SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
TRIAL EXAMINATIONS 2008

FORM VI

MATHEMATICS

Examination date

Tuesday 5th August 2008

Time allowed

3 hours

Instructions

All ten questions may be attempted.
All ten questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the ten questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 10 per boy. A total of 1250 booklets should be sufficient.
Candidature: 93 boys.

Examiner

SJE

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) The density of hydrogen in a certain container is 0·000 083 75 g/cm^3. Write this number in scientific notation, correct to two significant figures. [2]

(b) State the period of the curve $y = 2\cos\frac{x}{2}$. [1]

(c) Differentiate $y = (x+3)^2$. [1]

(d) Find $\int (e^{2x} - 1)\,dx$. [2]

(e) Solve $|2x - 1| = 5$. [2]

(f) (i) Write down the equation of the locus of a point P that is 2 units from the point $A(1, -3)$. [1]

(ii) How many times does this locus cut the x-axis? [1]

(g) Find the values of x for which the geometric series $2 + 4x + 8x^2 + \cdots$ has a limiting sum. [2]

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

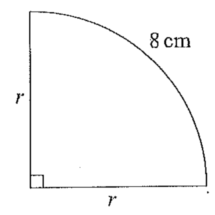
(a) Find rational numbers a and b such that $\dfrac{\sqrt{5}}{1+\sqrt{2}} = \sqrt{a} - \sqrt{b}$. [2]

(b) Differentiate:

(i) $y = \sin 3x$ [1]

(ii) $y = (e^x + 1)^2$ [2]

(c) [1]



The quadrant in the diagram above has an arc length of 8 cm. Find the exact value of the radius r.

(d) (i) Graph the curve $y = x(2-x)$, clearly showing all intercepts with the axes. [2]

(ii) Hence, or otherwise, solve $x(2-x) < 0$. [1]

(e) Find:

(i) $\displaystyle\int \frac{1}{\sqrt{x}}\,dx$ [1]

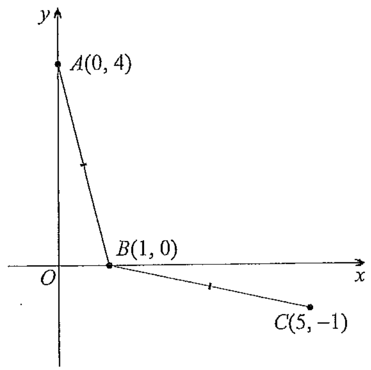
(ii) $\displaystyle\int_0^1 \frac{x^2}{x^3+1}\,dx$ [2]

Exam continues overleaf ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate $y = \dfrac{\log_e x}{x^2}$. [2]

(b)



In the diagram above, A is $(0, 4)$, B is $(1, 0)$ and C is $(5, -1)$. The intervals AB and BC are equal in length.

(i) Find the gradient of AC. [1]

(ii) Show that the equation of AC is $x + y - 4 = 0$. [1]

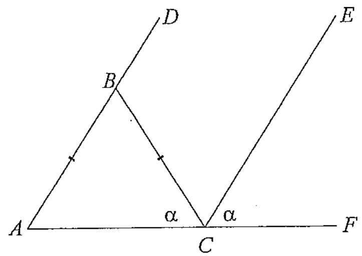
(iii) Find the perpendicular distance of B from AC. [2]

(iv) Show that $AC = 5\sqrt{2}$ units. [1]

(v) Given that the midpoint of AC is $(\frac{5}{2}, \frac{3}{2})$, find the coordinates of D so that $ABCD$ is a rhombus. [2]

(vi) Find the area of the rhombus. [1]

(c) [2]



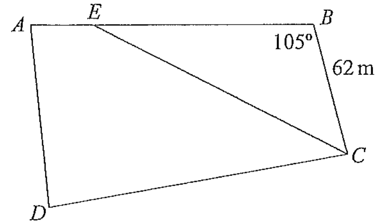
In the diagram above, $AB = BC$ and $\angle BCA = \angle ECF = \alpha$. Prove that $AD \| CE$.

Exam continues next page ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find the equation of the tangent to the curve $y = \tan x$ at the point $(\frac{\pi}{4}, 1)$. [3]

(b)



The diagram of a farmer's land $ABCD$ is drawn above. He wishes to split the land into two paddocks by building a fence from C to some point E on AB so that the triangular area EBC is $3000\,\text{m}^2$. He has measured $\angle B$ to be $105°$ and BC to be 62 metres.

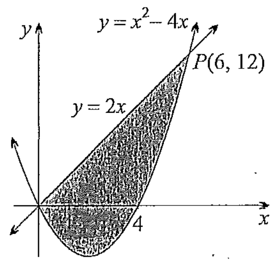
(i) Show that EB is 100 metres, correct to the nearest metre. [2]

(ii) Hence find the length of CE, correct to the nearest metre. [2]

(c) Use Simpson's rule with three function values to approximate $\displaystyle\int_1^3 \log_e x\,dx$. [3]

Write your approximation to two decimal places.

(d) [2]

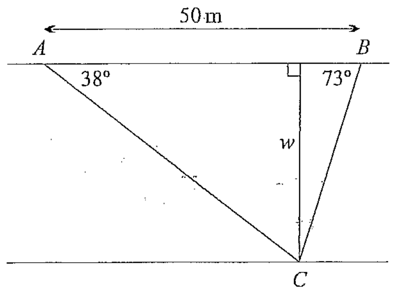


The graphs of $y = 2x$ and $y = x^2 - 4x$ are drawn above. They intersect at the origin and at the point $P(6, 12)$. Find the shaded area.

Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. Marks

(a) [3]



In the diagram above, two points A and B are 50 metres apart on the same side of a river. The point C is on the other side. A surveyor wants to determine the width w of the river. He measures $\angle BAC$ to be 38° and $\angle ABC$ to be 73°. Find the width of the river, correct to the nearest metre.

(b) Consider the function $y = 3x^4 - 4x^3 - 12x^2 + 10$.

(i) Find the coordinates of the stationary points and determine their nature. [5]

(ii) Show that the inflexion points occur where $x = \dfrac{1-\sqrt{7}}{3}$ and $x = \dfrac{1+\sqrt{7}}{3}$. [2]

(iii) Draw a neat sketch of the graph of the function indicating the above features and the y-intercept. Do not attempt to find the x-intercepts. [2]

QUESTION SIX (12 marks) Use a separate writing booklet. Marks

(a) If α and β are the roots of the quadratic equation $2x^2 - 5x + 1 = 0$, find the value of $\alpha^2 + \beta^2$. [3]

(b) A particle is moving on the x-axis. It starts from the origin O, and at time t seconds its velocity $v\ \text{ms}^{-1}$ is given by $v = 1 - 2\sin t$. Let $t = t_1$ and $t = t_2$ be the first two times that the particle comes to rest.

(i) Find t_1 and t_2. [2]

(ii) Sketch the velocity function for $0 \le t \le 2\pi$. [2]

(iii) Find the acceleration at $t = t_1$ and $t = t_2$. [2]

(iv) Find the displacement function. [2]

(v) Hence, or otherwise, find the distance travelled between $t = t_1$ and $t = t_2$. [1]

QUESTION SEVEN (12 marks) Use a separate writing booklet. Marks

(a) If a, b and c are consecutive terms of a geometric sequence, show that $\ln a$, $\ln b$ and $\ln c$ are consecutive terms of an arithmetic sequence. [2]

(b) A parabola in the coordinate plane is represented by the equation

$$x^2 - 10x - 16y - 7 = 0.$$

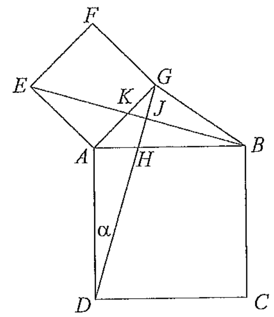
(i) By completing the square, find the coordinates of the vertex. [2]

(ii) Find the focal length. [1]

(iii) Find the equation of the directrix. [1]

(c) The sum of the first 8 terms of an arithmetic series is 88, and the sum of the first 18 terms is 558. Find the first three terms of the series. [3]

(d) [3]



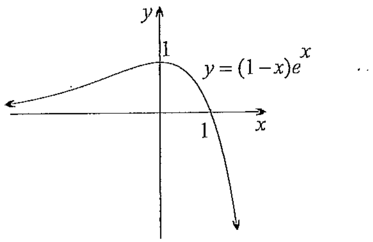
Two squares $ABCD$ and $AEFG$ are drawn above. Let $\angle ADG = \alpha$.

It is known that $\triangle ADG \equiv \triangle ABE$. Prove that $EB \perp DG$.

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) Show that $1+\cot^2\frac{\pi}{3}=\sec^2\frac{\pi}{6}$. **[2]**

(b)



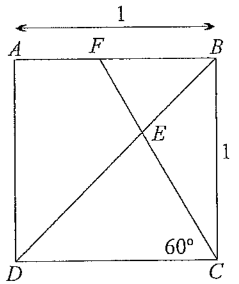
The diagram above shows the curve $y=(1-x)e^x$. There is a maximum turning point at $(0,1)$.

(i) Show that $y''=-e^x-xe^x$. **[2]**

(ii) Find the coordinates of any points of inflexion. **[2]**

(iii) For what values of c does the equation $(1-x)e^x=c$ have two solutions? **[1]**

(c)



A square $ABCD$ of side length 1 unit is shown above. The point F is drawn on AB such that $\angle DCF=60^\circ$. The diagonal DB intersects CF at E.

(i) Show that $\triangle DEC \,|||\, \triangle BEF$. **[3]**

(ii) Show that $FB=\dfrac{1}{\sqrt{3}}$. **[1]**

(iii) Hence, or otherwise, find the ratio **[1]**

$$\text{area}\,\triangle DEC : \text{area}\,\triangle BEF$$

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) The region between the curve $y=e^x+e^{-x}$ and the x-axis from $x=0$ to $x=1$ is rotated about the x-axis. Find the exact volume of the solid of revolution formed. **[3]**

(b) GenXYZ Home Loans is offering a special package for struggling first home buyers. The main details of this package, extracted from their brochure, are summarised below.

Stage	Term	Special Features	Interest Rate
Introductory Stage	0-2 years (2 years)	No monthly repayments	6% p.a. compounded monthly
Secondary Stage	2-10 years (8 years)	Monthly repayments commence. At the conclusion of this period, the amount owing must be reduced to the original size of the loan.	9% p.a. compounded monthly
Final Stage	Negotiable (not exceeding 20 years)	The size of the monthly repayment is determined by the borrower, provided that the loan is repaid within 20 years from the commencement of this Stage.	12% p.a. compounded monthly

Bernard and Esther have just borrowed \$500 000 for their first house, and they are willing to accept the terms of the package offered by GenXYZ Home Loans.

(i) Show that the amount owing at the end of the Introductory Stage is \$563 580. **[1]**

(ii) The size of the monthly repayment M required in the Secondary Stage can be calculated from the formula **[3]**

$$A_n=P(1+r)^n-M\Big(1+(1+r)+(1+r)^2+\ldots+(1+r)^{n-1}\Big)$$

where A_n is the amount owing after the nth repayment, P is the principal, and r is the relevant interest rate. (Do NOT show this.)

The principal for the Secondary Stage will be \$563 580. Find M so that the amount owing at the end of the Secondary Stage is \$500 000.

(iii) At the beginning of the Final Stage, Bernard and Esther calculate that they can now afford to repay \$6500 per month.

(α) Determine how many repayments of \$6500 it will take for the loan to be repaid in full. **[3]**

(β) The last monthly repayment of \$6500 is more than required. How much should be refunded to Bernard and Esther? **[2]**

QUESTION TEN (12 marks) Use a separate writing booklet. Marks

(a) Evaluate $\log_9 49 - \log_3 7$. [2]

(b) The rate of decay of radium-226 is proportional to the mass M present at that time, so that

$$\frac{dM}{dt} = -kM.$$

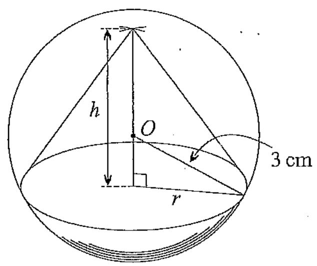
Radium-226 has a half-life of 1590 years. That is, the time taken for half the initial mass to decay is 1590 years. A sample of radium-226 begins to decay.

(i) Show that $M = M_0e^{-kt}$, where k and M_0 are constants, satisfies the differential equation above. [1]

(ii) Find the value of k. [2]

(iii) How many years will it take for 70% of the sample to decay? [2]

(c)



A right circular cone of height h and base radius r is inscribed in a sphere of radius 3 cm, as shown above.

(i) Show that the volume of the cone is given by $V = \frac{\pi}{3}(6h^2 - h^3)$. [1]

(ii) Find the dimensions of the cone so that its volume is maximised. [3]

(iii) What fraction of the sphere is occupied by this cone? [1]

END OF EXAMINATION

Form VI Mathematics Trial 2008

Question 1

(a) $0.00008375 \text{ g/cm}^3 \doteqdot 8.4 \times 10^{-5} \text{ g/cm}^3$

(b) Period $= 4\pi$

(c) $y' = 2(x+3)$

(d) $\int (e^{2x} - 1)\,dx = \dfrac{e^{2x}}{2} - x + c$

(e) $|2x-1| = 5$

$2x - 1 = 5$ or $2x - 1 = -5$

$x = 3$ or $x = -2$

(f) (i) $(x-1)^2 + (y+3)^2 = 4$

(ii) none

(g) $r = 2x$

$-1 < 2x < 1$

$\therefore -\frac{1}{2} < x < \frac{1}{2}$

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Question 2

(a) $\dfrac{\sqrt{5}}{1+\sqrt{2}} \times \dfrac{1-\sqrt{2}}{1-\sqrt{2}} = \dfrac{\sqrt{5} - \sqrt{10}}{1-2}$

$= \sqrt{10} - \sqrt{5}$

$\therefore a = 10, \; b = 5$

(b) (i) $y' = 3\cos 3x$

(ii) $y' = 2(e^x + 1) \times e^x$

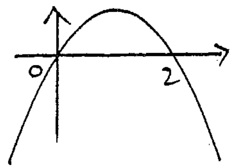
$= 2e^x(e^x + 1)$

(c) $l = r\theta$

$8 = r\dfrac{\pi}{2}$

$\therefore r = \dfrac{16}{\pi}$ cm

(d) (i)



(ii) $x(2-x) < 0$

$\therefore x < 0$ or $x > 2$

(e) (i) $\int \dfrac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx$

$= 2x^{1/2} + c$

(ii) $\int_0^1 \dfrac{x^2}{x^3+1}\,dx = \dfrac{1}{3}\int \dfrac{3x^2}{x^3+1}\,dx$

$= \dfrac{1}{3}\left[\log_e(x^3+1)\right]_0^1$

$= \dfrac{1}{3}(\log_e 2 - \log_e 1)$

$= \dfrac{1}{3}\log_e 2$

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Question 3

(a) $y = \dfrac{\log_e x}{x^2}$

$\dfrac{dy}{dx} = \dfrac{x^2\left(\frac{1}{x}\right) - \log_e x\,(2x)}{x^4}$

$= \dfrac{x - 2x\log_e x}{x^4}$

$= \dfrac{1 - 2\log_e x}{x^3}$

Question 3 (cont.)

(b) (i) gradient AC $= \frac{-1-4}{5-0}$ ✓

$= -1$

(ii) Eq'n AC is $y = -x + 4$ ✓

$x + y - 4 = 0$ as required.

(iii) $d = \frac{|1(1) + 1(0) - 4|}{\sqrt{1^2+1^2}}$ ✓

$= \frac{3}{\sqrt{2}}$ ✓

(iv) $AC^2 = (5-0)^2 + (1-4)^2$ ✓

$= 25 + 25$

$= 50$

$\therefore AC = \sqrt{50}$

$= 5\sqrt{2}$ as required

(v) Let D have coordinates (p, q).

$\frac{p+1}{2} = \frac{5}{2}$ $\quad$ $\frac{q}{2} = \frac{3}{2}$ ✓

$\therefore p = 4$ $\quad$ $\therefore q = 3$

So coordinates of D are $(4, 3)$ ✓

(vi) Area $= \frac{1}{2} AC \times BD$

$= \frac{1}{2} 5\sqrt{2} \times \frac{2 \times 3}{\sqrt{2}}$

$= 15 \ u^2$ ✓

(c) $\angle BAC = \alpha$ (angles opposite equal sides of a triangle) ✓

$AD \parallel CE$ (equal corresponding angles $\angle BAC = \angle ECF = \alpha$) ✓

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Question 4

(a) $y = \tan x$

$y' = \sec^2 x$ ✓

At $(\frac{\pi}{4}, 1)$ $\quad y' = \sec^2 \frac{\pi}{4}$

$= \frac{1}{\cos^2 \frac{\pi}{4}}$

$= 2$ ✓

Eq'n of tangent: $y - 1 = 2(x - \frac{\pi}{4})$

$y = 2x - \frac{\pi}{2} + 1$ ✓

(b) (i) Area $\triangle EBC = 3000$

$\therefore 3000 = \frac{1}{2} EB \times 62 \times \sin 105^\circ$ ✓

$EB = \frac{6000}{62 \sin 105^\circ}$ ✓

$= 100.188\ldots$

$\doteqdot 100$ m (nearest metre)

(ii) $CE^2 = EB^2 + BC^2 - 2.EB.BC.\cos 105^\circ$ ✓

$= 17097.028\ldots$

$\therefore CE = 131$ m (nearest metre) ✓

Question 4 (cont.)

(c) $\int_1^3 \log_e x \, dx \doteqdot \frac{3-1}{6}\left[\log_e 1 + 4\log_e 2 + \log_e 3\right]$

$\doteqdot \frac{1}{3}\left(0 + 4\log_e 2 + \log_e 3\right)$

$\doteqdot 1.29$ (2 decimal places)

(d) $\text{Area} = \int_0^6 \left(2x - (x^2 - 4x)\right) dx$

$= \int_0^6 (6x - x^2) \, dx$

$= \left[3x^2 - \frac{x^3}{3}\right]_0^6$

$= 108 - \frac{216}{3}$

$= 36 \ u^2$

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Question 5

(a) $\angle ACB = 69^\circ$

Using Sine Rule (in ΔABC)

$\frac{BC}{\sin 38^\circ} = \frac{50}{\sin 69^\circ}$

$BC = \frac{50 \sin 38^\circ}{\sin 69^\circ}$

$= 32.973 \ldots$

$\therefore w = BC \sin 73^\circ$

$= 31.532 \ldots$

$\doteqdot 32$ m (nearest metre)

(b) $y = 3x^4 - 4x^3 - 12x^2 + 10$

$y' = 12x^3 - 12x^2 - 24x$

$= 12x(x^2 - x - 2)$

$= 12x(x-2)(x+1)$

$y'' = 36x^2 - 24x - 24$

$= 12(3x^2 - 2x - 2)$

(i) Stationary Points at $x = 0$, $x = -1$, $x = 2$

x	-1	0	2
y''	36	-24	
conc.	$\smile$	$\frown$	$\smile$

$\therefore$ min. at $(-1, 5)$

max at $(0, 10)$

min at $(2, -22)$

(ii) $y'' = 0$ when $3x^2 - 2x - 2 = 0$

$x = \frac{2 \pm \sqrt{4 - -24}}{6}$

$= \frac{2 \pm 2\sqrt{7}}{6}$

$= \frac{1 \pm \sqrt{7}}{3}$ as required.

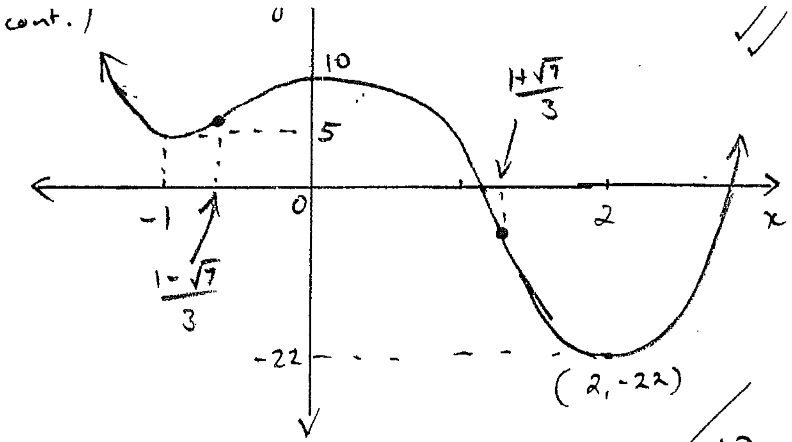
Now $x = \frac{1 - \sqrt{7}}{3} \doteqdot -0.82$ $x = \frac{1 + \sqrt{7}}{3} \doteqdot 1.22$

We can use the table above to confirm that there is a change of sign of y'' through these two points.

Hence these are points of inflexion

Question 5 (cont.)

(b) (iii)



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Question 6

(a) $2x^2 - 5x + 1 = 0$

$\alpha + \beta = --\frac{5}{2}$

$= \frac{5}{2}$ ✓

$\alpha\beta = \frac{1}{2}$ ✓

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{5}{2}\right)^2 - 2\left(\frac{1}{2}\right)$

$= \frac{25}{4} - 1$

$= \frac{21}{4}$ ✓

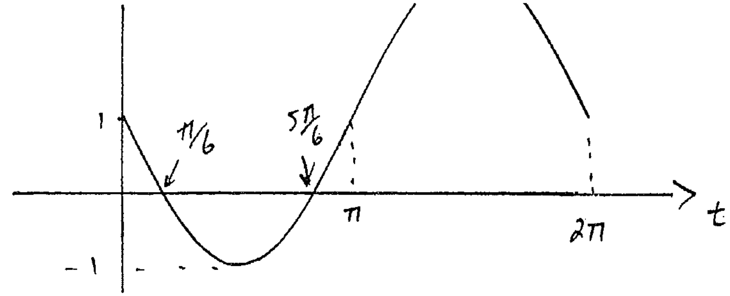
(b) (i) $v = 1 - 2\sin t$

$0 = 1 - 2\sin t$ ✓

$\frac{1}{2} = \sin t$

$\therefore t = \frac{\pi}{6}, \frac{5\pi}{6}$ ✓

(ii)



(iii) $a = \frac{dv}{dt}$

$= -2\cos t$ ✓

At $t_1 = \frac{\pi}{6}$, $a = -2\cos\frac{\pi}{6}$

$= -\sqrt{3}$ ms^{-2}

$t_2 = \frac{5\pi}{6}$ $a = -2\cos\frac{5\pi}{6}$

$= \sqrt{3}$ ms^{-2} ✓

(iv) $x = \int v\,dt$

$= t + 2\cos t + c$ ✓

$t = 0, x = 0$

$0 = 0 + 2 + c$

$\therefore c = -2$

So $x = t + 2\cos t - 2$ ✓

(v) $t = \pi/6$, $x = \frac{\pi}{6} + \sqrt{3} - 2$

$t = \frac{5\pi}{6}$, $x = \frac{5\pi}{6} - \sqrt{3} - 2$

So distance $= \left|\left(\frac{5\pi}{6} - \sqrt{3} - 2\right) - \left(\frac{\pi}{6} + \sqrt{3} - 2\right)\right|$

$= \left|\frac{2\pi}{3} - 2\sqrt{3}\right|$

$= 2\left(\sqrt{3} - \frac{\pi}{3}\right)$ m ✓

/12

Question 1

(a) a, b, c are in GP $\therefore \frac{b}{a} = \frac{c}{b}$ ✓

Taking logs of both sides

$$\ln \frac{b}{a} = \ln \frac{c}{b}$$

$$\ln b - \ln a = \ln c - \ln b$$ ✓

which is the condition for an AP

$\therefore \ln a, \ln b, \ln c$ are in AP.

and form an arithmetic sequence.

(b) (i)

$$x^2 - 10x - 16y - 7 = 0$$

$$x^2 - 10x = 16y + 7$$ ✓

$$x^2 - 10x + 25 = 16y + 32$$

$$(x-5)^2 = 16(y+2)$$

$\therefore$ Vertex is $(5, -2)$ ✓

(ii) Focal length is 4 units ✓

(iii) Eqn of directrix is $y = -6$ ✓

(c) $S_8 = 88 \quad \therefore \quad \frac{8}{2}(2a + 7d) = 88$

$$2a + 7d = 22 \quad (1)$$

$S_{18} = 558 \quad \therefore \quad \frac{18}{2}(2a + 17d) = 558$

$$2a + 17d = 62 \quad (2)$$ ✓

Now (2) − (1) $\quad 10d = 40$

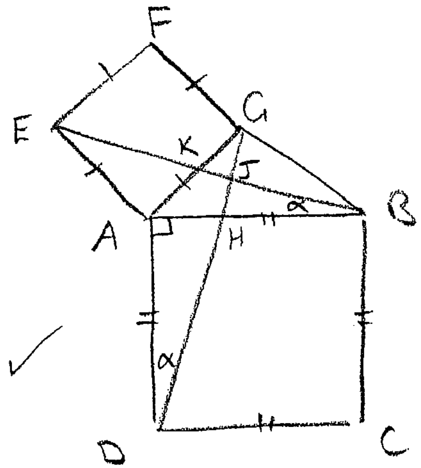
$$d = 4$$

Sub. $d = 4$ into (1) $\quad 2a + 28 = 22$

$$a = -3$$ ✓

$\therefore$ The first three terms are $-3, 1, 5$ ✓

(d)



$\triangle ADG \equiv \triangle ABE$

$\angle ABE = \angle ADG = \alpha$

(matching angles of congruent triangles ✓

$\angle HAD = 90°$ (angle of a square)

$\angle AHJ = 90° + \alpha$ (exterior angle of $\triangle AHD$) ✓

$\angle HJB + \alpha = \angle AHJ$ (exterior angle of $\triangle HJB$)

$= 90° + \alpha$

$\therefore \angle HJB = 90°$ ✓

Hence $EB \perp DG$

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Question 8

(a) $1 + \cot^2 \frac{\pi}{3} = \sec^2 \frac{\pi}{6}$

$$\text{LHS} = 1 + \frac{1}{\tan^2 \frac{\pi}{3}}$$

$$= 1 + \frac{1}{\left(\frac{\sqrt{3}}{1}\right)^2}$$ ✓

$$= \frac{4}{3}$$

$$\text{RHS} = \frac{1}{\cos^2 \frac{\pi}{6}}$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}$$ ✓

$$= \frac{4}{3}$$

$= \text{LHS}$ as required

Question … (cont.)

(b) (i) $y = (1-x)e^{x}$

$y' = -e^{x} + (1-x)e^{x}$ ✓

$= -xe^{x}$ ✓

$y'' = -e^{x} + -xe^{x}$

$= -e^{x} - xe^{x}$ as required

(ii) Point of inflexion when

$-e^{x} - xe^{x} = 0$

$x = -1$ ✓

x	-2	-1	0
y''	$\frac{1}{e^2}$	0	-1
conc	∪	·	∩

$\therefore$ Coordinates are $\left(-1, \frac{2}{e}\right)$ ✓

(iii) $0 < c < 1$ ✓

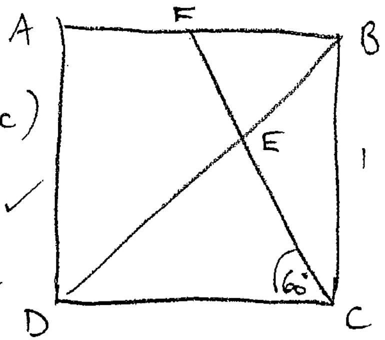
(c)

(i) In Δ's DEC and BEF

$\angle DCE = \angle BFE$ ✓
(alternate angles, $AB \parallel DC$)

$\angle \quad = \angle$
(vertically opposite) ✓

$\therefore \Delta DEC \parallel\!| \Delta BEF$ (A.A.) ✓



(ii) In ΔFBC $\frac{FB}{1} = \tan 30°$ ✓

$\therefore FB = \frac{1}{\sqrt{3}}$

(iii) Area ΔDEC : Area ΔBEF

$1^2 : \left(\frac{1}{\sqrt{3}}\right)^2$

(ratio of matching sides squared)

$1 : \frac{1}{3}$

$3 : 1$ ✓

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Question 9

(a) $V = \pi \int_0^1 (e^{x} + e^{-x})^2\, dx$ ✓

$= \pi \int_0^1 (e^{2x} + 2 + e^{-2x})\, dx$ ✓

$= \pi \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} \right]_0^1$

$= \pi \left[\left(\frac{1}{2}e^{2} + 2 - \frac{1}{2}e^{-2}\right) - \left(\frac{1}{2} - 0 - \frac{1}{2}\right) \right]$ ✓

$= \frac{\pi}{2}\left(e^{2} + 4 - e^{-2}\right)$ cubic units

(b) (i) 6% p.a = 0.5% per month.

$A = P(1+r)^{n}$

$= 500000(1 + 0.005)^{24}$ ✓

$= 563579.888\ldots$

$\doteqdot \$563\,579$ (nearest dollar)

Question 4 (cont.)

(b) (ii) 9% p.a. = 0.75% per month

$$A_n = P(1+r)^n - M\left(1 + (1+r) + (1+r)^2 + \ldots + (1+r)^{n-1}\right)$$

$$= P(1+r)^n - M\left(\frac{(1+r)^n - 1}{r}\right)$$ ✓

So $500\,000 = 563\,580(1.0075)^{96} - M\left(\frac{1.0075^{96}-1}{0.0075}\right)$ ✓

$$\therefore M = \frac{\left(563\,580(1.0075)^{96} - 500\,000\right)(0.0075)}{1.0075^{96} - 1}$$

$$= 4681.4599\ldots$$

$$= \$4681$$ ✓

(iii) (α) 12% p.a. = 1% per month

$$0 = 500\,000(1.01)^n - 6500\left(\frac{1.01^n - 1}{0.01}\right)$$ ✓

$$5000(1.01)^n = 6500(1.01)^n - 6500$$

$$(1.01)^n = \frac{6500}{1500}$$ ✓

$$= \frac{13}{3}$$

$$\therefore n = \frac{\log_e(13/3)}{\log_e 1.01}$$

$$= 147.3656\ldots$$ ✓

∴ 148 repayments are required

(β) $A_{148} = 500\,000(1.01)^{148} - 6500\left(\frac{1.01^{148}-1}{0.01}\right)$

$$= -4115.706\ldots$$ ✓

So they should be refunded \$4116 (approx.)

Question 10

(a) $\log_9 49 - \log_3 7 = \frac{\log_3 49}{\log_3 9} - \log_3 7$ ✓

$$= \frac{2\log_3 7}{2\log_3 3} - \log_3 7$$

$$= \log_3 7 - \log_3 7$$

$$= 0$$ ✓

(b) (i) $M = M_0 e^{-kt}$

$$\frac{dM}{dt} = -kM_0 e^{-kt}$$ ✓

$$= -kM.$$

(ii) $M = \frac{M_0}{2}$, $t = 1590$

$$\frac{M_0}{2} = M_0 e^{-k(1590)}$$ ✓

$$\frac{1}{2} = e^{-k(1590)}$$

$$-1590k = \ln\left(\tfrac{1}{2}\right)$$

$$= -\ln 2$$

$$k = \frac{\ln 2}{1590}$$ ✓

Question 10 (cont.)

b) (iii) $M = \frac{3}{10} M_0$

$\therefore \frac{3}{10} = e^{-kt}$

$\ln(\frac{3}{10}) = -kt$ ✓

$t = \frac{\ln(\frac{3}{10})}{-k}$

$= \frac{\ln(\frac{10}{3})}{\frac{\ln 2}{1590}}$

$= 2761.775\ldots$ years ✓

∴ It will take 2762 years.

(c) (i) $V_{cone} = \frac{1}{3}\pi r^2 h$

Now $r^2 = 3^2 - (h-3)^2$

$= 6h - h^2$

$\therefore V = \frac{1}{3}\pi(6h - h^2)h$ ✓

$= \frac{1}{3}\pi(6h^2 - h^3)$ as required.

(ii) $\frac{dV}{dh} = \frac{1}{3}\pi(12h - 3h^2)$

$= \frac{1}{3}h(12 - 3h)$ ✓

For $\frac{dV}{dh} = 0$, $h = 4$ (ignore $h = 0$)

Check there is a maximum at $h = 4$

$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(12 - 6h)$

$= \frac{1}{3}\pi(-12)$ when $h = 4$ ✓

< 0 ∴ maximum.

∴ Dimensions of cone are $h = 4$

$r = \sqrt{6(4) - 16}$

$= \sqrt{8}$

$= 2\sqrt{2}$ ✓

(iii) $V_{cone} = \frac{1}{3}\pi(8)(4)$

$= \frac{32\pi}{3}$

$V_{sphere} = \frac{4}{3}\pi 3^3$

$= 36\pi$ ✓

∴ Core occupies $\frac{\frac{32\pi}{3}}{36\pi}$ of the sphere

$= \frac{8}{27}$ of the sphere

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