there are always two real values of q satisfying the equation in part (iii). 1

(v) Deduce that three normals to the parabola, two of which are perpendicular to each other, pass through the point R . (You may assume that $p^2 \neq \frac{1}{2}$.) 2

QUESTION SEVEN (12 marks) Use a separate writing booklet.

Marks

(a)

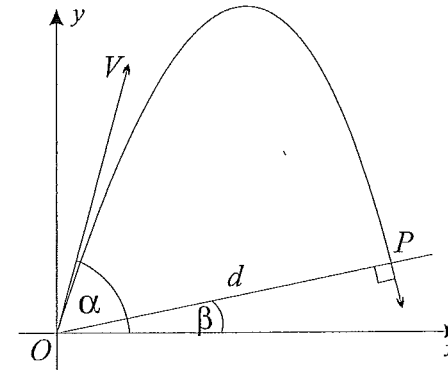


The diagram above shows a British 50 pence coin. The seven circular arcs AB , BC , \dots , GA are of equal length and their centres are E , F , \dots , D respectively. Each arc has radius a .

- (i) Show that the sector AEB has area $\frac{1}{14}\pi a^2$. 2
- (ii) Hence, or otherwise, show that the face of the coin has area $\frac{1}{2}a^2(\pi - 7 \tan \frac{\pi}{14})$. 2

Exam continues overleaf ...

(b)



The diagram above shows the parabolic path of a particle that is projected from the origin O with velocity V at an angle of α to the horizontal. It lands at the point P , which lies on a plane inclined at an angle of β to the horizontal. When the particle strikes the plane, it is travelling at 90° to the plane.

Let $OP = d$, and assume that the horizontal and vertical components of the displacement of the particle from O while it is moving on its parabolic path are given by

$$x = Vt \cos \alpha \quad \text{and} \quad y = Vt \sin \alpha - \frac{1}{2}gt^2,$$

where t is the time elapsed, and g is acceleration due to gravity.

- (i) Find the coordinates of P in terms of d and β . 1
- (ii) By substituting the coordinates of P found in part (i) into the displacement equations, show that 2

$$d = \frac{2V^2 \cos^2 \alpha}{g \cos^2 \beta} (\tan \alpha \cos \beta - \sin \beta).$$

- (iii) By resolving the horizontal and vertical components of the velocity at P , show that 3

$$\cot \beta = \frac{gd \cos \beta}{V^2 \cos^2 \alpha} - \tan \alpha.$$

- (iv) Hence show that $\tan \alpha = \cot \beta + 2 \tan \beta$. 2

END OF EXAMINATION

SOLUTIONS TO FORM VI EXTENSION 1

TRIAL HSC 2008

TOTAL = 12x7

(1)(a) $\frac{n!}{(n-1)!} = n$ ✓

(b) $\frac{-2x}{\sqrt{1-x^4}}$ ✓

(c) $\int \frac{1}{40+x^2} dx = \frac{1}{2\sqrt{10}} \tan^{-1} \frac{x}{2\sqrt{10}} + c$ ✓

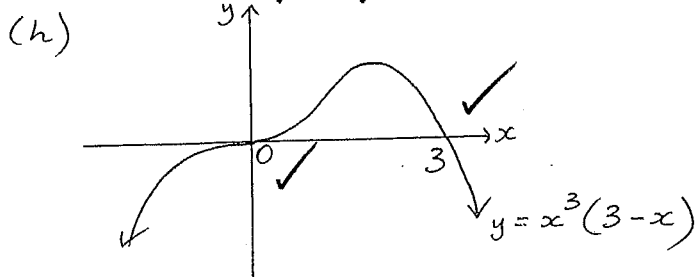
(d) $\ln e^{\frac{1}{2}} = \frac{1}{2} \ln e$
 $= \frac{1}{2}$ ✓

No penalty for omission of c.

(e) $\int 2xe^{x^2} dx = e^{x^2} + c$ ✓

(f) $\cos 2\theta = \frac{1-t^2}{1+t^2}$ ✓ (where $t = \tan \theta$)

(g) $P = \left(\frac{28-24}{11}, \frac{70+8}{11} \right)$
 $= \left(\frac{4}{11}, 7\frac{1}{11} \right)$



(i) Substitute $x=1$ into the identity: ✓

$$\sum_{r=0}^n {}^n C_r \cdot (1)^r = (1+1)^n$$

i.e. ${}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n$ ✓

(2) (a) $\int \frac{x}{(x+2)^2} dx = \int \frac{u-2}{u^2} du$ ✓
 $= \int \left(\frac{1}{u} - 2u^{-2} \right) du$ ✓
 $= \ln u + \frac{2}{u} + c$
 $= \ln(x+2) + \frac{2}{x+2} + c$ ✓

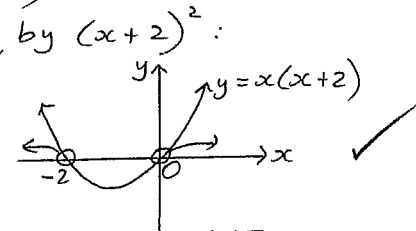
Let $x = u - 2$
 $\therefore \frac{dx}{du} = 1$
 $\therefore dx = du$

(b) $\frac{x}{x+2} > 0$ ($x \neq -2$)

Multiply both sides by $(x+2)^2$:

$x(x+2) > 0$ ✓

$x < -2$ or $x > 0$ ✓



(c) Let $\alpha = \tan^{-1} 2$ and $\beta = \tan^{-1} \sqrt{2}$.

$\therefore \tan \alpha = 2$, where $0 < \alpha < \frac{\pi}{2}$,

and $\tan \beta = \sqrt{2}$, where $0 < \beta < \frac{\pi}{2}$.

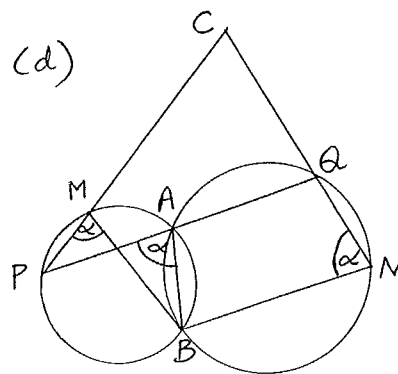
$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$ ✓

$= \frac{2 - \sqrt{2}}{1 + 2\sqrt{2}} \cdot \frac{1 - 2\sqrt{2}}{1 - 2\sqrt{2}}$ ✓

$= \frac{2 - 4\sqrt{2} - \sqrt{2} + 4}{1 - 8}$

$= \frac{6 - 5\sqrt{2}}{-7}$

$= \frac{5\sqrt{2} - 6}{7}$ ✓



(i) Exterior angle of cyclic quad $ABNQ$ is equal to the interior opposite angle. ✓

(ii) $\angle PMB = \alpha$ (angles at circumference standing on same arc)

$\therefore \angle PMB = \angle BNQ = \alpha$

\therefore quad $CMBN$ is cyclic (converse of reason in (i)) ✓

(3)(a) Let $V \text{ mm}^3$ be the volume of the ice-cube, and $x \text{ mm}$ its edge length.

We are given $\frac{dx}{dt} = -2 \text{ mm/min}$.

We want $\frac{dV}{dt}$ when $x = 15$.

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} \checkmark, \text{ where } V = x^3.$$

$$\therefore \frac{dV}{dt} = 3x^2 \cdot (-2) \checkmark$$

$$= -6x^2$$

$$\text{So when } x = 15, \frac{dV}{dt} = -6(15)^2 \checkmark$$

$$= -1350.$$

So when the edge is 15 mm , the volume is decreasing at $1350 \text{ mm}^3/\text{min}$. \checkmark

(b) Let the roots be α , $-\frac{2}{\alpha}$ and β .

(i) The product of the roots is $-\frac{d}{a} = -1$.

$$\therefore \alpha \cdot -\frac{2}{\alpha} \cdot \beta = -1 \checkmark$$

$$\therefore \beta = \frac{1}{2} \checkmark$$

So one of the roots is $\frac{1}{2}$.

(ii) The sum of the roots is $-\frac{b}{a} = \frac{17}{6}$.

$$\therefore \alpha - \frac{2}{\alpha} + \frac{1}{2} = \frac{17}{6} \checkmark$$

$$\alpha - \frac{2}{\alpha} = \frac{7}{3}$$

$$3\alpha^2 - 7\alpha - 6 = 0 \checkmark$$

$$(3\alpha + 2)(\alpha - 3) = 0$$

$$\alpha = -\frac{2}{3} \text{ or } 3 \checkmark$$

So the other two roots are $-\frac{2}{3}$ and 3 .

$$(c) \int_0^{\frac{\pi}{2}} (\cos x - \cos^2 x) dx = \int_0^{\frac{\pi}{2}} (\cos x - (\frac{1}{2} + \frac{1}{2} \cos 2x)) dx \checkmark$$

$$= [\sin x - \frac{1}{2}x - \frac{1}{4} \sin 2x]_0^{\frac{\pi}{2}} \checkmark$$

$$= \sin \frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{4} \sin \pi - (0 - 0 - 0) \checkmark$$

$$= 1 - \frac{\pi}{4} \checkmark$$

$$(4)(a) \text{ When } n=1, \text{ LHS} = 1 \times 2^2$$

$$= 4$$

$$\text{RHS} = \frac{1}{12} \times 1 \times 2 \times 3 \times 8 \checkmark$$

$$= 4$$

So the result is true for $n=1$.

Assume that the result is true for $n=k$, where k is a positive integer.

i.e. assume that $1 \times 2^2 + 2 \times 3^2 + \dots + k(k+1)^2 = \frac{1}{12} k(k+1)(k+2)(3k+5)$.

Prove that the result is true for $n=k+1$.

i.e. prove that

$$1 \times 2^2 + 2 \times 3^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2 = \frac{1}{12} (k+1)(k+2)(k+3)(3k+8)$$

$$\text{LHS} = 1 \times 2^2 + 2 \times 3^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2$$

$$= \frac{1}{12} k(k+1)(k+2)(3k+5) + (k+1)(k+2)^2 \checkmark$$

(using the assumption)

$$= \frac{1}{12} (k+1)(k+2)(k(3k+5) + 12(k+2)) \checkmark$$

$$= \frac{1}{12} (k+1)(k+2)(3k^2 + 17k + 24) \checkmark$$

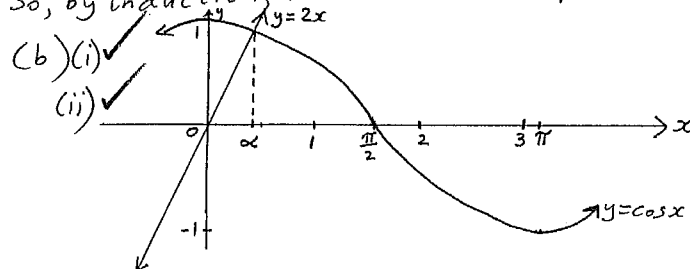
$$= \frac{1}{12} (k+1)(k+2)(k+3)(3k+8) \checkmark$$

$$= \text{RHS}$$

So the result is true for $n=k+1$ if it is true for $n=k$.

But the result is true for $n=1$.

So, by induction, it is true for all positive integer values of n .



(iii) Let $f(x) = 2x - \cos x$, so that $f'(x) = 2 + \sin x$. \checkmark

$$x_2 = 0.5 - \frac{1 - \cos 0.5}{2 + \sin 0.5} \checkmark$$

$$= 0.4506 \dots \checkmark$$

$$\doteq 0.45$$

$$(4)(c) \text{ RHS of identity} = (1+x)^{100}$$

$$= \sum_{r=0}^{100} \binom{100}{r} x^r$$

The coefficient of x^4 is $\binom{100}{4}$.

LHS of identity

$$= \left(\binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4 \right)$$

$$\cdot \left(\binom{96}{0} + \binom{96}{1}x + \binom{96}{2}x^2 + \binom{96}{3}x^3 + \binom{96}{4}x^4 + \dots + \binom{96}{96}x^{96} \right)$$

The coefficient of x^4 is

$$\binom{4}{0}\binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} + \binom{4}{4}\binom{96}{0}$$

$$= \binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} + 1,$$

since $\binom{4}{0} = \binom{4}{4} = \binom{96}{0} = 1$.

The coefficients of x^4 on both sides of the

identity are equal, so

$$\binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} = \binom{100}{4} - 1.$$

$$(5)(a) \text{ General term} = {}^5n C_r \cdot (ax^3)^{5n-r} \cdot (bx^{-2})^r$$

$$= {}^5n C_r \cdot a^{5n-r} \cdot b^r \cdot x^{15n-3r-2r}$$

$$= {}^5n C_r \cdot a^{5n-r} \cdot b^r \cdot x^{15n-5r}$$

We require $15n - 5r = 0$,

i.e. $r = 3n$.

So the constant term is

$${}^5n C_{3n} \cdot a^{2n} \cdot b^{3n}$$

$$(b)(i) \frac{dH}{dt} = -kAe^{-kt}$$

$$= -k(H - S)$$

(ii) When $t=0$, $H=80$.

$$\therefore 80 = A + 20$$

$$\therefore A = 60$$

When $t=5$, $H=70$.

$$\therefore 70 = 60e^{-5k} + 20$$

$$\frac{5}{6} = e^{-5k}$$

$$k = -\frac{1}{5} \ln \frac{5}{6}$$

$$\therefore H = 60e^{\frac{1}{5}t \ln \frac{5}{6}} + 20$$

$$= 20 + 60e^{\ln \left(\frac{5}{6}\right)^{\frac{t}{5}}}$$

$$= 20 + 60\left(\frac{5}{6}\right)^{\frac{t}{5}}, \text{ as required.}$$

(iii) When $t=60$,

$$H = 20 + 60\left(\frac{5}{6}\right)^{12}$$

$$= 26.729\dots$$

So after one hour, the temperature of the body is 26.7°C , correct to one decimal place.

$$(c) (i) \quad P(-b) = b^2(b+c) + b^2(c-b) + c^2(-b+b) - 2b^2c \checkmark$$

$$= b^3 + b^2c + b^2c - b^3 - 2b^2c$$

$$= 0$$

$\therefore a+b$ is a factor of $P(a)$ \checkmark

(ii) $P(a)$ is symmetric in a, b and c , so $b+c$ and $c+a$ are also factors of $P(a)$. \checkmark

$$\text{So } P(a) = (a+b)(b+c)(c+a).$$

Other methods, such as long division, are acceptable.

$$(b) (a) \quad \ddot{x} = -4\left(x + \frac{1}{x^3}\right)$$

$$\therefore \frac{1}{2}v^2 = -4 \int \left(x + x^{-3}\right) dx \quad \checkmark$$

$$= -4\left(\frac{x^2}{2} + \frac{x^{-2}}{-2}\right) + c$$

$$= -4\left(\frac{x^2}{2} - \frac{1}{2x^2}\right) + c \quad \checkmark$$

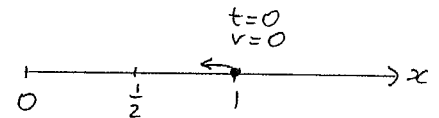
When $t=0$, $x=1$ and $v=0$. \checkmark

$$\therefore 0 = -4\left(\frac{1}{2} - \frac{1}{2}\right) + c$$

$$\therefore c = 0$$

$$\therefore v^2 = -8\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)$$

$$= -4x^2 + \frac{4}{x^2}$$



When $t = \frac{1}{2}$,

$$v^2 = -4 \cdot \frac{1}{4} + \frac{4}{\frac{1}{4}}$$

$$= 15 \quad \checkmark$$

$\therefore v = -\sqrt{15}$, because the particle is travelling in the negative direction.

$$(6)(b)(i) \quad y = \frac{x^2}{4a}$$

$$\therefore y' = \frac{x}{2a}$$

$$\text{When } x = 2ap,$$

$$y' = \frac{2ap}{2a}$$

$$= p$$

So the normal at P has gradient $-\frac{1}{p}$.

So the normal at P has equation

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap$$

$$x + py = 2ap + ap^3$$

(ii) When $x = -ap$ and $y = 3a + ap^2$,

$$\text{LHS} = x + py$$

$$= -ap + p(3a + ap^2)$$

$$= 2ap + ap^3$$

$$= \text{RHS}$$

So the normal at P passes through R.

(iii) The normal at Q has equation $x + qy = 2aq + aq^3$.

Substitute $x = -ap$ and $y = 3a + ap^2$:

$$-ap + 3aq + ap^2q = 2aq + aq^3$$

$$aq^3 - ap^2q - aq + ap = 0$$

$$aq(q^2 - p^2) - a(q - p) = 0$$

$\boxed{\div a}$

$$q(q - p)(q + p) - 1(q - p) = 0 \quad (a \neq 0)$$

$$(q - p)(q^2 + pq - 1) = 0$$

$q \neq p$ since P and Q are distinct points,
so $q^2 + pq - 1 = 0$.

(iv) Consider the equation $q^2 + pq - 1 = 0$ as a quadratic equation in q .

$\therefore \Delta = p^2 + 4 > 0$ for all real values of p .

So the equation ^{always} has two real roots.

(6)(b)(v) Consider again the quadratic equation

$q^2 + pq - 1 = 0$. Let the roots be q_1 and q_2 ($q_1 \neq q_2$).

The product of the roots is -1 .

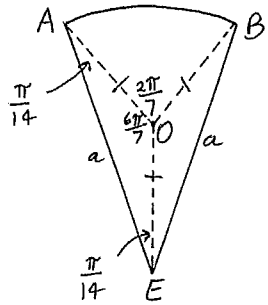
$$\therefore q_1 q_2 = -1$$

$$\therefore -\frac{1}{q_1} \cdot -\frac{1}{q_2} = -1$$

So the normals at the points $(2aq_1, aq_1^2)$ and $(2aq_2, aq_2^2)$ pass through R, and these normals (whose gradients are $-\frac{1}{q_1}$ and $-\frac{1}{q_2}$) are perpendicular.

From (ii), we also know that the normal at P passes through R.

(7)(a)



Let O be the centre of the coin.

$$\therefore OA = OB = OE$$

$$\begin{aligned} \angle AOB &= \frac{1}{7} \text{ of a revolution} \\ &= \frac{2\pi}{7} \end{aligned}$$

$$\therefore \angle AOE = \angle BOE = \frac{6\pi}{7} \text{ (angles at a point)}$$

$$\therefore \angle OAE = \angle OEA = \frac{\pi}{14} \text{ (angle sum of isosceles triangle)}$$

(i) $\angle AEB = \frac{\pi}{7}$ (with some justification) ✓

$$\begin{aligned} \text{So area of sector AEB} &= \frac{1}{2} r^2 \theta \\ &= \frac{1}{2} \cdot a^2 \cdot \frac{\pi}{7} \\ &= \frac{1}{14} \pi a^2 \end{aligned}$$

(ii) In $\triangle OAE$,

$$\frac{h}{\frac{1}{2}a} = \tan \frac{\pi}{14}$$

$$\therefore h = \frac{1}{2} a \tan \frac{\pi}{14}$$

$$\text{So } \triangle OAE \text{ has area } \frac{1}{4} a^2 \tan \frac{\pi}{14}$$

So area of portion AOB = area of sector AEB

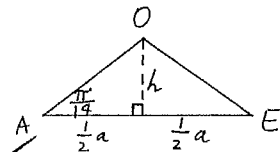
$$- 2 \times \text{area of } \triangle OAE$$

$$= \frac{1}{14} \pi a^2 - \frac{1}{2} a^2 \tan \frac{\pi}{14}$$

So area of coin is 7x area of AOB

$$= 7 \left(\frac{1}{14} \pi a^2 - \frac{1}{2} a^2 \tan \frac{\pi}{14} \right)$$

$$= \frac{1}{2} a^2 \left(\pi - 7 \tan \frac{\pi}{14} \right)$$



(7)(b)(i) P has coordinates $(d \cos \beta, d \sin \beta)$. ✓

(ii) This point lies on the parabola, so

$$d \cos \beta = V t \cos \alpha \quad (1) \text{ and } d \sin \beta = V t \sin \alpha - \frac{1}{2} g t^2 \quad (2)$$

$$\text{From (1), } t = \frac{d \cos \beta}{V \cos \alpha}$$

Substitute into (2):

$$d \sin \beta = V \sin \alpha \cdot \frac{d \cos \beta}{V \cos \alpha} - \frac{g}{2} \cdot \frac{d^2 \cos^2 \beta}{V^2 \cos^2 \alpha}$$

Dividing by d ($d \neq 0$, since $d = 0$ corresponds to the particle being at the origin),

$$\sin \beta = \tan \alpha \cos \beta - d \cdot \frac{g \cos^2 \beta}{2 V^2 \cos^2 \alpha}$$

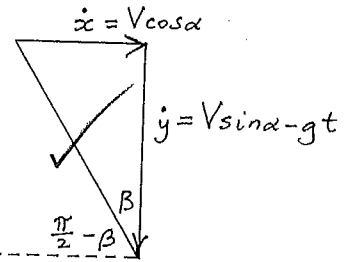
$$\therefore d = \frac{2 V^2 \cos^2 \alpha}{g \cos^2 \beta} (\tan \alpha \cos \beta - \sin \beta)$$

(iii) $\cot \beta = \frac{-\dot{y}}{\dot{x}}$ (\dot{y} is negative because the particle is moving downwards, \dot{x} is positive.)

$$= \frac{gt - V \sin \alpha}{V \cos \alpha}$$

$$= \frac{g}{V \cos \alpha} \cdot \frac{d \cos \beta}{V \cos \alpha} - \frac{V \sin \alpha}{V \cos \alpha}$$

$$= \frac{g d \cos \beta}{V^2 \cos^2 \alpha} - \tan \alpha$$



(iv) From (iii),

$$\tan \alpha = \frac{g d \cos \beta}{V^2 \cos^2 \alpha} - \cot \beta$$

Using (ii),

$$\tan \alpha = \frac{g \cos \beta}{V^2 \cos^2 \alpha} \cdot \frac{2 V^2 \cos^2 \alpha}{g \cos^2 \beta} (\tan \alpha \cos \beta - \sin \beta) - \cot \beta$$

$$= \frac{2}{\cos \beta} (\tan \alpha \cos \beta - \sin \beta) - \cot \beta$$

$$= 2 \tan \alpha - 2 \tan \beta - \cot \beta$$

$$\therefore \tan \alpha = \cot \beta + 2 \tan \beta$$