SYDNEY GRAMMAR SCHOOL

2008 Assessment Examination

FORM VI
MATHEMATICS EXTENSION 2

Wednesday 26th November 2008

General Instructions

- Writing time — Period 4
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.

Structure of the paper

- Total marks — 36
- All three questions may be attempted.
- All three questions are of equal value.

Collection

- Write your name, class and master clearly on each booklet.
- Hand in the three questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.

6A: DS 6B: MLS 6C: REP
6D: TCW 6E: PKH

Checklist

- Folded A3 booklets: 3 per boy.
- Candidature — 73 boys

Examiner
DS

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Express in the form $a + bi$, where a and b are real:

(i) $(3 - 2i)^2$ [2]

(ii) $\dfrac{17 + 4i}{2 - i}$ [2]

(b) If $w = u + iv$ and $z = x + iy$, where u, v, x and y are real, prove that [2]

$$\overline{w + z} = \overline{w} + \overline{z}\,.$$

(c) Solve the quadratic equation $z^2 + iz + 2 = 0$. [3]

(d) Given that $z_1 = \sqrt{2} - \sqrt{2}\,i$ and $z_2 = 1 + \sqrt{3}\,i$,

(i) write z_1 and z_2 in modulus–argument form, [2]

(ii) write $z_1 z_2$ in modulus–argument form. [1]

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. Marks

(a) Simplify i^{15}. [1]

(b) Sketch each of the following loci on separate diagrams:

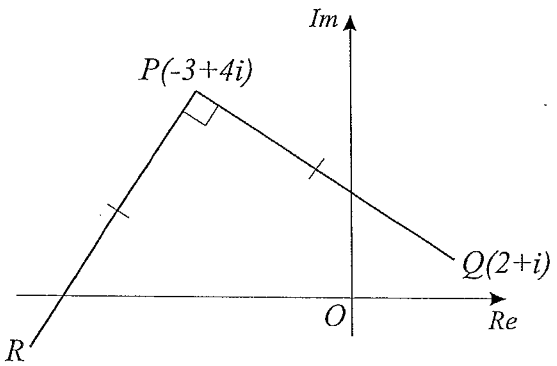
(i) $\arg(z+i) = \frac{3\pi}{4}$ [2]

(ii) $z\overline{z} < 1$ [2]

(c) (i) Sketch the locus defined by $|z-2| = 4$. [2]

(ii) Find the purely imaginary numbers that satisfy $|z-2| = 4$. [2]

(d)



In the diagram above, the points P and Q represent the complex numbers $-3+4i$ and $2+i$ respectively. The point R is the image of Q when Q is rotated clockwise through $\frac{\pi}{2}$ about P. Find the complex number that is represented by:

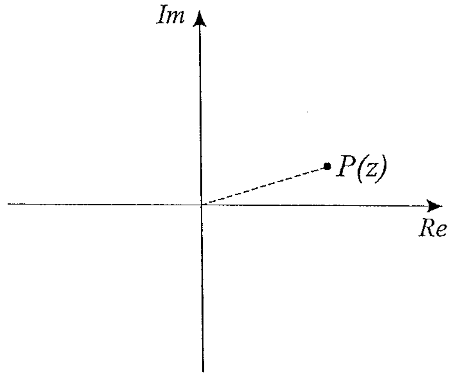
(i) the vector PQ, [1]

(ii) the point R. [2]

Exam continues overleaf ...

QUESTION THREE (12 marks) Use a separate writing booklet. Marks

(a)



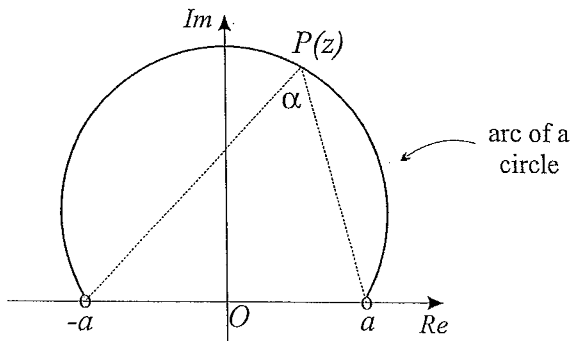
In the diagram above, the point P represents the complex number z, where $|z| = 2$ and $0 < \arg z < \frac{\pi}{4}$. Copy the diagram, and indicate clearly on it:

(i) the point Q representing $\overline{z}$, [1]

(ii) the point R representing $\dfrac{-3z}{2}$, [1]

(iii) the point S representing z^2. [2]

(b) [2]

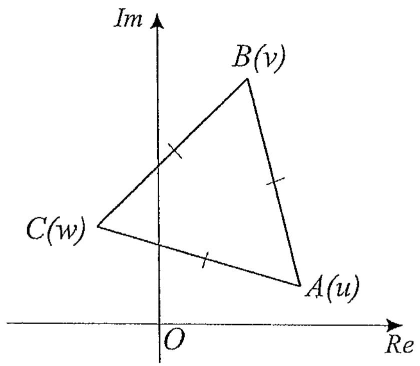


In the diagram above, the locus of the point P representing the complex number z is graphed. Write down a possible equation in terms of z, a and α for the locus of P. Note that the constants a and α are real.

Exam continues next page ...

QUESTION THREE (Continued)

(c)



In the Argand diagram above, the triangle ABC is equilateral. The points A, B and C represent the complex numbers u, v and w respectively.

(i) Explain why $w - u = (v - u)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. [2]

(ii) Write down a similar result involving $u - w$ and $v - w$. [1]

(iii) Deduce that $u^2 + v^2 + w^2 = uv + vw + wu$. [3]

END OF EXAMINATION

Extension 2 Assessment, November 2008

TOTAL: 12 × 3

(1)(a)(i) $(3-2i)^2 = 9 - 12i + 4i^2$
$= 5 - 12i$

(ii) $\dfrac{17+4i}{2-i} \cdot \dfrac{2+i}{2+i} = \dfrac{34 + 17i + 8i + 4i^2}{4 - i^2}$
$= \dfrac{30 + 25i}{5}$
$= 6 + 5i$

(b) LHS $= \overline{(u+iv) + (x+iy)}$
$= \overline{(u+x) + (v+y)i}$
$= (u+x) - (v+y)i$
$= (u - iv) + (x - iy)$
$= \bar{w} + \bar{z}$
$=$ RHS

Must be well set out.

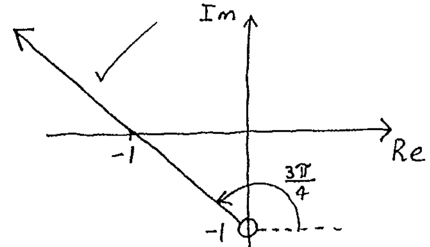
(c) $\Delta = i^2 - 4(1)(2)$
$= -9$
$\therefore z = \dfrac{-i \pm 3i}{2}$
$= -2i$ or i

(d)(i) $z_1 = \sqrt{2}(1 - i)$
$= 2\,\text{cis}\left(\frac{-\pi}{4}\right)$
and $z_2 = 1 + \sqrt{3}i$
$= 2\,\text{cis}\frac{\pi}{3}$

(ii) $z_1 z_2 = 2\,\text{cis}\left(-\frac{\pi}{4}\right) \cdot 2\,\text{cis}\frac{\pi}{3}$
$= 4\,\text{cis}\frac{\pi}{12}$

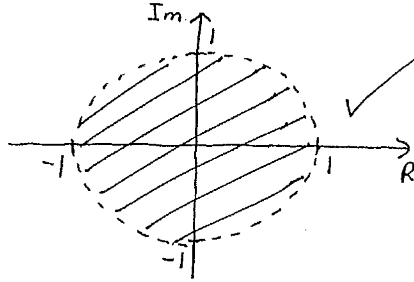
(2)(a) $i^{15} = (i^4)^3 \cdot i^3$
$= 1^3 \cdot -i$
$= -i$

(b)(i) $\arg(z - (-i)) = \dfrac{3\pi}{4}$

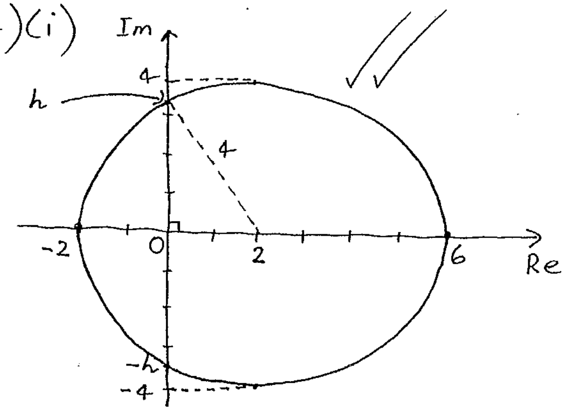


✓ for intercepts and open circle

(ii) Let $z = x + iy$.
$\therefore (x+iy)(x-iy) < 1$
$\therefore x^2 + y^2 < 1$

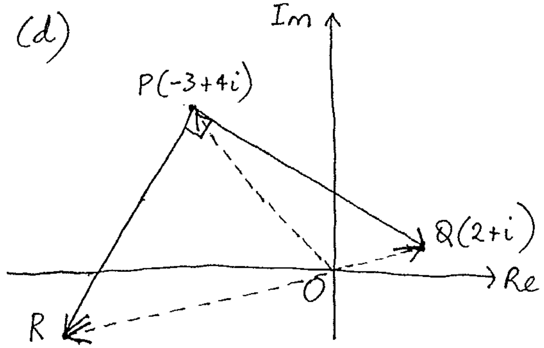


(c)(i)



(ii) By Pythagoras,
$h^2 + 2^2 = 4^2$
$\therefore h^2 = 12$
$\therefore h = 2\sqrt{3}$ $(h > 0)$
So $\pm 2\sqrt{3}i$ satisfy $|z - 2| = 4$.

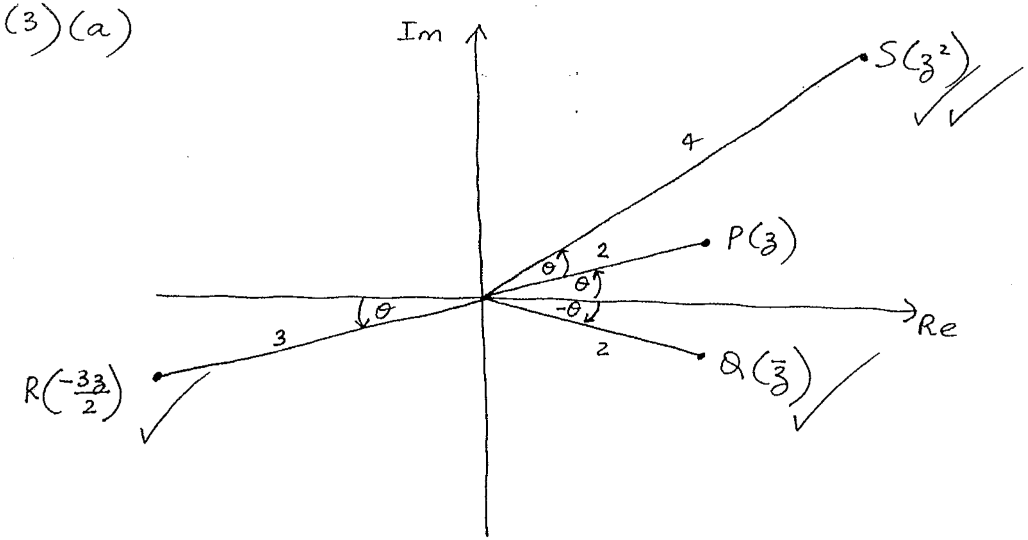
(d)



(i) $\overrightarrow{PQ}$ represents
$(2+i) - (-3+4i) = 5 - 3i$,
(since $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$).

(ii) $\overrightarrow{PR}$ represents
$-i(5 - 3i) = -3 - 5i$,
so $\overrightarrow{OR}$ (and hence R) represents $(-3+4i) + (-3-5i)$
$= -6 - i$
(since $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$).

(3) (a)



(b) $\arg\left(\frac{z-a}{z+a}\right) = \alpha$ ✓✓

(c) (i) $\vec{AC}$ represents $w-u$ and $\vec{AB}$ represents $v-u$.
$\vec{AC}$ is the image of $\vec{AB}$ under a rotation of $\frac{\pi}{3}$ in the anticlockwise direction about A, ✓
so $w-u = (v-u)\operatorname{cis}\frac{\pi}{3}$.

(ii) $v-w = (u-w)\operatorname{cis}\frac{\pi}{3}$. ✓

(iii) It follows from (i) and (ii) that
$$\frac{w-u}{v-u} = \frac{v-w}{u-w},$$ ✓
so $(w-u)(u-w) = (v-u)(v-w)$,
so $wu - w^2 - u^2 + wu = v^2 - vw - uv + wu$,
so $uv + vw + wu = u^2 + v^2 + w^2$. ✓✓