SYDNEY GRAMMAR SCHOOL

2010 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Tuesday 3rd August 2010

General Instructions
- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 120
- All eight questions may be attempted.
- All eight questions are of equal value.

Collection
- Write your candidate number clearly on each booklet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist
- SGS booklets — 8 per boy
- Candidature — 89 boys

Examiner
PKH

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. 2

(b) (i) Find the values of A and B such that $\frac{2x-1}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1}$. 2

(ii) Hence find $\int \frac{2x-1}{(x-1)^2}\,dx$. 2

(c) Find $\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$. 2

(d) Find $\int x\sec^2 x\,dx$. 3

(e) Use the substitution $x = 2\tan\theta$ to find $\int_0^2 \frac{1}{(4+x^2)^2}\,dx$. 4

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3+i$ and $w = 2-i$. Find in the form $x+iy$:

(i) wz 1

(ii) $\overline{w}\,\overline{z}$ 1

(iii) $\frac{w}{z}$ 1

(b) (i) Write $1 - i\sqrt{3}$ in modulus-argument form. 2

(ii) Find $(1-i\sqrt{3})^9$ in the form $a+ib$, where a and b are real. 2

(c) Simplify $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$. 2

(d) Sketch the region in the complex plane which simultaneously satisfies 3

$|z-3-4i| \le 5$ and $\frac{3\pi}{4} \le \arg(z-3-4i) \le \pi$.

(e) (i) By letting $z = x+iy$, find the locus of points in the complex plane which satisfy 2

$$\mathrm{Re}\left(z - \frac{1}{\overline{z}}\right) = 0.$$

(ii) Sketch the locus. 1

Exam continues next page ...

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

(i) Find the eccentricity of the ellipse. 1

(ii) Find the coordinates of the foci and the equations of the directrices. 2

(iii) Sketch the ellipse, showing the foci and directrices. 1

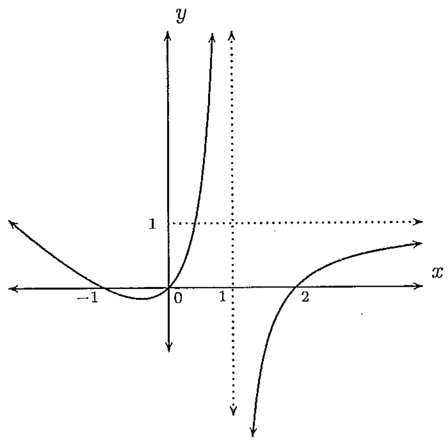
(b) Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with eccentricity e. Let $P(x_1, y_1)$ be a point on the ellipse in the first quadrant.

(i) Show that the normal to the ellipse at P has gradient $\frac{a^2 y_1}{b^2 x_1}$. 1

(ii) The normal at P cuts the x-axis at G. Show that the x-coordinate of G is $e^2 x_1$. 2

(iii) If S is the focus $(ae, 0)$, show that $SG = e\,PS$. 2

(c)



The diagram above shows the graph of $y = f(x)$, where the lines $x = 1$ and $y = 1$ are asymptotes. Draw separate one-third page sketches of the graphs of:

(i) $y = f(-x)$ 1

(ii) $y = f(|x|)$ 1

(iii) $y^2 = f(x)$ 2

(iv) $y = e^{f(x)}$ 2

Exam continues overleaf ...

QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

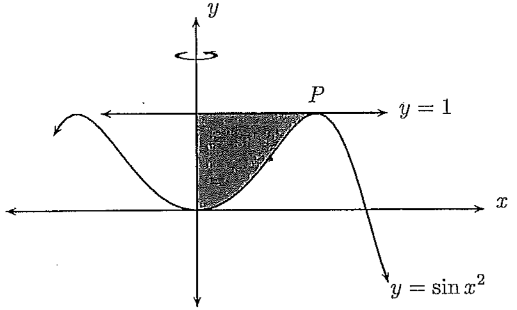
(a) Consider the curve $y = e^{-x}(1 - x)$.

(i) Find and classify the stationary point. [2]

(ii) Sketch the curve showing the main features. (You do not need to find the coordinates of the point of inflexion.) [3]

(iii) Suppose that the line $y = mx$ is a tangent to the curve in the fourth quadrant. Use your diagram to explain why m must be less than $-\dfrac{1}{2e^2}$. [2]

(b)



The region bounded by the curve $y = \sin x^2$, the y-axis and the line $y = 1$ is shaded in the diagram above.

(i) Find the x-coordinate of the point P. [1]

(ii) The shaded region is rotated about the y-axis. Use the method of cylindrical shells to find the volume of the solid generated. Leave your answer in terms of π. [3]

(c) Consider the polynomial $P(z) = z^3 + az^2 + bz + c$ where a, b and c are real. Suppose that ki is a zero of $P(z)$, where k is real.

(i) Show that $P(z)$ has one real zero. [2]

(ii) Show that $c = ab$. [2]

Exam continues next page ...

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

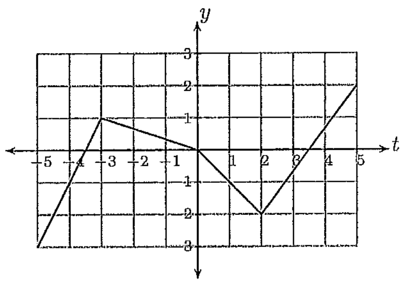
(a) The base of a solid S is the ellipse $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the x-axis are isosceles right-angled triangles with hypotenuse in the base. Find the volume of S. [4]

(b) (i) Show that $x^2\left(1+x^2\right)^{n-1} = \left(1+x^2\right)^n - \left(1+x^2\right)^{n-1}$. [1]

(ii) Suppose that $I_n = \displaystyle\int_0^1 \left(1+x^2\right)^n dx$, where n is a positive integer. [4]

Show that $I_n = \dfrac{1}{2n+1}\left(2^n + 2n\, I_{n-1}\right)$.

(c) [2]



The diagram above shows the graph of the function $y = f(t)$.

Let $F(x) = \displaystyle\int_{-5}^{x} f(t)\, dt$.

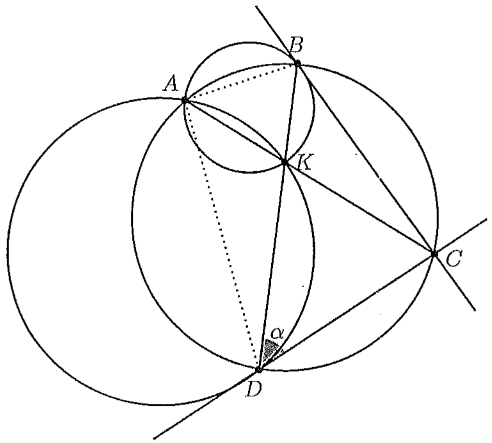
For which value of x does $F(x)$ achieve its absolute minimum value? Give reasons for your answer.

Question Five Continues Over the Page

Exam continues overleaf ...

QUESTION FIVE (Continued)

(d)



In the diagram above, $ABCD$ is a cyclic quadrilateral and diagonals AC and BD intersect at K. Circles AKD and AKB are drawn and it is known that CD is a tangent to circle AKD. Let $\angle BDC = \alpha$.

NOTE: You do not have to copy the diagram above. It has been reproduced for you on a tear-off sheet at the end of this paper. Insert this sheet into your answer booklet.

(i) Prove that $\triangle BCD$ is isosceles. **2**

(ii) Prove that CB is a tangent to circle AKB. **2**

Exam continues next page ...

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a) A particle of unit mass is projected vertically upwards against a constant gravitational force g and a resistance of magnitude $\dfrac{v}{10}$, where v is the velocity of the particle after t seconds. The initial velocity is 80 metres per second. Take g to be $10\,\text{m/s}^2$.

(i) Taking the positive direction of motion upwards, show that the equation of motion is $\ddot{x} = -\dfrac{v+100}{10}$. **1**

(ii) Show that the time T for the particle to reach its greatest height is given by $T = 10\ln 1{\cdot}8$ seconds. **3**

(iii) Show that the maximum height H attained by the particle is approximately 212 metres. **3**

(iv) From its maximum height the particle falls to its original position under gravity and under the same resistance.
Determine whether the speed at which the particle returns to its starting point is greater than or less than the speed of projection. (Take the positive direction of motion downwards in this part.) **4**

(b) (i) Suppose that x is a positive real number and that n is a positive integer. **1**
Show that $\dfrac{1}{1+x^n} < 1$.

(ii) Let $I_n = \displaystyle\int_0^1 \frac{1}{1+x^n}\,dx$ where n is a positive integer and $n \geq 2$. **3**
Show that $\dfrac{\pi}{4} \leq I_n < 1$.

Exam continues overleaf ...

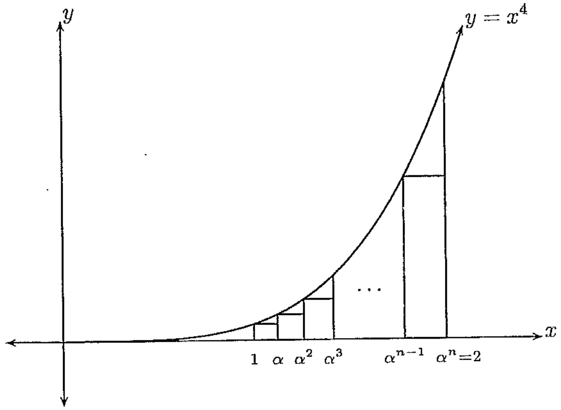
QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that the normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \dfrac{c}{t}\right)$ has equation $ty - t^3x = c(1 - t^4)$. 2

(ii) Show that from a point $(0, k)$ on the y-axis, exactly two normals can be drawn to the hyperbola. 4

(iii) Show that there can never be more than four normals drawn to the hyperbola from an arbitrary point in the plane. 1

(b)



In the diagram above the curve $y = x^4$ is graphed. The interval from $x = 1$ to $x = 2$ is divided into n unequal subintervals. The first subinterval is $1 \le x \le \alpha$, the second is $\alpha \le x \le \alpha^2$, and so on, where $\alpha = 2^{\frac{1}{n}}$.

A rectangle is constructed on each subinterval, as shown in the diagram.

(i) Write down the value of $\lim\limits_{n\to\infty} \alpha$. 1

(ii) Show that the sum S_n of the areas of the n rectangles is given by 4

$$S_n = \frac{\alpha^{5n} - 1}{1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4}.$$

(iii) Hence evaluate $\lim\limits_{n\to\infty} S_n$. 2

(iv) What is a simple geometrical interpretation of your answer to part (iii)? 1

Exam continues next page ...

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that $(\alpha^2 - 1)(\alpha^4 - 14\alpha^2 + 1) = \alpha^6 - 15\alpha^4 + 15\alpha^2 - 1$. 1

(ii) Use de Moivre's theorem to show that $\cot 6\theta = \dfrac{(\alpha^2 - 1)(\alpha^4 - 14\alpha^2 + 1)}{2\alpha(3\alpha^4 - 10\alpha^2 + 3)}$, where $\alpha = \cot\theta$ (and 6θ is not a multiple of π). 3

(iii) Hence show that $\cot^2 \dfrac{\pi}{12} + \cot^2 \dfrac{5\pi}{12} = 14$. 2

(iv) Deduce that $\cot \dfrac{\pi}{12} + \tan \dfrac{\pi}{12} = 4$. 2

(b) It can be shown that $\ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$, for $-1 < x < 1$.

(i) Assuming this result, show that $\ln\left(\dfrac{1+x}{1-x}\right) = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \cdots\right)$, where $-1 < x < 1$. 2

(ii) Hence show that $\ln 2 = 2\left(\dfrac{1}{3} + \dfrac{1}{3 \times 3^3} + \dfrac{1}{5 \times 3^5} + \dfrac{1}{7 \times 3^7} + \cdots\right)$. 2

(iii) Consider the approximation $\ln 2 \doteqdot 2\left(\dfrac{1}{3} + \dfrac{1}{3 \times 3^3} + \dfrac{1}{5 \times 3^5}\right)$. 3

Deduce that the error in this approximation is less than $\dfrac{1}{7 \times 2^2 \times 3^5}$.

END OF EXAMINATION

Question One

(a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$

$= 2\int e^{u}\,du$

$= 2e^{\sqrt{x}} + C$

(b) $\dfrac{2x-1}{(x-1)^2} = \dfrac{A}{(x-1)^2} + \dfrac{B}{x-1}$

$2x-1 = A + B(x-1)$

Subst $x=1$: $1 = A$

coeff of x: $2 = B$

(ii) $\displaystyle\int \frac{2x-1}{(x-1)^2}\,dx$

$= \displaystyle\int \frac{1}{(x-1)^2} + \frac{2}{x-1}\,dx$

$= -\dfrac{1}{x-1} + 2\ln|x-1| + C$

(c) $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$

$3+2x-x^2$
$= -(x^2-2x+1)+4$
$= 4-(x-1)^2$

$= \displaystyle\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx$

$= \sin^{-1}\left(\dfrac{x-1}{2}\right) + C$

(d) $\displaystyle\int x\sec^2 x\,dx$

$= \displaystyle\int x\,\frac{d}{dx}(\tan x)\,dx$

$= x\tan x - \displaystyle\int \tan x \cdot 1\,dx$

$= x\tan x - \displaystyle\int \frac{\sin x}{\cos x}\,dx$

$= x\tan x + \ln|\cos x| + C$

(e) $\displaystyle\int_0^2 \frac{1}{(4+x^2)^2}\,dx$ Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$

x	0	2
θ	0	$\frac{\pi}{4}$

$= \displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{(4+4\tan^2\theta)^2}\,2\sec^2\theta\,d\theta$

$= \dfrac{2}{4^2}\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$= \dfrac{1}{8}\displaystyle\int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$

$= \dfrac{1}{8}\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta$

$= \dfrac{1}{16}\left[\theta + \dfrac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$

$= \dfrac{1}{16}\left[\dfrac{\pi}{4} + \dfrac{1}{2}\right]$

$= \dfrac{\pi}{64} + \dfrac{1}{32}$

Question Two

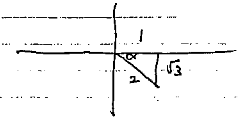
(a) $z = 3+i$ and $w = 2-i$

(i) $wz = (3+i)(2-i)$
$= 6-3i+2i+1$
$= 7-i$ ✓

(ii) $\bar{w}\,\bar{z} = \overline{wz} = 7+i$ ✓

(iii) $\dfrac{w}{z} = \dfrac{2-i}{3+i} \times \dfrac{3-i}{3-i}$
$= \dfrac{6-2i-3i-1}{10}$
$= \dfrac{5-5i}{10}$ ✓
$= \frac{1}{2} - \frac{1}{2}i$

(b) (i) $1-i\sqrt{3}$
$= 2\,\text{cis}\left(-\frac{\pi}{3}\right)$
✓ ✓

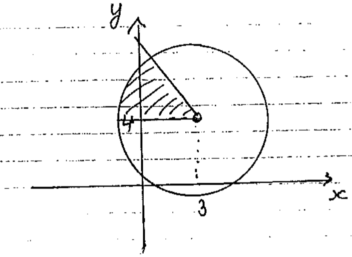


$\tan\alpha = \sqrt{3}$
$\alpha = \frac{\pi}{3}$

(ii) $(1-i\sqrt{3})^9 = \left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^9$
$= 512\,\text{cis}(-3\pi)$ ✓
$= 512\,\text{cis}\,\pi$
$= -512$ ✓

(c) $\dfrac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$
$= \dfrac{\cos 3\theta + i\sin 3\theta}{\cos(-2\theta) + i\sin(-2\theta)}$ ✓ $= \cos 5\theta + i\sin 5\theta$ ✓

(d)



✓ circle
✓ wedge
✓ correct part shaded

(e) $\text{Re}\left(z - \frac{1}{z}\right) = 0$

Let $z = x+iy$

$z - \frac{1}{z} = x+iy - \dfrac{1}{x-iy} \times \dfrac{x+iy}{x+iy}$

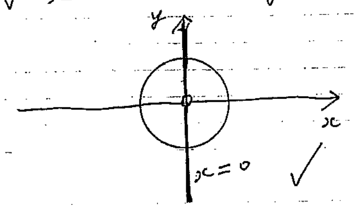
$= x+iy - \dfrac{x}{x^2+y^2} - \dfrac{y}{x^2+y^2}i$

$\text{Re}\left(z+\frac{1}{z}\right) = 0$

$x - \dfrac{x}{x^2+y^2} = 0$

$x\left(1 - \dfrac{1}{x^2+y^2}\right) = 0$ ✓

✓ $x = 0$ or $x^2+y^2 = 1$



✓

Note: $z = 0$ is excluded.

Question Three

(a) (i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$b^2 = a^2(1-e^2)$

$9 = 25(1-e^2)$

$\frac{9}{25} = 1-e^2$

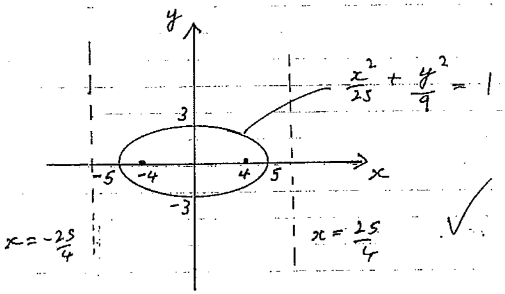
$e^2 = \frac{16}{25}$

$e = \frac{4}{5}$ ✓

(ii) Foci are $(ae, 0)$ $(-ae, 0)$
ie $(4, 0)$ and $(-4, 0)$ ✓

Directrices are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ ✓
ie $x = \frac{25}{4}$ and $x = \frac{-25}{4}$

(iii)



✓

(b) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Differentiate implicitly with respect to x

$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}$

$= -\frac{b^2}{a^2}\frac{x}{y}$

At $P(x_1, y_1)$ $m = -\frac{b^2}{a^2}\frac{x_1}{y_1}$ ✓

Gradient of the normal is $\frac{a^2}{b^2}\frac{y_1}{x_1}$

(ii) Eqn of the normal is

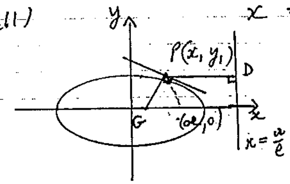
$y - y_1 = \frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1)$

$a^2 y_1 x = (a^2 - b^2) x_1 y_1$ ✓

$x = \frac{a^2 - b^2}{a^2} x_1$ ✓

$x = e^2 x_1$

(iii)



$PS = e\,PD$ (distance to directrix)

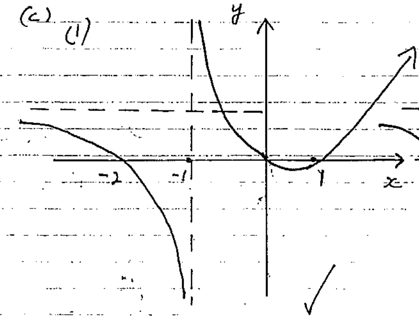
$PS = e\left(\frac{a}{e} - x_1\right)$ ✓

$= a - e x_1$

$SG = ae - e^2 x_1$ ✓

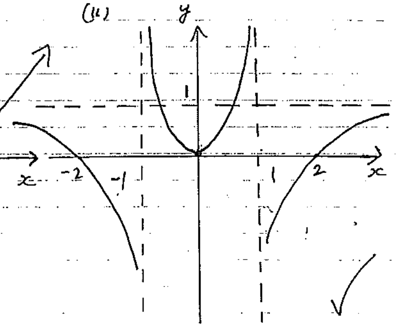
So $SG = e\,PS$

(c) (i)



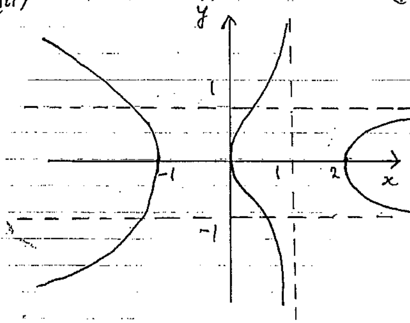
✓

(ii)

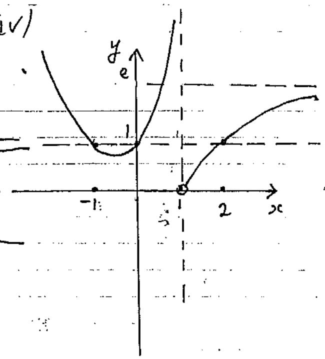


✓

(iii)



(iv)



✓ shape ✓

✓ vertical tangent on x axis. ✓

Question Four

(a) $y = e^{-x}(1-x)$

(i) $y' = e^{-x} \cdot -1 + (1-x) \cdot -e^{-x}$

$= e^{-x}(-1-1+x)$ ✓

$= e^{-x}(-2+x)$

Stationary pt where $y' = 0$

$x = 2, \; y = -\frac{1}{e^2}$ ✓

Table of values for y'

x	1	2	3
y'	$-\frac{1}{e}$	0	$\frac{1}{e^3}$

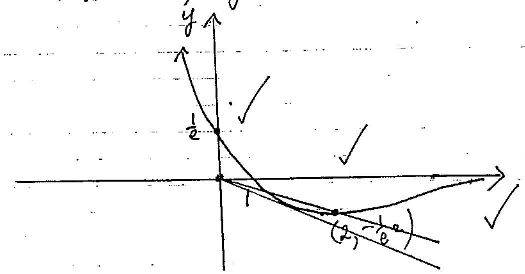
Minimum stat pt at $(2, -\frac{1}{e^2})$

(iii) When $x = 0$, $y = \frac{1}{e}$

x intercept at $x = 1$

As $x \to \infty$, $y \to 0^-$

$x \to -\infty$, $y \to \infty$



✓ ✓ ✓

(iv) gradient of the tangent through the origin < gradient of the line through origin and the minimum pt.

So $m < \dfrac{-\frac{1}{e^2}}{2}$

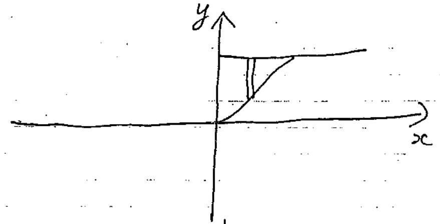
$\therefore \; m < -\dfrac{1}{2e^2}$

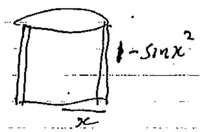
(b) (i)

$\sin x^2 = 1$

$x^2 = \dfrac{\pi}{2}$

$x = \sqrt{\dfrac{\pi}{2}}$





$V = \int_0^b 2\pi x y \, dx$

$V = 2\pi \int_0^{\sqrt{\frac{\pi}{2}}} x(1-\sin x^2)\, dx$

$= 2\pi \int_0^{\sqrt{\frac{\pi}{2}}} x - x \sin x^2 \, dx$

$= 2\pi \left[\dfrac{x^2}{2} - \dfrac{1}{2}\cos x^2\right]_0^{\sqrt{\frac{\pi}{2}}}$

$= 2\pi\left(\left(\dfrac{\pi}{4} - 0\right) - \left(0 + \dfrac{1}{2}\right)\right)$

$= \left(\dfrac{\pi^2}{2} - \pi\right)$ units^3

(c) $P(z) = z^3 + az^2 + bz + c$

(i) ki is a root

$\therefore -ki$ is a root since the coeffs are real.

Let α be the third root

Σ roots $= ki - ki + \alpha = -a$ — ①

$\alpha = -a$

$\therefore$ Third root is real

(ii) Σ roots two at a time $= \alpha ki - \alpha ki + k^2 = b$

$k^2 = b$ — ②

product of the roots $= ki \times -ki \times \alpha = -c$

$k^2\alpha = -c$ — ③

Subst ① and ② into ③

$b \times -a = -c$

$\therefore c = ab.$

Alternative solution:

$\alpha = -a$ is a root

substitute into the polynomial

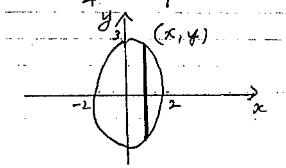
$(-a)^3 + a^3 - ab + c = 0$

$c = ab$

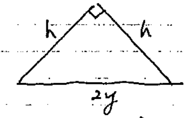
Question Five

(a) $9x^2 + 4y^2 = 36$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$



Cross-section:



$4y^2 = 2h^2$

$A(x) = \frac{1}{2}h^2$

$= y^2$

$= \frac{36-9x^2}{4}$

Volume $= \frac{1}{2}\int_0^2 36 - 9x^2 \, dx$

$= \frac{1}{2}\left[36x - 3x^3\right]_0^2$

$= \frac{1}{2}(72 - 24)$

$= 24 \text{ units}^3$

(b) (i) $x^2(1+x^2)^{n-1} = (1+x^2)^n - (1+x^2)^{n-1}$

RHS $= (1+x^2)^{n-1}(1+x^2-1)$

$= x^2(1+x^2)^{n-1} =$ LHS

(ii) $I_n = \int_0^1 (1+x^2)^n \frac{d}{dx}(x) \, dx$

$= \left[(1+x^2)^n \times x\right]_0^1 - \int x \times n(1+x^2)^{n-1} \times 2x \, dx$

$= 2^n - 2n\int x^2(1+x^2)^{n-1} \, dx$

$= 2^n - 2n\int (1+x^2)^n - (1+x^2)^{n-1} \, dx$ from (i)

$\therefore \quad I_n = 2^n - 2nI_n + 2nI_{n-1}$

$I_n(1+2n) = 2^n + 2nI_{n-1}$

$I_n = \frac{1}{2n+1}\left(2^n + 2nI_{n-1}\right)$

(c) $F(-3\tfrac{1}{2}) = -\left(\frac{1}{2} \times 3 \times \frac{3}{2}\right) = -\frac{9}{4}$

$F(0) = -\frac{9}{4} + \frac{1}{2} \times \frac{7}{2} \times 1$

$= -\frac{1}{2}$

$F(3\tfrac{1}{2}) = -\frac{1}{2} - \frac{1}{2} \times \frac{7}{2} \times 2$

$= -4$

Absolute minimum occurs at $x = 3\tfrac{1}{2}$

(d)(i) $\angle DAK = \alpha$ (alt segment th in circle AKD)

$\angle DBC = \alpha$ (angles standing on the same arc DC)

$CD = CB$ (sides opposite equal angles)

(ii) $\angle BAC = \alpha$ (angles standing on the same arc BC)

$\therefore \quad \angle KBC = \angle KAB$

$\therefore$ BC is a tangent to circle ABK by converse to the alt. segment Th

Question 6

(a) (i) +ve ↑ ↓g ↓$\frac{v}{10}$

$$\ddot{x} = -g - \frac{v}{10}$$

Note: unit mass

$$\ddot{x} = -\frac{v+10g}{10}$$

(ii)
$$\frac{dv}{dt} = -\frac{v+100}{10}$$
$$\frac{dt}{dv} = -\frac{10}{v+100}$$
$$t = -10\ln(v+100) + C$$

When $t=0$, $v=80$: $0 = -10\ln 180 + C$

$$C = 10\ln 180$$
$$t = 10\ln\left(\frac{180}{v+100}\right)$$

When $v=0$, $t = 10\ln 1.8$ seconds

(iii)
$$v\frac{dv}{dx} = -\frac{v+100}{10}$$
$$\frac{dx}{dv} = -\frac{10v}{v+100}$$
$$\frac{dx}{dv} = -10\,\frac{v+100-100}{v+100}$$
$$\frac{dx}{dv} = -10 + \frac{1000}{v+100}$$
$$x = -10v + 1000\ln(v+100) + C$$

When $x=0$, $v=80$: $C = 800 - 1000\ln 180$

$$x = -10v + 1000\ln(v+100) + 800 - 1000\ln 180$$

Max height H when $v=0$

$$H = 1000\ln 1000 + 800 - 1000\ln 180$$
$$= 800 - 1000\ln 1.8$$
$$\doteqdot 212 \text{ metres.}$$

(iv) +ve ↓ ↓g ↑$\frac{v}{10}$

$$\ddot{x} = 10 - \frac{v}{10}$$

The acceleration is positive.

We need to see how far a particle needs to fall in order to attain a speed of 80 m/sec

$$v\frac{dv}{dx} = \frac{100-v}{10}$$
$$\frac{dx}{dv} = \frac{10v}{100-v}$$
$$\frac{dx}{dv} = -10\,\frac{-v+100-100}{100-v}$$
$$\frac{dx}{dv} = -10 + \frac{1000}{100-v}$$
$$x = -10v - 1000\ln(100-v) + C$$

When $x=0$, $v=0$: $0 = 0 - 1000\ln 100 + C$

$$C = 1000\ln 100$$
$$x = -10v - 1000\ln(100-v) + 1000\ln 100$$

When $v = 80$, $x = -800 - 1000\ln 20 + 1000\ln 100$

$$\doteqdot 809 \text{ metres.}$$

So the particle hits the ground at a slower speed since it has only fallen 212 metres.

(b) (i) x is positive

$$x^n > 0$$
$$1 + x^n > 1$$
$$\frac{1}{1+x^n} < 1$$

(ii)
$$\int_0^1 \frac{1}{1+x^n}\,dx < \int_0^1 1\,dx$$
$$\therefore \int \frac{1}{1+x^n}\,dx < 1$$

For $0 < x < 1$,
$$x^n < x^2$$
$$1+x^n < 1+x^2$$
$$\frac{1}{1+x^n} > \frac{1}{1+x^2}$$
$$\therefore \int_0^1 \frac{1}{1+x^n}\,dx > \int_0^1 \frac{1}{1+x^2}\,dx$$
$$\int_0^1 \frac{1}{1+x^n}\,dx > \left[\tan^{-1}x\right]_0^1$$
$$\int_0^1 \frac{1}{1+x^n}\,dx > \frac{\pi}{4}$$

So $\frac{\pi}{4} < \int_0^1 \frac{1}{1+x^n}\,dx < 1$

Question Seven

(a) (i)
$$y = c^2 x^{-1}$$
$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

Gradient of normal $= \frac{c^2 t^2}{c^2} = t^2$

Equ of normal is
$$y - \frac{c}{t} = t^2(x - ct)$$
$$ty - c = t^3 x - ct^4$$
$$ty - t^3 x = c(1 - t^4)$$

(ii) The normal passes through $(0, K)$
$$Kt = c(1 - t^4)$$
$$ct^4 + Kt - c = 0$$

Let $f(t) = ct^4 + Kt - c$
$$f'(t) = 4ct^3 + K$$

Turning pts where $f'(t) = 0$

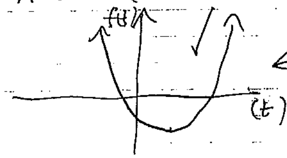
Note: $t = \sqrt[3]{\frac{K}{4c}} \times \sqrt[3]{\ }$

$$t^3 = \frac{-K}{4c}$$

There exists only one turning point.

Also $f(0) = -c$ and as $x \to \pm\infty$, $y \to \infty$

Graph loooks like.



So there exist 2 real roots and hence two normals from $(0, K)$

(iii) Let (h, k) be the arbitrary point.

So $+k - c = t^3 h - ct^4$

$ct^4 - ht^3 + kt - c = 0$

This a polynomial of degree 4 }
It has at most 4 real roots }

So there can be at most 4 normals from an arbitrary point.

(b) (i) $\lim_{n\to\infty} \sqrt[n]{2} = 1$

(ii) $S_n = (\alpha-1)\times 1 + (\alpha^2-\alpha)\alpha^4 + (\alpha^3-\alpha^2)\alpha^8 + \cdots + (\alpha^n - \alpha^{n-1})\alpha^{4n-4}$

$= (\alpha-1)(1 + \alpha^5 + \alpha^{10} + \alpha^{15} + \cdots \alpha^{5n-5})$

$= (\alpha-1)\dfrac{a(r^n-1)}{r-1}$

$= \dfrac{(\alpha-1)(\alpha^{5n}-1)}{\alpha^5-1}$

$= \dfrac{(\alpha-1)(\alpha^{5n}-1)}{(\alpha-1)(\alpha^4+\alpha^3+\alpha^2+\alpha+1)}$

$= \dfrac{\alpha^{5n}-1}{\alpha^4+\alpha^3+\alpha^2+\alpha+1}$ Note: $\alpha^n = 2$

(iii) $\lim S_n = \lim_{n\to\infty} \dfrac{(\alpha^n)^5 - 1}{\alpha^4+\alpha^3+\alpha^2+\alpha+1}$

$= \dfrac{2^5 - 1}{1+1+1+1+1} = \dfrac{31}{5}$

(iv) We are finding the area under the curve.

Question Eight

(a) (i) $(\alpha^2-1)(\alpha^4 - 14\alpha^2 + 1) = \alpha^6 - 15\alpha^4 + 15\alpha^2 - 1$

LHS $= \alpha^2(\alpha^4 - 14\alpha^2 + 1) - (\alpha^4 - 14\alpha^2 + 1)$

$= \alpha^6 - 14\alpha^4 + \alpha^2 - \alpha^4 + 14\alpha^2 - 1$

$= \alpha^6 - 15\alpha^4 + 15\alpha^2 + 1$

$=$ RHS

(ii) $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

$\cos 6\theta + i\sin 6\theta = \cos^6\theta + 6\cos^5\theta\sin\theta\, i + 15\cos^4\theta\sin^2\theta \times -1 + 20\cos^3\theta\sin^3\theta(-i) + 15\cos^2\theta\sin^4\theta + 6\cos\theta\sin^5\theta\, i - \sin^6\theta$

Now $\cot 6\theta = \dfrac{\cos 6\theta}{\sin 6\theta}$

Equating the real and imaginary parts

$\cot 6\theta = \dfrac{\cos^6\theta - 15\cos^4\theta + 15\cot^2\theta - 1}{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}$

Divide top and bottom by $\sin^6\theta$:

$\cot 6\theta = \dfrac{\alpha^6 - 15\alpha^4 - 15\alpha^2 - 1}{6\alpha^5 - 20\alpha^3 + 6\alpha}$

$= \dfrac{(\alpha^2-1)(\alpha^4 - 14\alpha^2 + 1)}{2\alpha(3\alpha^4 - 10\alpha^2 + 1}$ from (i)

(iii) $\cot 6\theta = 0$ has solutions $\frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}$

Now $\theta = \frac{3\pi}{12} = \frac{\pi}{4}$
and $\theta = \frac{9\pi}{12} = \frac{3\pi}{4}$ are the solutions to $\alpha^2 - 1 = 0$

The other solutions correspond to

$$\alpha^4 - 14\alpha^2 + 1 = 0$$

Consider this as a quadratic in $\cot^2\theta$

Its solutions are $\cot^2\frac{\pi}{12}$ and $\cot^2\frac{5\pi}{12}$

$$\Sigma \text{ roots} = \cot^2\frac{\pi}{12} + \cot^2\frac{5\pi}{12} = 14 .$$

(iv) $$\left(\cot\frac{\pi}{12} + \cot\frac{5\pi}{12}\right)^2 = \cot^2\frac{\pi}{12} + \cot^2\frac{5\pi}{12} - 2\cot\frac{\pi}{12}\cot\frac{5\pi}{12}$$

$$\left(\cot\frac{\pi}{12} + \cot\frac{5\pi}{12}\right)^2 = 14 - 2 \text{ product roots}$$
$$= 14 - 2(-1)$$
$$= 16$$

$$\cot\frac{\pi}{12} + \cot\frac{5\pi}{12} = \pm 4$$

$$\underset{\text{(complement)}}{\cot\frac{\pi}{12} + \tan\frac{\pi}{12}} = 4 \quad \text{since } \cot\frac{\pi}{12} + \tan\frac{\pi}{12} > 0$$

(b) (i) $\ln\left(\frac{1+x}{1-x}\right)$

$$= \ln(1+x) - \ln(1+(-x))$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$
$$- \left(-x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \dots\right)$$
$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right)$$

(ii) $$\frac{1+x}{1-x} = 2$$
$$1 + x = 2 - 2x$$
$$x = \tfrac{1}{3}$$

$$\ln 2 = \ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$$
$$= 2\left(\frac{1}{3} + \frac{1}{3\times 3^3} + \frac{1}{5\times 3^5} + \frac{1}{7\times 3^7} + \dots\right)$$

(iii) Error (E) $= 2\left(\frac{1}{7\times 3^7} + \frac{1}{9\times 3^9} + \frac{1}{11\times 3^{11}} + \dots\right)$

$$E < 2\left(\frac{1}{7\times 3^7} + \frac{1}{7\times 3^9} + \frac{1}{7\times 3^{11}} + \dots\right)$$
$$E \leqslant \frac{2}{7}\left(\frac{1}{3^7} + \frac{1}{3^9} + \frac{1}{3^{11}} + \dots\right)$$
$$E \leqslant \frac{2}{7} \cdot \frac{a}{1-r}$$
$$E \leqslant \frac{2}{7} \frac{\frac{1}{3^7}}{1-\frac{1}{3^2}}$$
$$E \leqslant \frac{2}{7} \frac{\frac{1}{3^7}}{\frac{8}{9}}$$
$$E \leqslant \frac{2}{7} \times \frac{1}{3^7} \times \frac{3^2}{2^3}$$
$$E \leqslant \frac{1}{2^3 \times 3^5 \times 7}$$

So Error $< \frac{1}{7\times 2^3 \times 3^5}$