Sydney Grammar School
Mathematics Department
Trial Examinations 2007

FORM VI

MATHEMATICS

Examination date

Wednesday 1st August 2007

Time allowed

3 hours (plus 5 minutes reading time)

Instructions

All ten questions may be attempted.
All ten questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the ten questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.
Bundle the separate sheet with Question 10.

Checklist

Folded A3 booklets: 10 per boy. A total of 1250 booklets should be sufficient.
Candidature: 93 boys.

Examiner

SJE/LYL

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\sqrt[3]{43{\cdot}1 - 16{\cdot}98}$ correct to one decimal place. $\boxed{2}$

(b) Solve $x^2 = 3x$. $\boxed{2}$

(c) Solve $|x+1| = 5$. $\boxed{2}$

(d) Differentiate $3x + \cos 2x$. $\boxed{2}$

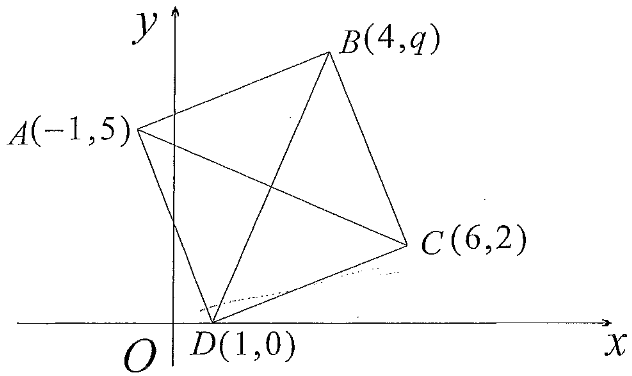
(e) Write down the exact value of $\tan \frac{3\pi}{4}$. $\boxed{1}$

(f) Find a primitive function of $\dfrac{5}{x}$. $\boxed{1}$

(g) Find a and b if $a + b\sqrt{2} = (3+\sqrt{2})(2-\sqrt{2})$. $\boxed{2}$

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above $ABCD$ is a square.

(i) Find the gradient of AC. $\boxed{1}$

(ii) Show that the equation of BD is $7x - 3y - 7 = 0$. $\boxed{2}$

(iii) Find q, the y-coordinate of B. $\boxed{1}$

(iv) Find the length of AC. $\boxed{1}$

(v) Hence, or otherwise, find the area of $ABCD$. $\boxed{1}$

Exam continues next page ...

(b) (i) Evaluate $\displaystyle\sum_{n=2}^{4} \frac{n}{n+1}$. **1**

(ii) Evaluate $\displaystyle\int_1^2 e\,dx$. **1**

(c) Find the limiting sum of the geometric series $500 + 100 + 20 + 4 + \cdots$. **2**

(d) Find $\displaystyle\int \frac{x}{2x^2+3}\,dx$. **2**

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate with respect to x:

(i) e^{x^2-9} **1**

(ii) $x^2 \tan 5x$ **2**

(b) Consider the parabola $(x-1)^2 = -6(y+4)$.

(i) Write down coordinates of the vertex. **1**

(ii) Find the equation of the directrix. **2**

(c) Evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{3\pi}{2}} 2\cos x\,dx$, leaving your answer as an exact value. **2**

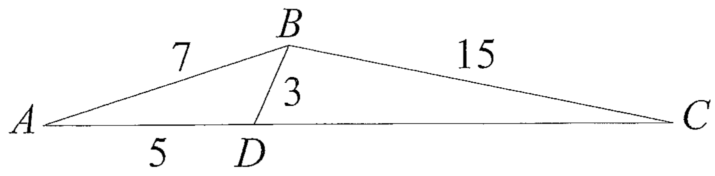
(d) Find the sum of the arithmetic series $1 + 4 + 7 + \cdots + 226$. **2**

(e) Find k such that $\displaystyle\int_1^k \frac{dx}{x} = 2$. **2**

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, triangle ABC has dimensions $AB = 7$ cm and $BC = 15$ cm. The point D lies on AC such that $AD = 5$ cm and $BD = 3$ cm.

(i) Use the cosine rule to show that $\angle ADB = 120°$. **2**

(ii) Show that $\angle BCD = 10°$ (rounded to the nearest degree). **2**

(iii) Find the length of DC, correct to the nearest millimetre. **2**

(b) The roots of the quadratic equation $px^2 - x + q = 0$ are -1 and 3. Find p and q. **3**

(c) Find the equation of the normal to the curve $y = (2-x)^3$ at the point where $x = 0$. **3**

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) A particle moves in a straight line. At time t seconds, its displacement x metres from the origin is given by $x = 3\cos\frac{t}{2}$, where $0 \le t \le 4\pi$.

(i) Sketch the graph of x as a function of t. **2**

(ii) Find the times at which the particle is at rest. **2**

(iii) What is the particle's initial displacement and acceleration? **3**

(iv) Find the total distance travelled by the particle. **1**

(b) Consider the function $f(x) = \dfrac{x^2}{1+x^2}$.

(i) Show that $f''(x) = \dfrac{2(1-3x^2)}{(1+x^2)^3}$ **3**

(ii) For what values of x is the function concave up? **1**

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. Marks

(a) The table below shows the value of a function $f(x)$ for three values of x. [2]

x	3	4	5
$f(x)$	$\sqrt{7}$	$\sqrt{14}$	$\sqrt{23}$

Use the trapezoidal rule with the three given function values to find an approximation of $\int_3^5 f(x)\,dx$. Give your answer correct to one decimal place.

(b) A ball is rolled up an inclined plane and is subject to an acceleration of $a = -6\,\text{m/s}^2$. Initially the ball has a velocity of $v = 12\,\text{m/s}$ and its displacement, measured from the bottom of the plane, is 36 m.

(i) Show that the velocity function is $v = 12 - 6t$. [1]

(ii) Find the displacement as a function of time. [2]

(iii) When does the ball reach the bottom of the plane and what is its speed then? [2]

(c) The fruit bat population in the Sydney Botanical Gardens has been increasing according to the equation $P = Ae^{kt}$, where A and k are constants. On 1st April 2005 there were 4800 bats, and by 1st April 2007 there were 10 800.

(i) Show that $k = \log_e \frac{3}{2}$. [2]

(ii) If the trend continues without any intervention, how many bats will inhabit the gardens by 1st April 2010? [1]

(iii) During what year is the bat population increasing at a rate of 6500 bats per year? [2]

Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. Marks

(a) Find the area between the curves $y = x^2$ and $x = y^2$. [3]

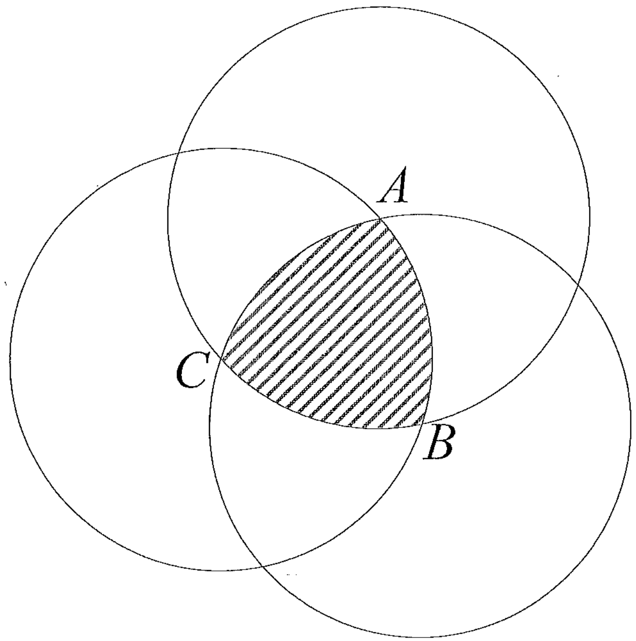
(b) Alex decides to invest \$30 000 into an investment fund offering 9% p.a. interest compounded monthly. How many months will it be before his money has doubled? Give your answer correct to the nearest month. [3]

(c) Gabriel's Horn is formed by rotating the area enclosed by the curve $y = \frac{1}{x}$ and the x-axis, between $x = 1$ and $x = a$, around the x-axis.

(i) Find the volume of the horn when $a = 5$. [2]

(ii) Find the limiting value of the volume as a gets larger. [1]

(d)



In the diagram above, ABC is a Reuleaux Triangle. Its sides are equal arcs of congruent circles centred at A, B and C. The radius of each circle is 12 cm. Find:

(i) the perimeter of the Reuleaux Triangle, [1]

(ii) the exact area of the Reuleaux Triangle. [2]

Exam continues next page ...

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) Karen borrows \$15 000 from the bank. The loan plus interest and charges are to be repaid at the end of each month in equal monthly instalments of $\$M$ over 5 years. Interest is charged at 6% p.a. and is calculated on the balance owing at the beginning of each month. Furthermore, at the end of each month a bank charge of \$15 is added to the account.
Let A_n be the amount owing after n months.

(i) Write down expressions for A_1 and A_2 and show that the amount owing after three months is given by $\boxed{3}$

$$A_3 = 15\,000 \times 1{\cdot}005^3 - (M - 15)(1 + 1{\cdot}005 + 1{\cdot}005^2).$$

(ii) Hence write an expression of A_n. $\boxed{1}$

(iii) Find the monthly instalment, correct to the nearest cent. $\boxed{3}$

(b) The line $x - 4y + 2 = 0$ is a tangent to the parabola $x = Ay^2$, where A is a constant.

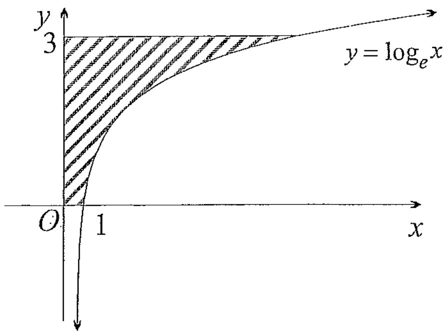
(i) Form a quadratic equation and hence show that $A = 2$. $\boxed{2}$

(ii) Draw a neat sketch of the parabola and the tangent, showing the point of contact. $\boxed{3}$

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that $\dfrac{d}{dx}(x \ln x - x) = \ln x$. $\boxed{1}$

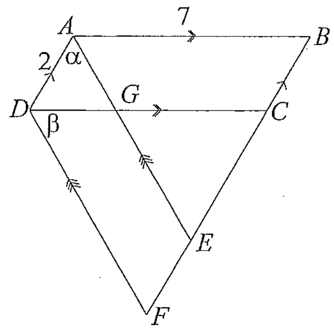
(ii)



NOT TO SCALE

The shaded region in the diagram above is bounded by the curve $y = \log_e x$, the line $y = 3$ and the cordinate axes. Using the result in part (i), or otherwise, find the exact area of the region. $\boxed{2}$

(b)



In the diagram above, $ABCD$ and $AEFD$ are parallelograms. AE bisects $\angle DAB$ and DC bisects $\angle FDA$. Let $\angle DAG = \alpha$ and $\angle FDG = \beta$.

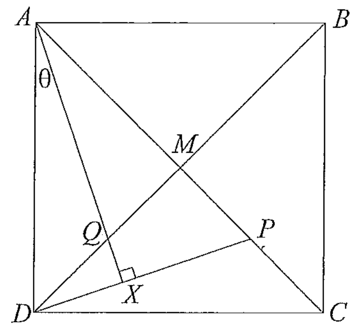
(i) Show that $\triangle AGD$ is equilateral, giving reasons. $\boxed{3}$

(ii) If $AD = 2$ units and $AB = 7$ units, show that the area of the the trapezium $ABFD$ is $\dfrac{77\sqrt{3}}{4}$ square units. $\boxed{3}$

(c) Show that if $y = \dfrac{e^x + e^{-x}}{2}$ then $y'' = \sqrt{1 + (y')^2}$. $\boxed{3}$

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a)



The square $ABCD$ is shown above. The diagonals AC and BD intersect at M. P is a point on the diagonal AC between M and C, and P is joined to D. The point X is chosen on DP so that $AX \perp DP$, and AX intersects the diagonal DB at Q. Let $\angle DAQ = \theta$.

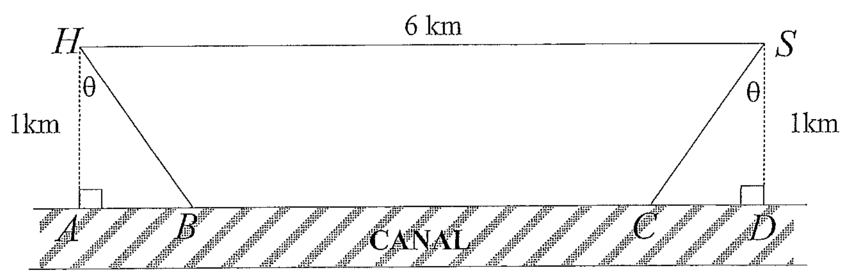
The diagram has been reproduced on a separate sheet which can be used for your solution to this question. Insert this sheet with the rest of Question 10.

(i) Show that $\angle PDC = \theta$. $\boxed{1}$

(ii) Hence show that $\triangle ADQ \equiv \triangle DCP$. $\boxed{3}$

(iii) Deduce that $\triangle DXQ \parallel\!| \triangle AMQ$. $\boxed{2}$

(b)



The diagram above shows that the distance between a boy's home H and his school S is 6 km. A canal $ABCD$ is 1 km from both his home and school. In winter the canal is frozen, so he take an alternate route $HBCS$, walking HB, skating BC and walking CS. His walking speed is 4 km/h and his skating speed is 12 km/h. Let $\angle AHB = \angle DSC = \theta$.

(i) Show that the time taken for this alternate route is $T = \dfrac{1}{2\cos\theta} + \dfrac{1}{2} - \dfrac{\tan\theta}{6}$. $\boxed{2}$

(ii) Find, to the nearest minute, the value of θ which minimises the time taken for the journey to school. $\boxed{4}$

END OF EXAMINATION

Q1

a) $\sqrt[3]{26 \cdot 12} \doteq 2 \cdot 967$ (2.9, 296, etc) 1 mark
$\doteq 3 \cdot 0$ (1dp) ✓✓

b) $x^2 = 3x$
$x^2 - 3x = 0$
$x(x-3) = 0$
$x = 0$ ✓ $x - 3 = 0$
$x = 3$ ✓

c) $|x+1| = 5$
$x+1 = 5$ or $x+1 = -5$
$x = 4$ ✓ or $x = -6$ ✓

d) $3 - 2\sin 2x$ ✓ ✓

e) $\tan \frac{3\pi}{4} = -1$ ✓

f) $5\log_e x + c$ ✓

g) $(3+\sqrt{2})(2-\sqrt{2})$
$= 6 - 3\sqrt{2} + 2\sqrt{2} - 2$ ✓
$= 4 - \sqrt{2}$
$a = 4 \quad b = -1$ ✓

Q2

a) i) gradient $AC = \frac{2-5}{6--1}$
$= -\frac{3}{7}$ ✓

ii) gradient of $BD = \frac{7}{3}$ ✓

equation of BD
$\frac{y-0}{x-1} = \frac{7}{3}$ ✓
$y = \frac{7(x-1)}{3}$
$7x - 3y - 7 = 0$

iii) Sub $x = 4$ into $7x - 3y - 7 = 0$
$28 - 3y - 7 = 0$
$21 = 3y$
$y = 7$ ✓

iv) $AC^2 = (6--1)^2 + (2-5)^2$
$= 49 + 9$
$= 58$
$AC = \sqrt{58}$ units ✓

v) Area $ABCD = \frac{1}{2} \times (\sqrt{58})^2$ ✓
$= 29$ square units

b) i) $\sum_{n=2}^{4} \frac{n}{n+1} = \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$
$= \frac{133}{60}$ or $2\frac{13}{60}$ ✓

ii) $\int_1^2 e\,dx = [ex]_1^2$
$= 2e - e$
$= e$ ✓

c) $a = 500$, $r = \frac{1}{5}$ } either ✓
$S_\infty = \frac{a}{1-r}$
$= \frac{500}{1-\frac{1}{5}}$
$= 625$ ✓

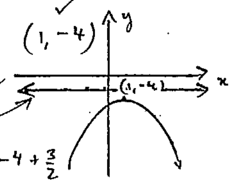
d) $\int \frac{x}{2x^2+3}\,dx = \frac{1}{4}\int \frac{4x}{2x^2+3}\,dx$ ✓
$= \frac{1}{4}\log_e(2x^2+3) + C$ ✓

Question 3

a) (i) $\frac{d}{dx}\left(e^{x^2-9}\right) = 2xe^{x^2-9}$ ✓

(ii) $\frac{d}{dx}\left(x^2\tan 5x\right) = 2x\tan 5x + x^2 \cdot 5\sec^2 5x$ ✓
$= 2x\tan 5x + 5x^2\sec^2 5x$ ✓

b) $(x-1)^2 = -6(y+4)$ ✓
(i) Vertex is $(1,-4)$
(ii) $4a = 6$
$a = \frac{3}{2}$ ✓



directrix is $y = -4 + \frac{3}{2}$
$y = -\frac{5}{2}$ ✓

c) $\int_{\pi/3}^{3\pi/2} 2\cos x\,dx = \left[2\sin x\right]_{\pi/3}^{3\pi/2}$ ✓
$= -2 - 2\frac{\sqrt{3}}{2}$
$= -2 - \sqrt{3}$ ✓

d) $1 + 4 + 7 + \ldots + 226$
$a = 1$
$d = 3$
$T_n = a + (n-1)d$
So $226 = 1 + (n-1)3$
$n - 1 = 75$ ✓
$n = 76$
Sum of series is $= \frac{n}{2}(a + l)$
$= \frac{76}{2}(1 + 226)$
$= 8626$ ✓

e) $\int_1^k \frac{dx}{x} = 2$
$\left[\log_e x\right]_1^k = 2$
$\log_e k - \log_e 1 = 2$ ✓
$\log_e k = 2$
$\therefore k = e^2$ ✓

a) i) $\cos\angle ADB = \dfrac{5^2+3^2-7^2}{2\times5\times3}$ ✓✓

$\angle ADB = 120^\circ$

ii) $\dfrac{\sin BCD}{3} = \dfrac{\sin 60}{15}$ ✓

$\sin BCD = \dfrac{3\times\sin 60^\circ}{15}$ ✓

$BCD \doteqdot 10^\circ$

(170° impossible as $170^\circ + 60 > 180^\circ$)

iii) $\dfrac{DC}{\sin 110^\circ} = \dfrac{3}{\sin 10^\circ}$ ✓

$DC = \dfrac{3\sin 110^\circ}{\sin 10^\circ}$

$\doteqdot 16.2$ cm ✓

b) $-3 = \dfrac{q}{p}$ ✓

$2 = \dfrac{1}{p}$

$p = \dfrac{1}{2}$ ✓

$q = -\dfrac{3}{2}$ ✓

c) $y = (2-x)^3$

$y' = -3(2-x)^2$ ✓

$m = -12$

normal $m = \dfrac{1}{12}$ ✓

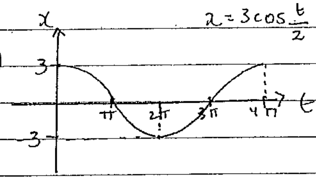
when $x = 0$ $y = 8$

$y - 8 = \dfrac{1}{12}(x)$

$12y - 96 = x$

$x - 12y + 96 = 0$ ✓

5. a) i)



✓ correct shape, amplitude or period.

✓ remaining information correct and presented on the graph

ii) At rest when $t = 0$ ✓ one correct

$t = 2\pi$ or $t = 4\pi$ ✓ all three correct

iii) Initial displacement $x = 3$m ✓

$x = 3\cos\dfrac{t}{2}$

$\dot{x} = -\dfrac{3}{2}\sin\dfrac{t}{2}$

$\ddot{x} = -\dfrac{3}{4}\cos\dfrac{t}{2}$ ✓ ✓

$\therefore$ at $t = 0$ acceleration $= -\dfrac{3}{4}\ ms^{-2}$

iv) Total distance travelled = 12m ✓

b) $f(x) = \dfrac{x^2}{1+x^2}$

$f'(x) = \dfrac{(1+x^2)2x - x^2(2x)}{(1+x^2)^2}$ ✓

$= \dfrac{2x(1+x^2-x^2)}{(1+x^2)^2}$

$= \dfrac{2x}{(1+x^2)^2}$

$f''(x) = \dfrac{(1+x^2)^2 . 2 - 2x . 2 . 2x(1+x^2)}{(1+x^2)^4}$ ✓

$= \dfrac{2(1+x^2)\left[(1+x^2) - 4x^2\right]}{(1+x^2)^{\cancel{4}\,3}}$

$= \dfrac{2(1-3x^2)}{(1+x^2)^3}$ ✓

ii) Concave up when $f''(x) > 0$

$1 - 3x^2 > 0$

$x^2 < \dfrac{1}{3}$ ✓

$-\dfrac{1}{\sqrt3} < x < \dfrac{1}{\sqrt3}$

6

) $\displaystyle\int_a^b f(x)\,dx \doteqdot \frac{b-a}{2}\left(f(a)+f(b)\right)$

$\displaystyle\int_3^5 f(x)\,dx = \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx$

$\doteqdot \frac{1}{2}\left(f(3)+f(4)\right) + \frac{1}{2}\left(f(4)+f(5)\right)$

$\doteqdot \frac{1}{2}\left(\sqrt7 + 2\sqrt{14} + \sqrt{23}\right)$ ✓

$\doteqdot 7.5$ (1dp) ✓

) $a = -6$

) $\dfrac{dv}{dt} = -6$

$v = -6t + c$

$t = 0$ $v = 12$

$c = 12$ ✓

$v = -6t + 12$

) $\dfrac{dx}{dt} = -6t + 12$

$x = \dfrac{-6t^2}{2} + 12t + k$ ✓

$t = 0$ $x = 36$

$k = 36$ ✓

$x = -3t^2 + 12t + 36$

) $x = 0$

$-3t^2 + 12t + 36 = 0$

$3t^2 - 12t - 36 = 0$

$t^2 - 4t - 12 = 0$

$(t+2)(t-6) = 0$

$t \neq -2$ $t = 6$ ✓

$t = 6$

$v = 12 - 36$

$= -24$ m/s

$\therefore$ speed $= 24$ m/s ✓

c) i) $P = Ae^{kt}$

$A = 4800$

$t = 2$ $P = 10\,800$

$10\,800 = 4800\,e^{2k}$ ✓

$\dfrac{108}{48} = e^{2k}$

$\dfrac{9}{4} = e^{2k}$

$\log\left(\dfrac{3}{2}\right)^2 = \log_e e^{2k}$ ✓ must show

$2\log\dfrac{3}{2} = 2k$

$k = \log\dfrac{3}{2}$

ii) $t = 5$

$P = 4800\,e^{k5}$

$= 36450$ ✓

iii) 6500 bats / year

$\dfrac{dP}{dt} = kP$

$6500 = kP$

$P = \dfrac{6500}{k}$

$P = Ae^{kt}$

$\dfrac{6500}{k} = 4800e^{kt}$ ✓

$\dfrac{65\cancel{00}}{48\cancel{00}k} = e^{kt}$

$\log_e\left(\dfrac{65}{48k}\right) = kt$

$t = \dfrac{\log_e\left(\frac{65}{48k}\right)}{k}$

$\doteqdot 2.97$

$\therefore$ During 2008 the bat population is 6500 bats per year. ✓

$t = 0$ 2005 (1st April)

$t = 1$ 2006 1st year

$t = 2$ 2007 2nd year

$t = 3$ 2008 3rd year

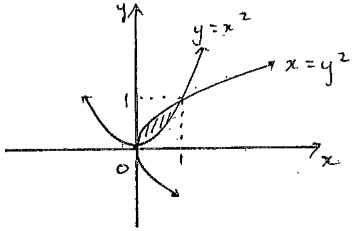
Question 7

(a) $\text{Area} = \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx$ ✓

$= \left[\frac{2}{3}x^{3/2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$ ✓

$= \frac{2}{3} - \frac{1}{3}$

$= \frac{1}{3}$ square unit. ✓



(b) $A = P(1+r)^n$

$60000 = 30000(1.0075)^n$ ✓

$1.0075^n = 2$

$n = \dfrac{\log_e 2}{\log_e 1.0075}$

$\doteqdot 92.77$ ✓

$\therefore$ His money has doubled after 93 months

$P = \$30\,000$

$r = 9\%$ p.a

$= 0.0075$ ✓

$A = \$60\,000$ (doubled)

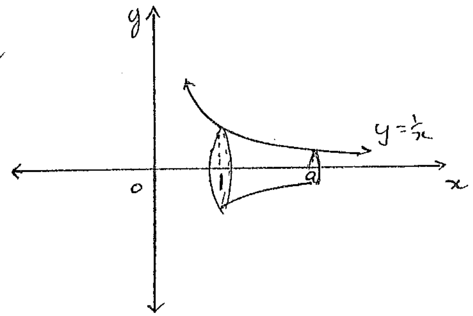
(c) (i) $V = \pi \int_1^5 \left(\frac{1}{x}\right)^2 dx$ ✓

$= \pi \int_1^5 x^{-2}\,dx$

$= \pi \left[-x^{-1}\right]_1^5$

$= \pi\left[-\frac{1}{5} - -1\right]$ ✓

$= \frac{4}{5}\pi$ cubic units



(ii) $V = \pi\left(-\frac{1}{a} - -1\right)$

$= \pi\left(1 - \frac{1}{a}\right)$

as $a \to \infty$, $V \to \pi$ ✓ $\therefore$ Limiting Value is π cubic units.

(d) (i) $\text{Perimeter} = 3 \times r\theta$

$= 3 \times 12 \frac{\pi}{3}$

$= 12\pi$ cm ✓

(ii) Area = Area Equilateral Triangle + 3 segments.

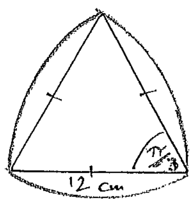
$= \frac{1}{2}r^2 \sin\theta + 3\left(\frac{1}{2}r^2(\theta - \sin\theta)\right)$ ✓

$= \frac{144}{2}\sin\frac{\pi}{3} + \frac{3}{2}\,144 \cdot \frac{\pi}{3} - \frac{3}{2}(144)\sin\frac{\pi}{3}$

$= 72\frac{\sqrt{3}}{2} + 72\pi - 3(72)\frac{\sqrt{3}}{2}$

$= 72\pi - 72\sqrt{3}$

$= 72(\pi - \sqrt{3})$ cm² ✓



Question 8

a) 6% p.a $= 0.005$ per month

(i) $A_1 = 15000(1.005) - M + 15$ ✓

$A_2 = 15000(1.005)^2 - M(1.005) + 15(1.005) - M + 15$ ✓

$A_3 = 15000(1.005)^3 - M(1.005)^2 - M(1.005) - M + 15(1.005)^2 + 15(1.005) + 15$

$= 15000(1.005)^3 - M(1.005^2 + 1.005 + 1) + 15(1.005^2 + 1.005 + 1)$

$= 15000(1.005)^3 - (M-15)(1.005^2 + 1.005 + 1)$ ✓

$= 15000(1.005)^3 - (M-15)(1 + 1.005 + 1.005^2)$ as required.

(ii) $A_n = 15000(1.005)^n - (M-15)(1 + 1.005 + 1.005^2 + \ldots + 1.005^{n-1})$ ✓

(iii) $A_{60} = 0$ ✓

So $(M-15)\Big[\underbrace{1 + 1.005 + 1.005^2 + \ldots + 1.005^{n-1}}_{\text{GP} \quad a=1,\ r=1.005}\Big] = 15000(1.005)^{60}$

$\therefore (M-15)\left[\dfrac{1.005^{60} - 1}{1.005 - 1}\right] = 15000(1.005)^{60}$ ✓

$M = \dfrac{15000(1.005)^{60} \times 0.005}{1.005^{60} - 1} + 15$

$= \$304.99$ ✓

b) (i) $x - 4y + 2 = 0$

$x = Ay^2$

$Ay^2 - 4y + 2 = 0$ ✓

For tangent $\Delta = 0$

So $(-4)^2 - 4A(2) = 0$

$16 - 8A = 0$ ✓

$A = 2$

(ii) At point of intersection

$2y^2 - 4y + 2 = 0$

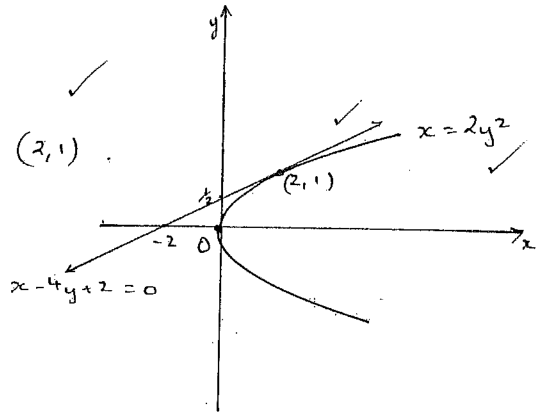
$y^2 - 2y + 1 = 0$

$(y-1)^2 = 0$

$y = 1$

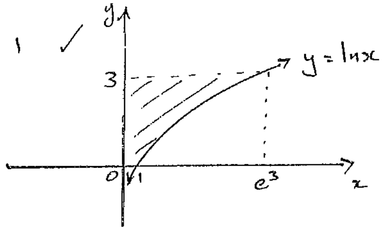
and $x = 2$ ✓

Coords. of point of intersection $(2,1)$.



Question 9

a) (i) $\frac{d}{dx}\left(x\ln x - x\right) = x\cdot\frac{1}{x} + \ln x - 1$ ✓
$= \ln x$



(ii) Area $= 3e^3 - \int_1^{e^3} \ln x \, dx$ ✓

$= 3e^3 - \left[x\ln x - x\right]_1^{e^3}$

$= 3e^3 - \left[(e^3\ln e^3 - e^3) - (0-1)\right]$

$= 3e^3 - \left[3e^3 - e^3 + 1\right]$ ✓

$= e^3 - 1$ square units.

Alternate approach:

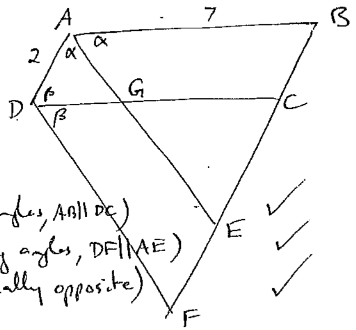
Area $= \int_0^3 x \, dy$

$= \int_0^3 e^y \, dy$

$= \left[e^y\right]_0^3$

$= e^3 - e^0$

$= e^3 - 1$ sq. units



b) i) $\angle GAB = \alpha$

$\angle ADG = \beta$

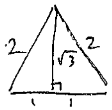
$\angle AGD = \alpha$ (alternate angles, AB ∥ DC) ✓

$\angle AGC = \beta$ (corresponding angles, DF ∥ AE) ✓

$\angle AGD = \angle CGE$ (vertically opposite) ✓

$\therefore \alpha = \beta$

$\therefore \triangle ADG$ is equilateral.



ii) Area = Area ABCD + $\triangle DGF$

$= 7\sqrt{3} + \frac{1}{2}7^2\sin 60°$ ✓✓

$= 7\sqrt{3} + \frac{49}{2}\cdot\frac{\sqrt{3}}{2}$

$= \frac{28\sqrt{3} + 49\sqrt{3}}{4}$

$= \frac{77\sqrt{3}}{4}$ sq. units ✓

Height of parallelogram ABCD: $\sqrt{3}$

Alternate Method 1

Area of trapezium $= \frac{1}{2}h(a+b)$

$= \frac{1}{2}\cdot\frac{7\sqrt{3}}{2}(2+9)$

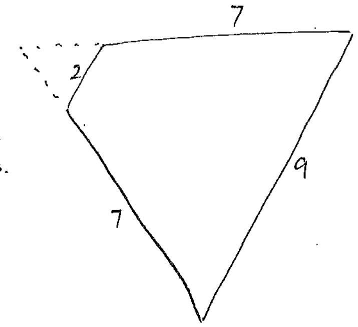
$= \frac{77\sqrt{3}}{4}$ sq. units

$h = \frac{7\sqrt{3}}{2}$

Question 9 (cont.)

Alternate Method 2

Consider area of equilateral triangle side lengths 9 units.



Area $= \frac{1}{2}9^2\sin 60° - \frac{1}{2}2^2\sin 60°$

$= \frac{81}{2}\cdot\frac{\sqrt{3}}{2} - \frac{4}{2}\cdot\frac{\sqrt{3}}{2}$

$= \frac{77\sqrt{3}}{4}$ sq. units.

(c) $y = \frac{e^x + e^{-x}}{2}$

$y' = \frac{1}{2}(e^x - e^{-x})$ ✓

$y'' = \frac{1}{2}(e^x + e^{-x})$

We need to show $y'' = \sqrt{1 + (y')^2}$

LHS $= \frac{1}{2}(e^x + e^{-x})$

RHS $= \sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2}$

$= \sqrt{1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})}$ ✓

$= \frac{1}{2}\sqrt{4 + e^{2x} - 2 + e^{-2x}}$

$= \frac{1}{2}\sqrt{e^{2x} + 2 + e^{-2x}}$

$= \frac{1}{2}\sqrt{(e^x + e^{-x})^2}$ ✓

$= \frac{1}{2}(e^x + e^{-x})$

$=$ LHS

Question 10

(a) (i) $\angle ADX = 180° - 90° - \theta$ (angle sum of $\triangle ADX$)

$= 90° - \theta$

$\therefore \angle PDC = 90° - (90° - \theta)$ (property of a square)

$= \theta$ as required.

(ii) In triangles ADQ and DCP

$AD = DC$ (sides of a square)

$\angle PDC = \angle QAD = \theta$ (from (i) above)

$\angle ADQ = \angle DCP = 45°$ (diagonals of a square bisect vertex)

$\therefore \triangle ADQ \equiv \triangle DCP$ (AAS)

(iii) In triangles DXQ and AMQ

$\angle AMQ = 90°$ (given)

$\angle DXQ = 90°$ (diagonals of a square intersect at right angles)

$\angle AQM = \angle DQX$ (vertically opposite angles)

$\therefore \triangle DXQ \parallel\!| \triangle AMQ$ (A.A.)

b) (i) In $\triangle HAB$, $\cos\theta = \frac{1}{HB}$ and $\tan\theta = \frac{AB}{1}$

$HB = \frac{1}{\cos\theta}$ $AB = \tan\theta$.

Distance walked $= 2HB$, $\therefore$ Time walked $= \frac{\frac{2}{\cos\theta}}{4}$

$= \frac{2}{\cos\theta}$ $= \frac{1}{2\cos\theta}$ hours

Distance skated $= 6 - 2\tan\theta$, $\therefore$ Time skated $= \frac{6-2\tan\theta}{12}$ hours

So $T = \frac{1}{2\cos\theta} + \frac{6-2\tan\theta}{12}$

$= \frac{1}{2\cos\theta} + \frac{1}{2} - \frac{\tan\theta}{6}$ as required.

(ii) $T = \frac{(\cos\theta)^{-1}}{2} + \frac{1}{2} - \frac{\tan\theta}{6}$

$\frac{dT}{d\theta} = \frac{-(-\sin\theta)(\cos\theta)^{-2}}{2} - \frac{\sec^2\theta}{6}$

$= \frac{\sin\theta}{2\cos^2\theta} - \frac{1}{6\cos^2\theta}$

$= \frac{1}{\cos^2\theta}(3\sin\theta - 1)$

For minimum $\frac{dT}{d\theta} = 0$ So $3\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{3}$

$\theta \doteqdot 19°28'$

and

θ	10°	19°28'	30°
$\frac{dT}{d\theta}$	-0.94	0	$\frac{1}{9}$

$\therefore \theta = 19°28'$ is a minimum.

Test boundaries: $\theta = 0$ $T = \frac{1}{4} + \frac{6}{12} + \frac{1}{4}$ $= 1$ hour

$\theta = 90°$ $T = \frac{6}{4}$ $= 1\frac{1}{2}$ hours

and $\theta = 19°28'$ $T = \frac{1}{2\cos 19°28'} + \frac{1}{2} - \frac{\tan 19°28'}{6}$

$\doteqdot 58.28$ minutes.