SYDNEY GRAMMAR SCHOOL

2011 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Tuesday 2nd August 2011

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 120
- All eight questions may be attempted.
- All eight questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 8 per boy
- Candidature — 85 boys

Examiner
DS

QUESTION ONE (15 marks) Use a separate writing booklet. Marks

(a) Find the exact value of $\displaystyle\int_0^1 xe^{-x^2}\,dx$. $\boxed{2}$

(b) Find $\displaystyle\int \frac{1}{\sqrt{x^2-12x+61}}\,dx$. $\boxed{2}$

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan x\,dx$. $\boxed{3}$

(d) Use the substitution $x=\sqrt{2}\sin\theta$ to find the exact value of $\displaystyle\int_0^1 \frac{x^2}{\sqrt{2-x^2}}\,dx$. $\boxed{4}$

(e) Find $\displaystyle\int \frac{x(x+9)}{(x+3)(x^2+9)}\,dx$. $\boxed{4}$

Exam continues next page ...

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Express $\dfrac{23-14i}{3-4i}$ in the form $a+bi$, where a and b are real. **2**

(b) Find the two square roots of $-16+30i$. **2**

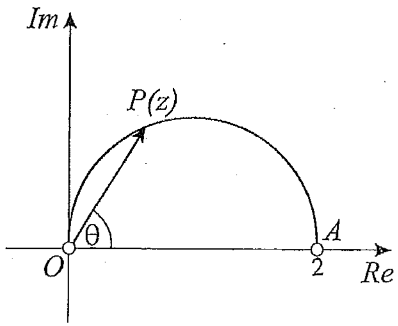
(c) Let $w=-\sqrt{3}+i$.

(i) Express w in modulus–argument form. **2**

(ii) Show that $w^9+512i=0$. **2**

(d) Shade the region in the complex plane where $|z+2|\le 2$ and $-\frac{\pi}{6}\le \arg(z+3)\le \frac{\pi}{3}$ are simultaneously satisfied. **3**

(e)



The diagram above shows the semicircular locus of the point P that represents the complex number z.
Let $\arg z=\theta$, as shown on the diagram.

(i) Copy the diagram and on it show a vector representing $z-2$. **1**

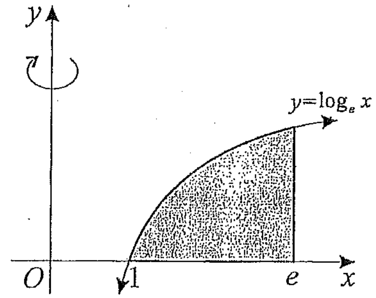
(ii) Explain why $\left|\dfrac{z-2}{z}\right|=\tan\theta$. **1**

(iii) Show that $\arg\left(\dfrac{z-2}{z}\right)=\dfrac{\pi}{2}$. **2**

Exam continues overleaf ...

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the region bounded by the curve $y=\log_e x$, the x-axis and the vertical line $x=e$. The region is rotated about the y-axis to form a solid.

(i) Find the volume of the solid by slicing perpendicular to the axis of rotation. **3**

(ii) Find the volume of the solid by the method of cylindrical shells. **4**

(b) It is known that $5+6i$ is a zero of the polynomial $P(x)=2x^3-19x^2+112x+d$, where d is real.

(i) What are the other two zeroes of $P(x)$? **2**

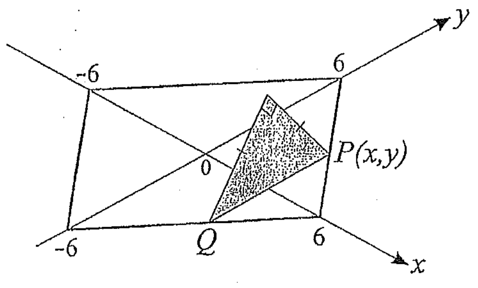
(ii) Find the value of d. **2**

(c) The polynomial equation $2x^3-x^2+5=0$ has roots α, β and γ. Find a polynomial equation with integer coefficients whose roots are α^3, β^3 and γ^3. **4**

Exam continues next page ...

QUESTION FOUR (15 marks) Use a separate writing booklet. Marks

(a)



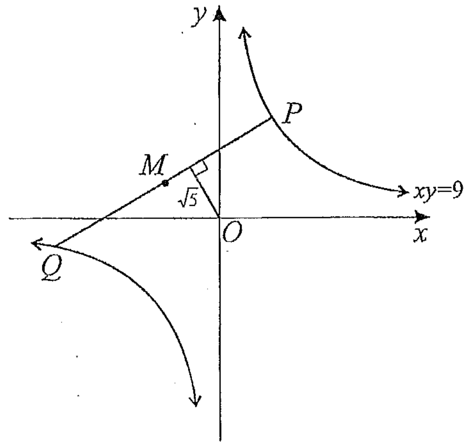
The diagram above shows the horizontal square base of a solid. Vertical cross-sections of the solid perpendicular to the x-axis are right-angled isosceles triangles with hypotenuse in the base.

(i) Find, as a function of x, the area of the typical cross-section standing on the interval PQ. **2**

(ii) Find the volume of the solid. **2**

QUESTION FOUR (Continued)

(b)



In the diagram above, $P\left(3p, \dfrac{3}{p}\right)$ and $Q\left(3q, \dfrac{3}{q}\right)$ are variable points on the rectangular hyperbola $xy = 9$. The perpendicular distance from the origin to the chord PQ is $\sqrt{5}$ units. Let M be the midpoint of the chord PQ.

(i) Show that the chord PQ has equation $x + pqy = 3(p + q)$. **2**

(ii) Using the perpendicular distance formula, or otherwise, show that **1**

$$9(p+q)^2 = 5(1 + p^2q^2).$$

(iii) Show that the locus of M has Cartesian equation $y^2 = \dfrac{5x^2}{4x^2 - 5}$. **3**

(c) Suppose that $H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, for $n = 1, 2, 3, \ldots$. **5**

So $H(1) = 1$, $H(2) = 1 + \frac{1}{2}$, $H(3) = 1 + \frac{1}{2} + \frac{1}{3}$, and so on.

Prove by mathematical induction that

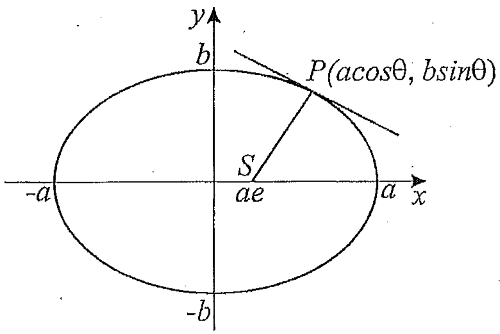
$$n + H(1) + H(2) + H(3) + \cdots + H(n-1) = nH(n)$$

for $n = 2, 3, 4, \ldots$.

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a) Solve the inequation $1 + 2x - x^2 > \dfrac{2}{x}$. $\boxed{4}$

(b)



The diagram above shows the variable point $P(a\cos\theta, b\sin\theta)$ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(i) Find the gradient of the tangent at P. $\boxed{1}$

(ii) Show that the product of the gradient of the interval SP and the gradient of the tangent at P is $\boxed{2}$

$$\frac{\cos\theta(1 - e^2)}{e - \cos\theta}.$$

(iii) Prove that SP is never perpendicular to the tangent at P, provided that $\theta \neq 0$ or π. $\boxed{2}$

(c) (i) Use de Moivre's theorem to find expressions for $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$. $\boxed{2}$

(ii) Show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$. $\boxed{1}$

(iii) By letting $\theta = \frac{\pi}{12}$ in part (ii), show that $\tan\frac{\pi}{12}$ is a root of the equation $\boxed{1}$

$$x^3 - 3x^2 - 3x + 1 = 0.$$

(iv) <u>Hence</u> find the exact value of $\tan\frac{\pi}{12}$. $\boxed{2}$

Exam continues overleaf ...

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

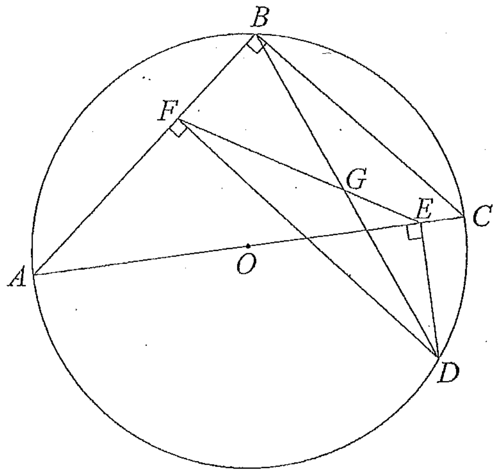
(a) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$.

(i) Use integration by parts to show that $I_n = (n-1)\displaystyle\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \cos^2\theta \, d\theta$. $\boxed{2}$

(ii) Hence show that $I_n = \dfrac{n-1}{n} I_{n-2}$, for $n = 2, 3, 4, \ldots$. $\boxed{1}$

(iii) Find the exact value of $I_9 \times I_{10}$. $\boxed{2}$

(b)



In the diagram above, triangle ABC is right-angled at B. Its circumcircle is drawn, with centre O. A point D is chosen on the circumcircle, then DE and DF are drawn perpendicular to AC and AB respectively. The point G is the intersection of DB and EF.

NOTE: You do not have to copy the diagram. It has been reproduced for you on a tear-off sheet at the end of the paper. Insert the tear-off sheet into your answer booklet.

(i) Explain why $ADEF$ is a cyclic quadrilateral. $\boxed{1}$

(ii) Let $\angle DAE = \theta$.
Prove that $\triangle FGB$ is isosceles. $\boxed{2}$

(iii) Prove that $ODEG$ is a cyclic quadrilateral. $\boxed{2}$

(iv) Deduce that OG is perpendicular to BD. $\boxed{1}$

Exam continues next page ...

QUESTION SIX (Continued)

(c) Let $P(x)$ be a polynomial of degree n, where n is odd.
It is known that $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$.

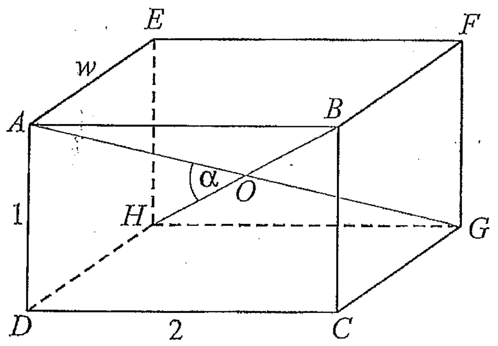
(i) Write down the zeroes of the polynomial $(x+1)P(x) - x$. [1]

(ii) Let A be the leading coefficient of the polynomial $(x+1)P(x) - x$. Factorise the polynomial, and hence find A. [2]

(iii) Find $P(n+1)$. [1]

QUESTION SEVEN (15 marks) Use a separate writing booklet. Marks

(a)



In the rectangular prism above $DC = 2$, $AD = 1$ and $AE = w$. Angle α is the acute angle between the diagonals AG and BH, which intersect at O. Let r be the ratio of the volume of the prism to its surface area.

(i) Show that $AG^2 = 5 + w^2$. [1]

(ii) Show that $\cos\alpha = \dfrac{|3 - w^2|}{5 + w^2}$. [2]

(iii) Show that $r < \frac{1}{3}$ for all possible values of w. [2]

(iv) If $r \geq \frac{1}{4}$, prove that $\alpha \leq \cos^{-1}\frac{1}{9}$. [2]

QUESTION SEVEN (Continued)

(b) A particle of mass 2 kg experiences a resistive force, in Newtons, of 10% of the square of its velocity v metres per second when it moves through the air. The particle is projected vertically upwards from a point A with velocity u metres per second. The highest point reached is B, directly above A. Assume that $g = 10\,\text{ms}^{-2}$, and take upwards as the positive direction.

(i) Show that the acceleration of the particle as it rises is given by [1]

$$\ddot{x} = -\frac{v^2 + 200}{20}.$$

(ii) Show that the distance x metres of the particle from A as it rises is given by [2]

$$x = 10 \log_e \left(\frac{200 + u^2}{200 + v^2} \right).$$

(iii) Show that the time t seconds that the particle takes to reach a velocity of v metres per second is given by [2]

$$t = \sqrt{2}\left(\tan^{-1}\frac{u}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}} \right).$$

(iv) Now suppose that we take two of the 2 kg particles described above. [3]
One of the particles is projected upwards from A with initial velocity $10\sqrt{2}\,\text{ms}^{-1}$, then, $\frac{3\sqrt{2}}{5}$ seconds later, the other particle is projected upwards from A with initial velocity $30\sqrt{2}\,\text{ms}^{-1}$. Will the second particle catch up to the first particle before the first particle reaches its maximum height? You must explain your reasoning and show your working.

QUESTION EIGHT (15 marks) Use a separate writing booklet. Marks

(a) Show that $\dfrac{1+\cos\alpha}{\sin\alpha} = \tan\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2}\right)$. [2]

(b) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\sin x}\,dx$, where $0 < \alpha < \frac{\pi}{2}$.

(i) Use the substitution $t = \tan\frac{x}{2}$ to show that [3]

$$I = \int_0^1 \frac{2}{(t+\cos\alpha)^2 + \sin^2\alpha}\,dt\,.$$

(ii) Use the further substitution $t + \cos\alpha = \sin\alpha\tan u$ and the result in part (a) above to show that $I = \dfrac{\alpha}{\sin\alpha}$. [4]

(c) (i) Find, in modulus–argument form, the roots of the equation $z^{2n+1} = 1$. [2]

(ii) Hence factorise $z^{2n} + z^{2n-1} + \cdots + z^2 + z + 1$ into quadratic factors with real coefficients. [2]

(iii) Deduce that [2]

$$2^n \sin\tfrac{\pi}{2n+1}\sin\tfrac{2\pi}{2n+1}\cdots\sin\tfrac{n\pi}{2n+1} = \sqrt{2n+1}\,.$$

END OF EXAMINATION

2011 Extension 2 Trial HSC Solutions

(1)(a) $\int_0^1 xe^{-x^2}dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1$ ✓

$= -\frac{1}{2}(e^{-1}-1)$

$= \frac{1}{2}(1-e^{-1})$ ✓

(b) $\int \frac{1}{\sqrt{x^2-12x+61}}dx = \int \frac{1}{\sqrt{(x-6)^2+25}}dx$ ✓

$= \ln\left(x-6+\sqrt{x^2-12x+61}\right)+c$ ✓

(c) $\int_0^{\frac{\pi}{4}} \sec^4 x \tan x\, dx$

$= \int_0^{\frac{\pi}{4}} \sec^2 x(1+\tan^2 x)\tan x\, dx$ ✓

$\left.\begin{aligned} &= \int_0^1 (u+u^3)\,du \\ &= \left[\frac{u^2}{2}+\frac{u^4}{4}\right]_0^1 \\ &= \frac{3}{4}\end{aligned}\right\}$ ✓

Let $u = \tan x$

$\therefore du = \sec^2 x\, dx$ ✓

x	0	$\frac{\pi}{4}$
u	0	1

(d) $\int_0^1 \frac{x^2}{\sqrt{2-x^2}}dx$

$\left.\begin{aligned} &= \int_0^{\frac{\pi}{4}} \frac{2\sin^2\theta}{\sqrt{2(1-\sin^2\theta)}}.\sqrt{2}\cos\theta\, d\theta \\ &= \int_0^{\frac{\pi}{4}} 2\sin^2\theta\, d\theta\end{aligned}\right\}$ ✓

$= 2\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta$ ✓

$\left.\begin{aligned} &= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2} \\ &= \frac{\pi}{4}-\frac{1}{2}\end{aligned}\right\}$ ✓

Let $x = \sqrt{2}\sin\theta$

$\therefore dx = \sqrt{2}\cos\theta\, d\theta$ ✓

x	0	1
θ	0	$\frac{\pi}{4}$

1)(e) Let $\frac{x^2+9x}{(x+3)(x^2+9)} = \frac{A}{x+3}+\frac{Bx+C}{x^2+9}$. ✓

$\therefore x^2+9x = A(x^2+9)+(Bx+C)(x+3)$

$\left.\begin{aligned} &\text{Let } x=-3. \\ &\therefore -18 = 18A \\ &\therefore A = -1 \\ &\text{Let } x = 0. \\ &\therefore 0 = -9+3C \\ &\therefore C = 3 \\ &\text{Let } x = 1. \\ &\therefore 10 = -10+4(B+3) \\ &\therefore B+3 = 5 \\ &\therefore B = 2\end{aligned}\right\}$ ✓

$\therefore \int \frac{x(x+9)}{(x+3)(x^2+9)}dx = \int\frac{-1}{x+3}dx + \int\frac{2x}{x^2+9}dx + \int\frac{3}{9+x^2}dx$

$= -\ln|x+3| + \ln(x^2+9) + \tan^{-1}\left(\frac{x}{3}\right) + c$

✓✓

(2)(a) $\dfrac{23-14i}{3-4i}\cdot\dfrac{3+4i}{3+4i}=\dfrac{69+56-42i+92i}{9-16i^2}$ ✓

$=\dfrac{125+50i}{25}$ ✓

$=5+2i$

(b) Let $-16+30i=(a+ib)^2$.

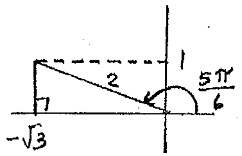
$\therefore\ a^2-b^2=-16$ and $ab=15$ ✓

By inspection, $(a,b)=(3,5)$ or $(-3,-5)$. ✓

So the two square roots are $3+5i$ and $-3-5i$.

(c)(i) $w=-\sqrt{3}+i$

$=2\,\text{cis}\,\frac{5\pi}{6}$ ✓ ✓



(ii) $w^9=\left(2\,\text{cis}\,\frac{5\pi}{6}\right)^9$

$=2^9\,\text{cis}\,\frac{15\pi}{2}$

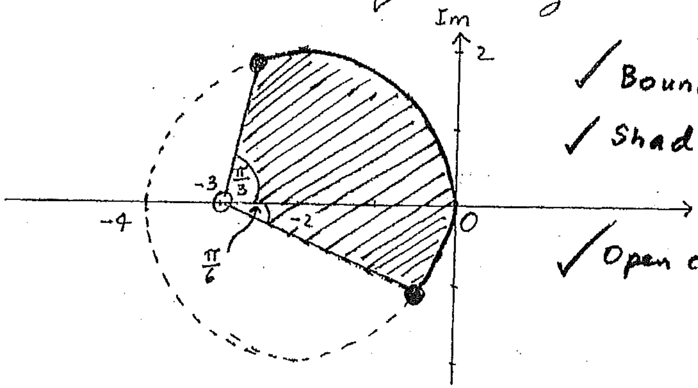
$=512\,\text{cis}\,\frac{3\pi}{2}$ ✓

$=512(0-i)$

$=-512i$, so $w^9+512i=0$.

So w is a root of the equation $z^9+512i=0$. ✓

(d)

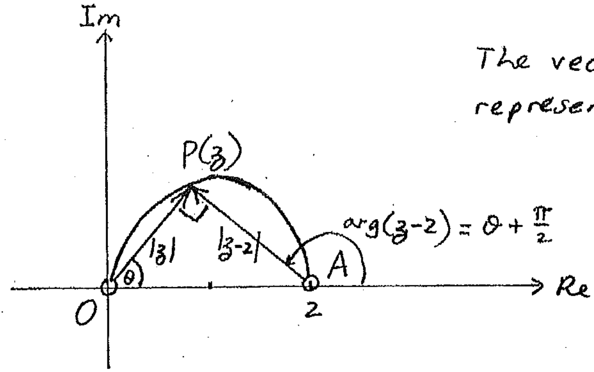


✓ Boundaries

✓ Shading

✓ Open circle

(2)(e)(i)



The vector $\overrightarrow{AP}$ represents $z-2$. ✓

(ii) $\angle APO=\frac{\pi}{2}$ (angle in a semicircle)

So in ΔAPO,

$\left|\dfrac{z-2}{z}\right|=\dfrac{|z-2|}{|z|}$

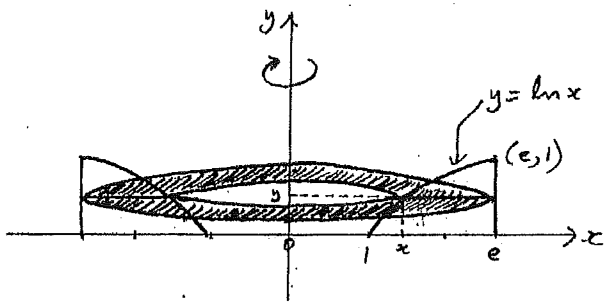
$=\tan\theta$. ✓

(iii) $\arg\left(\dfrac{z-2}{z}\right)=\arg(z-2)-\arg z$ ✓

$=\left(\theta+\frac{\pi}{2}\right)-\theta$ ($\arg(z-2)$ is an exterior angle of ΔAPO)

$=\frac{\pi}{2}$ ✓

(3)(a)(i)

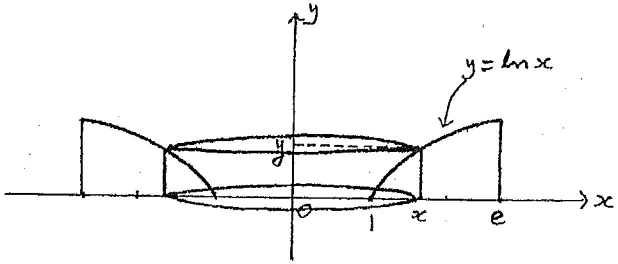


$R = e$
$r = x$

$A(y) = \pi(R^2 - r^2)$
$= \pi(e^2 - x^2)$, where $x = e^y$
$= \pi(e^2 - e^{2y})$

So $V = \int_{y=0}^{1} \pi(e^2 - e^{2y})\,dy$
$= \pi\left[e^2 y - \frac{1}{2}e^{2y}\right]_0^1$
$= \pi\left(e^2 - \frac{1}{2}e^2 - 0 + \frac{1}{2}\right)$
$= \pi\left(\frac{1}{2}e^2 + \frac{1}{2}\right)$
$= \frac{\pi}{2}(e^2 + 1)$ u^3

(ii)



$A(x) = 2\pi r h$
$= 2\pi . x . y$
$= 2\pi x \ln x$

So $V = \int_{x=1}^{e} 2\pi x \ln x\,dx$
$= 2\pi\left[\frac{1}{2}x^2 \ln x\right]_1^e - 2\pi\int_1^e \frac{1}{2}x^2 . \frac{1}{x}\,dx$
$= \pi e^2 \ln e - \pi\int_1^e x\,dx$
$= \pi e^2 - \pi\left(\frac{1}{2}e^2 - \frac{1}{2}\right)$
$= \frac{\pi}{2}(e^2+1)$ u^3

Integration by parts:
Let $u = \ln x$
$\therefore u' = \frac{1}{x}$
Let $v' = x$
$\therefore v = \frac{1}{2}x^2$

(3)(b)(i) $\overline{5+6i} = 5-6i$ is a zero, because all the coefficients of $P(x)$ are real.

Let α be the 3rd zero.

$\therefore (5+6i) + (5-6i) + \alpha = \frac{19}{2}$

$\therefore \alpha = -\frac{1}{2}$

So the zeroes of $P(x)$ are $5+6i$, $5-6i$, $-\frac{1}{2}$.

(ii) $(5+6i)(5-6i)\left(-\frac{1}{2}\right) = -\frac{d}{2}$

$\therefore d = 25 + 36$
$= 61$

(c) Let $u = x^3$, so that $x = u^{\frac{1}{3}}$ (or simply replace x with $x^{\frac{1}{3}}$)

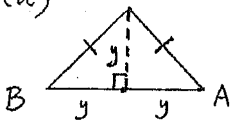
The new equation is

$2u - u^{\frac{2}{3}} + 5 = 0$
$u^{\frac{2}{3}} = 2u + 5$
$u^2 = (2u+5)^3$
$u^2 = 8u^3 + 60u^2 + 150u + 125$
$8u^3 + 59u^2 + 150u + 125 = 0$

Since u is a dummy variable, the new equation can also be written as

$8x^3 + 59x^2 + 150x + 125 = 0.$

(4)(a)



(i) $A(x) = y^2$, where $x + y = 6$

$= (6-x)^2$

(ii) $V = \int_{x=-6}^{6} (6-x)^2\,dx$

$= 2\left[\dfrac{(6-x)^3}{-3}\right]_0^6$

$= -\dfrac{2}{3}(0 - 6^3)$

$= 144\ u^3$

(b)(i) $m_{PQ} = \dfrac{\frac{3}{p} - \frac{3}{q}}{3p - 3q}$

$= \dfrac{3(q-p)}{3pq(p-q)}$

$= \dfrac{-1}{pq}$

So the chord PQ has equation

$y - \dfrac{3}{p} = \dfrac{-1}{pq}(x - 3p)$

$pqy - 3q = -x + 3p$

$x + pqy = 3(p+q)$

(ii) The perpendicular distance from $(0,0)$ to the line $x + pqy - 3(p+q) = 0$ is $\sqrt{5}$ units.

So $\left|\dfrac{1(0) + pq(0) - 3(p+q)}{\sqrt{1^2 + (pq)^2}}\right| = \sqrt{5}$,

so $|-3(p+q)| = \sqrt{5(1+p^2q^2)}$,

so $9(p+q)^2 = 5(1+p^2q^2)$.

(4)(b)(iii) M is the point $\left(\dfrac{3p+3q}{2}, \dfrac{\frac{3}{p}+\frac{3}{q}}{2}\right)$

$= \left(\dfrac{3(p+q)}{2}, \dfrac{3(p+q)}{2pq}\right)$.

So the locus of M has parametric equations

$x = \dfrac{3(p+q)}{2}$ (1) and $y = \dfrac{3(p+q)}{2pq}$ (2)

From (1), $p + q = \dfrac{2x}{3}$.

Substitute into (2): $y = \dfrac{x}{pq}$, so $pq = \dfrac{x}{y}$.

" " part (ii) to get the Cartesian equation:

$9\left(\dfrac{2x}{3}\right)^2 = 5\left(1 + \dfrac{x^2}{y^2}\right)$

$\dfrac{4x^2}{5} = 1 + \dfrac{x^2}{y^2}$

$\dfrac{x^2}{y^2} = \dfrac{4x^2 - 5}{5}$

$y^2 = \dfrac{5x^2}{4x^2 - 5}$

(c) When $n = 2$, $LHS = 2 + H(1)$ and $RHS = 2H(2)$

$= 2 + 1$ $\quad = 2(1 + \frac{1}{2})$

$= 3$ $\quad = 3$

So the result is true for $n = 2$.

Assume that the result is true for the integer $n = k$.

i.e. assume that $k + H(1) + H(2) + \dots + H(k-1) = k\,H(k)$.

Prove that the result is true for $n = k+1$.

i.e. prove that $(k+1) + H(1) + H(2) + \dots + H(k-1) + H(k) = (k+1)H(k+1)$

$LHS = 1 + (k + H(1) + H(2) + \dots + H(k-1)) + H(k)$

$= 1 + k\,H(k) + H(k)$ (using the assumption)

$= 1 + (k+1)H(k)$

$= \dfrac{k+1}{k+1} + (k+1)\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right)$

$= (k+1)\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1}\right)$

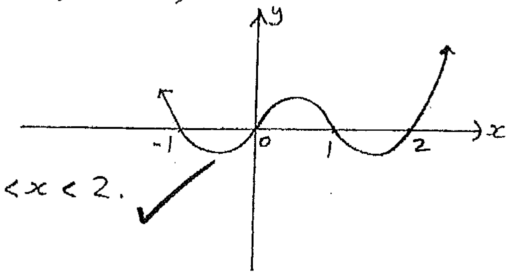
$= (k+1)H(k+1) = RHS$.

So, by induction, the result is true for $n = 2, 3, 4, \dots$

(5)(a) $1+2x-x^2 > \frac{2}{x}$, $x \neq 0$

Multiply both sides by x^2:

$$x^2+2x^3-x^4 > 2x$$
$$x^4-2x^3-x^2+2x<0$$
$$x(x^3-2x^2-x+2)<0$$
$$x\left(x^2(x-2)-1(x-2)\right)<0$$
$$x(x^2-1)(x-2)<0$$
$$x(x+1)(x-1)(x-2)<0$$



The solution is

$-1<x<0$ or $1<x<2$.

(b)(i) At P, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$= \frac{b\cos\theta}{-a\sin\theta} \text{ or } \frac{-b\cos\theta}{a\sin\theta}$$

(ii) $m_{SP}\cdot m_{tangent} = \frac{b\sin\theta}{a\cos\theta-ae} \cdot \frac{+b\cos\theta}{-a\sin\theta}$

$$= \frac{b^2\cos\theta\cancel{\sin\theta}}{a^2\cancel{\sin\theta}(e-\cos\theta)}, \text{ where } b^2=a^2(1-e^2)$$

$$= \frac{\cos\theta(1-e^2)}{e-\cos\theta}$$

(iii) Suppose $m_{SP}\cdot m_{tangent} = -1$.

Then $\cos\theta(1-e^2) = \cos\theta - e$

$$\cancel{\cos\theta} - e^2\cos\theta = \cancel{\cos\theta} - e$$
$$e\cos\theta = 1$$
$$\cos\theta = \frac{1}{e}, \text{ where } 0<e<1, \text{ so that } \frac{1}{e}>1.$$

This is impossible, because $-1 \le \cos\theta \le 1$ for all real θ, so, provided $\theta \neq 0$ or π, SP cannot be perpendicular to the tangent.

(c)(i)

$$\cos 3\theta + i\sin 3\theta$$
$$= (\cos\theta + i\sin\theta)^3 \quad \text{(de Moivre)}$$
$$= \cos^3\theta + 3\cos^2\theta . i\sin\theta + 3\cos\theta . i^2\sin^2\theta + i^3\sin^3\theta$$

Equating real and imaginary parts,

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

(ii) $\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$

$$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}, \text{ after dividing top and bottom by } \cos^3\theta.$$

(iii) Let $\theta = \frac{\pi}{12}$ in (ii).

$$\therefore \tan\frac{\pi}{4} = \frac{3\tan\frac{\pi}{12} - \tan^3\frac{\pi}{12}}{1-3\tan^2\frac{\pi}{12}}$$

It follows that $x = \tan\frac{\pi}{12}$ is a root of the equation

$$1 = \frac{3x-x^3}{1-3x^2}$$

i.e. $1-3x^2 = 3x - x^3$

i.e. $x^3-3x^2-3x+1=0$

(iv) By inspection, $x=-1$ is a root of the equation.

So $(x+1)$ is a factor of the LHS.

$$\begin{array}{r|l} & x^2-4x+1 \\ x+1 & x^3-3x^2-3x+1 \\ & x^3+x^2 \\ \hline & -4x^2-3x \\ & -4x^2-4x \\ \hline & x+1 \end{array}$$

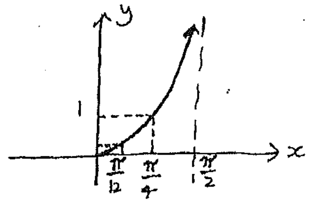
So the equation can be written

$(x+1)(x^2-4x+1)=0$.

So $\tan\frac{\pi}{12}$ is one of the roots of $x^2-4x+1=0$

i.e. $(x-2)^2=3$,

from which $x = 2 \pm \sqrt{3}$.



But $\tan\frac{\pi}{12} < \tan\frac{\pi}{4} = 1$,

so $\tan\frac{\pi}{12} = 2-\sqrt{3}$.

(6)(a)(i)

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \sin\theta \, d\theta$$
$$= \left[-\cos\theta \sin^{n-1}\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos\theta.(n-1)\sin^{n-2}\theta \cos\theta \, d\theta$$
$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \cos^2\theta \, d\theta$$

Let $u = \sin^{n-1}\theta$ (11)

$\therefore u' = (n-1)\sin^{n-2}\theta \cos\theta$

Let $v' = \sin\theta$

$\therefore v = -\cos\theta$

(ii) From (i),

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta(1-\sin^2\theta)\, d\theta$$
$$I_n = (n-1)\left(\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta - \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta\right)$$
$$I_n = (n-1)(I_{n-2} - I_n)$$
$$(n-1)I_n + I_n = (n-1)I_{n-2}$$
$$nI_n = (n-1)I_{n-2}$$
$$\therefore I_n = \frac{n-1}{n} I_{n-2}$$

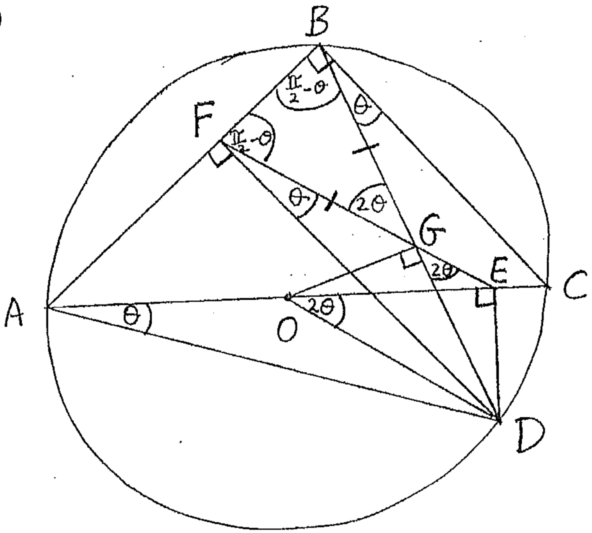
(iii)

$$I_9 = \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times I_1, \text{ where } I_1 = \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = \left[-\cos\theta\right]_0^{\frac{\pi}{2}} = 1$$

and $$I_{10} = \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times I_0, \text{ where } I_0 = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}.$$

So $$I_9 \times I_{10} = \frac{9!}{10!} \times \frac{\pi}{2} = \frac{\pi}{20}.$$

6)(b)



(i) $\angle AFD = \angle AED = \frac{\pi}{2}$ (given), so A, F, E and D are concyclic (converse of angle in a semicircle).

(ii) Let $\angle DAE = \theta$.
$\therefore \angle DFE = \angle DAE = \theta$ (angles in the same segment of circle ADEF)
and $\angle DBC = \angle DAC = \angle DAE = \theta$ (angles in same segment of circle ADCB)
So $\angle GFB = \angle GBF = \frac{\pi}{2} - \theta$ (complementary angles), so ΔFGB is isosceles (two equal angles).

(iii) $\angle FGB = 2\theta$ (angle sum of ΔFGB), so $\angle DGE = 2\theta$ (vertically opposite)
Also $\angle DOE = 2\theta$ (angle at centre of circle ADCB is twice $\angle DAE$ at the circumference)
So O, D, E and G are concyclic (converse of angles in the same segment)

(iv) $\angle OGD = \angle OED = \frac{\pi}{2}$ (angles in the same segment of circle ODEG)
So $OG \perp BD$.

(6)(c)(i) $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$.

So $(k+1)P(k) - k = 0$ for $k = 0, 1, 2, \ldots, n$.

So $x = 0, 1, 2, \ldots, n$ are zeroes of the polynomial $(x+1)P(x) - x$.

(ii) From (i), it follows that

$$(x+1)P(x) - x = A\,x(x-1)(x-2)\ldots(x-n), \quad (*)$$

where "A" is a constant.

leading coefficient

Let $x = -1$.

$\therefore\ 1 = A(-1)(-2)(-3)\ldots(-1-n)$

$\therefore\ 1 = A(-1)^{n+1}(n+1)!$

$\therefore\ A = \dfrac{1}{(n+1)!}$ $\left((-1)^{n+1} = 1,\right.$ since n is odd$\left.\right)$

(iii) Let $x = n+1$ in $(*)$.

$\therefore\ (n+2)P(n+1) - (n+1) = \dfrac{1}{(n+1)!}(n+1)(n)(n-1)\ldots 3.2.1$

$\therefore\ P(n+1) = \dfrac{1 + (n+1)}{n+2}$

$= 1$

(7)(a)(i) By Pythagoras, $DG^2 = DC^2 + CG^2$,

so $DG^2 = 4 + w^2$.

By Pythagoras, $AG^2 = AD^2 + DG^2$

$= 1^2 + (4 + w^2)$

$= 5 + w^2$.

(ii) By Pythagoras, $AH^2 = AD^2 + DH^2 = 1 + w^2$.

Also, $OA^2 = \left(\frac{1}{2}AG\right)^2 = \frac{1}{4}(5 + w^2) = OH^2$.

So in ΔAOH, by the cosine rule,

$$\cos\alpha = \frac{OA^2 + OH^2 - AH^2}{2 \times OA \times OH}$$

$$= \frac{\left|\frac{1}{4}(5+w^2) + \frac{1}{4}(5+w^2) - (1+w^2)\right|}{2 \times \frac{1}{4}(5+w^2)} \cdot \frac{2}{2}$$

$$= \frac{|5 + w^2 - 2 - 2w^2|}{5 + w^2}$$

$$= \frac{|3 - w^2|}{5 + w^2}.$$

(the absolute value is needed because α is acute, so $\cos\alpha > 0$)

(iii) $V = 2w$, $S = 2(2w + 1w + 2)$

$= 6w + 4$

So $r = \dfrac{V}{S} = \dfrac{2w}{6w+4}$

$= \dfrac{1}{3 + \frac{2}{w}}$.

So as $w \to 0^+$, $r \to 0^+$,

and as $w \to \infty$, $r \to \left(\frac{1}{3}\right)^-$. (this is the important part of the solution)

So $0 < r < \frac{1}{3}$ for all values of w.

(7)(a)(iv) If $r \geq \frac{1}{4}$,

then $\dfrac{w}{3w+2} \geq \dfrac{1}{4}$

$$4w \geq 3w + 2$$
$$w \geq 2.$$

From (ii), as $w \to \infty$, $\alpha \to \cos^{-1} 1 = 0$.

So the maximum value of $\cos\alpha$ is $\left|\dfrac{3-2^2}{5+2^2}\right| = \dfrac{1}{9}$.

So $\alpha \leq \cos^{-1}\frac{1}{9}$.

(b)(i) $F = -mg - 10\%$ of v^2

$$\therefore m\ddot{x} = -mg - \frac{v^2}{10}$$
$$2\ddot{x} = -2 \times 10 - \frac{v^2}{10}$$
$$\ddot{x} = -10 - \frac{v^2}{20}$$
$$= -\frac{200 + v^2}{20}$$

(ii) $v\dfrac{dv}{dx} = -\dfrac{200+v^2}{20}$

$$\frac{dx}{dv} = \frac{-20v}{200+v^2}$$
$$x = -10\int \frac{2v}{200+v^2}\,dv$$
$$= -10\ln(200+v^2) + c_1$$

When $x = 0$, $v = u$,

so $c_1 = 10\ln(200+u^2)$.

So $x = 10\ln(200+u^2) - 10\ln(200+v^2)$

$$x = 10\ln\left(\frac{200+u^2}{200+v^2}\right).$$

(7)(b)(iii) $\dfrac{dv}{dt} = -\dfrac{200+v^2}{20}$

$$\frac{dt}{dv} = \frac{-20}{200+v^2}$$
$$t = -20\int \frac{1}{200+v^2}\,dv$$
$$= -20 \cdot \frac{1}{10\sqrt{2}} \tan^{-1}\frac{v}{10\sqrt{2}} + c_2$$

When $t = 0$, $v = u$, so $c_2 = \dfrac{2}{\sqrt{2}} \tan^{-1}\dfrac{u}{10\sqrt{2}}$.

So $t = \sqrt{2}\left(\tan^{-1}\dfrac{u}{10\sqrt{2}} - \tan^{-1}\dfrac{v}{10\sqrt{2}}\right)$

(iv) Find the distance AB and the time taken for the first particle.

When $u = 10\sqrt{2}$ and $v = 0$,

$$x = 10\ln\left(\frac{200 + (10\sqrt{2})^2}{200 + 0^2}\right)$$
$$= 10\ln 2 \text{ metres.}$$

When $u = 10\sqrt{2}$ and $v = 0$,

$$t = \sqrt{2}(\tan^{-1} 1 - \tan^{-1} 0)$$
$$= \frac{\pi\sqrt{2}}{4} \text{ seconds } (= 1{\cdot}11\ldots \text{ seconds}).$$

Now consider the second particle, for which $u = 30\sqrt{2}$.

Its velocity when it reaches B is given by

$$10\ln 2 = 10\ln\left(\frac{200 + (30\sqrt{2})^2}{200 + v^2}\right)$$
$$2 = \frac{2000}{200+v^2}$$
$$v^2 + 200 = 1000$$
$$v = 20\sqrt{2} \text{ ms}^{-1} \quad (v > 0).$$

Now find the time taken for the second particle to reach B. When $v = 20\sqrt{2}$,

$$t = \sqrt{2}(\tan^{-1} 3 - \tan^{-1} 2)$$
$$= 0{\cdot}20067\ldots \text{ seconds.}$$

Now, $\sqrt{2}(\tan^{-1}3 - \tan^{-1}2) + \dfrac{3\sqrt{2}}{5} < \dfrac{\pi\sqrt{2}}{4}$

(i.e. $1{\cdot}0492\ldots < 1{\cdot}11\ldots$).

So the second particle reaches B before the first particle.

So the second particle overtakes the first particle while they are both rising.

(8)(a) $LHS = \dfrac{1+\cos\alpha}{\sin\alpha}$

$= \dfrac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$ ✓

$= \cot\frac{\alpha}{2}$

$= \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$ ✓

$= RHS$

(b)(i)

$I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\sin x}\,dx$

$= \displaystyle\int_0^1 \frac{1}{1+\cos\alpha\cdot\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$

$= \displaystyle\int_0^1 \frac{2}{1+t^2+2t\cos\alpha}\,dt$ ✓

$t = \tan\frac{x}{2}$

$\therefore x = 2\tan^{-1}t$

$\therefore dx = \dfrac{2}{1+t^2}\,dt$

x	0	$\frac{\pi}{2}$
t	0	1

✓

$= \displaystyle\int_0^1 \frac{2}{(t^2+2t\cos\alpha+\cos^2\alpha)+\sin^2\alpha}\,dt$

$= \displaystyle\int_0^1 \frac{2}{(t+\cos\alpha)^2+\sin^2\alpha}\,dt$ ✓

(13)

(8)(b)(ii)

$I = \displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} \frac{2}{\sin^2\alpha\tan^2 u+\sin^2\alpha}\cdot\sin\alpha\sec^2 u\,du$

$= \displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} \frac{2\sec^2 u}{\sin\alpha(\tan^2 u+1)}\,du$

$= \dfrac{2}{\sin\alpha}\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} du$ ✓

$= \dfrac{2}{\sin\alpha}\Big[u\Big]_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}}$

$= \dfrac{2}{\sin\alpha}\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right)$

$= \dfrac{2}{\sin\alpha}\cdot\dfrac{\alpha}{2}$

$= \dfrac{\alpha}{\sin\alpha}$ ✓

$t+\cos\alpha = \sin\alpha\tan u$

$\therefore dt = \sin\alpha\sec^2 u\,du$ ✓

When $t=0$,

$\tan u = \cot\alpha$

$\tan u = \tan\left(\frac{\pi}{2}-\alpha\right)$

$u = \frac{\pi}{2}-\alpha$ ✓

When $t=1$,

$\tan u = \dfrac{1+\cos\alpha}{\sin\alpha}$

$= \tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$

$u = \frac{\pi}{2}-\frac{\alpha}{2}$

(18)

(8)(c)(i) $z^{2n+1} = 1$

Let $z = cis\,\theta$.

$\therefore cis(2n+1)\theta = cis(2k\pi)$, where $k \in \mathbb{Z}$

$\therefore \quad \theta = \frac{2k\pi}{2n+1}$ for $k = 0, 1, 2, \dots, 2n$

So the roots are

$z = cis\,0, cis\frac{2\pi}{2n+1}, cis\frac{4\pi}{2n+1}, \dots, cis\frac{4n\pi}{2n+1}$.

(ii) $z^{2n+1} - 1 = (z-1)(z^{2n} + z^{2n-1} + \dots + z^2 + z + 1)$

So $(z^{2n} + z^{2n-1} + \dots + z^2 + z + 1)$

$$= \left(z - cis\frac{2\pi}{2n+1}\right)\left(z - cis\frac{4n\pi}{2n+1}\right)\left(z - cis\frac{4\pi}{2n+1}\right)\left(z - cis\frac{(4n-2)\pi}{2n+1}\right)$$

$$\cdots\left(z - cis\frac{2n\pi}{2n+1}\right)\left(z - cis\frac{(2n+2)\pi}{2n+1}\right)$$

$$= \left(z - cis\frac{2\pi}{2n+1}\right)\left(z - \overline{cis\frac{2\pi}{2n+1}}\right)\left(z - cis\frac{4\pi}{2n+1}\right)\left(z - \overline{cis\frac{4\pi}{2n+1}}\right)$$

$$\cdots\left(z - cis\frac{2n\pi}{2n+1}\right)\left(z - \overline{cis\frac{2n\pi}{2n+1}}\right)$$

$$= \left(z^2 - \left(2\cos\frac{2\pi}{2n+1}\right)z + 1\right)\left(z^2 - \left(2\cos\frac{4\pi}{2n+1}\right)z + 1\right)\cdots\left(z^2 - \left(2\cos\frac{2n\pi}{2n+1}\right)z + 1\right)$$

(iii) Let $x = 1$ in the identity in (ii):

$2n+1 = 2\left(1 - \cos\frac{2\pi}{2n+1}\right) . 2\left(1 - \cos\frac{4\pi}{2n+1}\right) \dots 2\left(1 - \cos\frac{2n\pi}{2n+1}\right)$

$2n+1 = 2^n . 2\sin^2\frac{\pi}{2n+1} \cdot 2\sin^2\frac{2\pi}{2n+1} \dots 2\sin^2\frac{n\pi}{2n+1}$

(since $1 - \cos 2\theta = 2\sin^2\theta$)

$\therefore 2^{2n} \sin^2\frac{\pi}{2n+1} \sin^2\frac{2\pi}{2n+1} \dots \sin^2\frac{n\pi}{2n+1} = 2n+1$

$\therefore 2^n \sin\frac{\pi}{2n+1} \sin\frac{2\pi}{2n+1} \dots \sin\frac{n\pi}{2n+1} = \sqrt{2n+1}$