SOUTH SYDENY HIGH SCHOOL MATHEMATICS ASSESMENT TASK 4 2013

**Section 1 (5 marks) MULTIPLE CHOICE**

1 What is the value of $\int_0^1 xe^{-x^2}\,dx$?

(A) $\frac{1-e}{2e}$ (B) $\frac{e-1}{2e}$

(C) $\frac{2e-1}{e}$ (D) $\frac{1-2e}{e}$

2 Which of the following is an expression for $\int \frac{1}{\sqrt{7-6x-x^2}}dx$?

(A) $\sin^{-1}\left(\frac{x-3}{2}\right)+c$ (B) $\sin^{-1}\left(\frac{x+3}{2}\right)+c$

(C) $\sin^{-1}\left(\frac{x-3}{4}\right)+c$ (D) $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

3 Which of the following is an expression for $\int \frac{x}{\sqrt{16-x^2}}dx$?

(A) $-2\sqrt{16-x^2}+c$ (B) $-\sqrt{16-x^2}+c$

(C) $\frac{1}{2}\sqrt{16-x^2}+c$ (D) $-\frac{1}{2}\sqrt{16-x^2}+c$

4 What is the value of $\int_1^3 x(x-2)^5\,dx$? Use the substitution $u=x-2$.

(A) $\frac{1}{7}$ (B) $\frac{2}{7}$

(C) $\frac{1}{3}$ (D) $\frac{2}{3}$

5 Let $I_n=\int x^n e^{ax}dx$. Which of the following is the correct expression for $I_n$?

(A) $I_n=\frac{x^n e^{ax}}{a}-nI_{n-1}$

(B) $I_n=\frac{x^n e^{ax}}{a}-\frac{n}{a}I_{n-1}$

(C) $I_n=\frac{x^n e^{ax}}{a}+nI_{n-1}$

(D) $I_n=\frac{x^n e^{ax}}{a}+\frac{n}{a}I_{n-1}$

# TABLE OF STANDARD INTEGRALS

$\int x^n dx = \frac{1}{n+1}x^{n+1};\quad n\neq -1$

$\int \frac{1}{x}dx = \ln x,\quad x>0$ Note: $\ln x=\log_e x,\quad x>0$

$\int e^{ax}dx = \frac{1}{a}e^{ax},\quad a\neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$

$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\quad a\neq 0$

$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$

$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

**Question 6** (15 marks) **Marks**

(a) Use integration by parts to find $\int 3xe^x dx$ **2**

(b) (i) Find real numbers $a$, b, and $c$ such that

$$\frac{7x+4}{(x^2+1)(x+2)}=\frac{ax+b}{x^2+1}+\frac{c}{x+2}$$ **3**

(ii) Hence find $\int \frac{7x+4}{(x^2+1)(x+2)}dx$ **2**

(c) Find $\int \cos^3 x dx$ **2**

(d) Find $\int \frac{x^2}{x^2+4}dx$ **2**

(e) Use the substitution $t=\tan\frac{x}{2}$, or otherwise, to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta+2\sin\theta+3}d\theta$ **4**

**Question 7** (15 marks)

(a) (i) Let $I_n=\int_0^{\frac{\pi}{2}} \cos^n t dt$, **3**

Show that $I_n=\left(\frac{n-1}{n}\right)I_{n-2}$ with $n\geq 2$.

(ii) Hence, or otherwise, find the exact value of $I_4$. **2**

(b) Consider the point $P\left(ct,\frac{c}{t}\right)$, where $t\neq 1$, which lies on the rectangular hyperbola $xy=c^2$.

(i) Show that the equation of the tangent to the hyperbola at $P$ is $x+t^2y=2ct$. **2**

(ii) Let the tangent to the hyperbola at $P$ intersect the coordinate axes at $A$ and $B$. Show that $PA=PB$. **2**

(ii) Let the normal to hyperbola at $P$ meet the axes of symmetry of the hyperbola at $C$ and $D$. **3**

Show that $PC=PD=PA$.
[You mat assume that the equation of the normal is $t^3x-ty=c(t^4-1)$.]

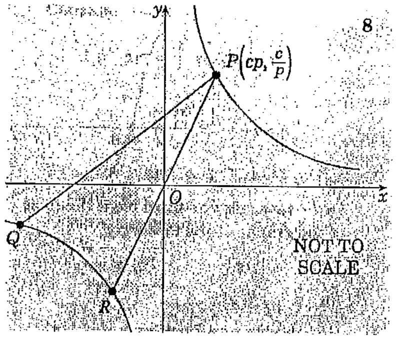
(iii) Sketch a graph of the hyperbola showing the results proved so far. **1**

(iii) Explain why $A$, $B$, $C$ and $D$ must be the vertices of a square **1**

(c) Find $\int \frac{\tan^{-1}x}{1+x^2}dx$ **1**

**Question 8** (15 marks)

(a)




The point $P\left(cp,\frac{c}{p}\right)$ where $p\neq\pm1$, is a point on the hyperbola $xy=c^2$, and the normal to the hyperbola at $P$ intersects the second branch at $Q$. The line through $P$ and the origin $O$ intersects the second branch at $R$.

(i) Show that the equation of the normal is at $P$ is $py-c=p^3(x-cp)$. **2**

(ii) Show that the $x$ coordinates of $P$ and $Q$ satisfy the equation **2**

$$x^2-c\left(p-\frac{1}{p^3}\right)x-\frac{c^2}{p^2}=0$$

(iii) Find the coordinates of $Q$, and deduce that the $\angle QRP$ is a right angle. **3**

(b) (i) Use the substitution $u=a-x$, where a is a constant, to show that **2**

$$\int_0^a f(x)dx=\int_0^a f(a-x)dx$$

(ii) Hence, or otherwise show that $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}dx=\frac{\pi}{4}$. **2**

(c) (i) Use the substitution $u=-x$, to show that $\int_{-a}^{a} f(x)dx=\int_0^a [f(x)+f(-x)]dx$ **2**

(ii) Hence or otherwise, show that $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1+\sin x}=2\int_0^{\frac{\pi}{4}} \sec^2 x dx=2$ **2**

# Extension 2 Task 4 Solutions 2013

**Multiple Choice**

1. B  2. D  3. B  4. D  5. B

**Question 6**

(a) $\int 3xe^{x}\,dx = uv - \int vu'$ ($u = 3x$, $v' = e^x$)

$= 3xe^{x} - 3\int e^{x}\,dx$

$= 3xe^{x} - 3e^{x} + C$

$= \underline{3e^{x}(x-1) + C}$ 2

b) i) $\dfrac{7x+4}{(x^2+1)(x+2)} = \dfrac{ax+b}{x^2+1} + \dfrac{c}{x+2}$

$7x+4 = (ax+b)(x+2) + c(x^2+1)$

when $x=-2$: $-14+4 = 5c$, $-10 = 5c \Longrightarrow \boxed{c=-2}$

when $x=0$: $4 = 2b-2$, $6 = 2b \Longrightarrow \boxed{b=3}$

when $x=1$: $11 = (a+3)3 - 4$, $11 = 3a+9-4$, $6 = 3a \Longrightarrow \boxed{a=2}$ 3

ii) $\int \dfrac{7x+4}{(x^2+1)(x+2)}\,dx = \int \dfrac{2x+3}{x^2+1}\,dx - \int \dfrac{2}{x+2}\,dx$

$= \ln(x^2+1) + 3\tan^{-1}x - 2\ln(x+1) + C$ 2

c) $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$

let $u = \sin x$, $du = \cos x\,dx$

$= \int (1-\sin^2 x)\cos x\,dx$

$= \int (1-u^2)\,du$

$= u - \dfrac{u^3}{3} + C$

$= \underline{\sin x - \dfrac{\sin^3 x}{3} + C}$ 2

d) $\int \dfrac{x^2}{x^2+4}\,dx = \int \dfrac{x^2+4-4}{x^2+4}\,dx$

$= \int \left(1 - \dfrac{4}{x^2+4}\right)dx$

$= x - \dfrac{4}{2}\tan^{-1}\dfrac{x}{2} + C$

$= \underline{x - 2\tan^{-1}\dfrac{x}{2} + C}$ 2

e) $\displaystyle\int_0^{\pi/2} \frac{1}{\cos\theta + 2\sin\theta + 3}\,d\theta = \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + \frac{4t}{1+t^2} + 3}\cdot\frac{2}{1+t^2}\,dt$

$t = \tan\dfrac{\theta}{2}$

$\dfrac{dt}{d\theta} = \dfrac{1}{2}\sec^2\dfrac{\theta}{2}$

$\dfrac{d\theta}{dt} = \dfrac{2}{\sec^2\frac{\theta}{2}}$

$d\theta = \dfrac{2}{1+t^2}\,dt$

(since $\sec^2\frac{\theta}{2} = 1 + \tan^2\frac{\theta}{2}$)

when $\theta = \frac{\pi}{2}$, $t = 1$; $\theta = 0$, $t = 0$

$\displaystyle = \int_0^1 \frac{2/(1+t^2)}{\frac{1-t^2+4t+3+3t^2}{1+t^2}}$

$\displaystyle = \int_0^1 \frac{2}{4+4t+2t^2}\,dt$

$\displaystyle = \int_0^1 \frac{2}{2(t^2+2t+2)}\,dt$

$\displaystyle = \int_0^1 \frac{dt}{t^2+2t+2}$

$\displaystyle = \int_0^1 \frac{dt}{(t+1)^2+1}$

$= \left[\tan^{-1}(t+1)\right]_0^1$

$= \tan^{-1}2 - \tan^{-1}1$

$= \underline{\tan^{-1}2 - \frac{\pi}{4}}$ 4

**Question 7**

a) i) $\displaystyle I_n = \int_0^{\pi/2} \cos^n t\,dt$

$\displaystyle = \int_0^{\pi/2} \cos t \cdot \cos^{n-1} t\,dt$ ($v' = \cos t$, $u = \cos^{n-1}t$)

$= uv - \int vu'$

$\displaystyle = \left[\sin t \cos^{n-1} t\right]_0^{\pi/2} - \int_0^{\pi/2} \sin t\,(n-1)\cos^{n-2}t\,(-\sin t)$

$\displaystyle = (n-1)\int_0^{\pi/2} \sin^2 t \cos^{n-2} t\,dt$

$\displaystyle I_n = (n-1)\int_0^{\pi/2} (1-\cos^2 t)\cos^{n-2} t\,dt$

$\displaystyle I_n = (n-1)\int_0^{\pi/2} (\cos^{n-2} t - \cos^n t)\,dt$

$I_n = (n-1)\left[I_{n-2} - I_n\right]$

$I_n = nI_{n-2} - nI_n - I_{n-2} + I_n$

$0 = nI_{n-2} - nI_n - I_{n-2}$

$nI_n = I_{n-2}(n-1)$

$I_n = \dfrac{n-1}{n} I_{n-2}$ 3

ii) $I_4 = \dfrac{3}{4} I_2$

$\displaystyle = \frac{3}{4}\int_0^{\pi/2} \cos^2 t\,dt$

$\displaystyle = \frac{3}{4}\int_0^{\pi/2} \frac{1}{2}(1+\cos 2t)\,dt$

$= \dfrac{3}{8}\left[t + \dfrac{1}{2}\sin 2t\right]_0^{\pi/2}$

$= \dfrac{3}{8}\left[\dfrac{\pi}{2} + 0 - 0\right] = \underline{\dfrac{3\pi}{16}}$ 2

b) (i) $x = ct$, $y = \dfrac{c}{t}$

$\dfrac{dx}{dt} = c$, $\dfrac{dy}{dt} = -\dfrac{c}{t^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx}$

$= -\dfrac{c}{t^2}\cdot\dfrac{1}{c}$

$= -\dfrac{1}{t^2}$

$m_T = -\dfrac{1}{t^2}$

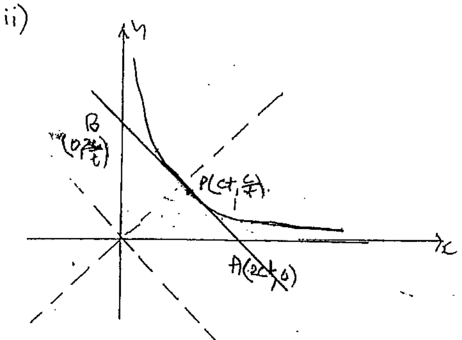
$\therefore$ Reqd eqn: $y - y_1 = m(x - x_1)$

$y - \dfrac{c}{t} = -\dfrac{1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$

$\underline{x + t^2y = 2ct}$ 2

ii)

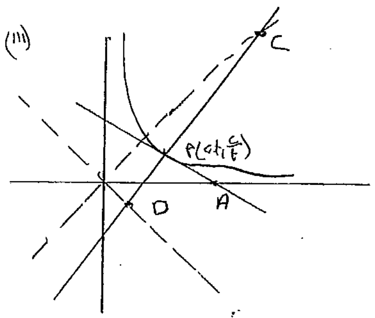



At A: $y = 0$, $\therefore x = 2ct$ At B: $x = 0$, $\therefore y = \dfrac{2c}{t}$

Now $PA^2 = (ct)^2 + \dfrac{c^2}{t^2}$ $PB^2 = (ct)^2 + \dfrac{c^2}{t^2}$

$\therefore$ $\underline{PA = PB}$ 2

(iii)




Eqn of PC is $t^3x - ty = c(t^4 - 1)$

C lies on $y = x$

$\therefore t^3x - tx = c(t^4-1)$

$x = \dfrac{c(t^4-1)}{t(t^2-1)}$

$= \dfrac{c(t^2+1)}{t}$

$\therefore C\left[\dfrac{c(t^2+1)}{t}, \dfrac{c(t^2+1)}{t}\right]$

D lies on $y = -x$

$\therefore t^3x + tx = c(t^4-1)$

$x = \dfrac{c(t^2-1)}{t}$

$\therefore D\left[\dfrac{c(t^2-1)}{t}, \dfrac{c(t^2+1)}{t}\right]$

$$PC^2 = \left[ct - \frac{c(t^2+1)}{t}\right]^2 + \left[\frac{c}{t} - \frac{c(t^2+1)}{t}\right]^2$$

$$= \left[\frac{ct^2 - ct^2 - c}{t}\right]^2 + \left[\frac{c - ct^2 - c}{t}\right]^2$$

$$\therefore = \frac{c^2}{t^2} + c^2t^2$$

$$PD^2 = \left[ct - \frac{c(t^2-1)}{t}\right]^2 + \left[\frac{c}{t} - \frac{c(t^2-1)}{t}\right]^2$$

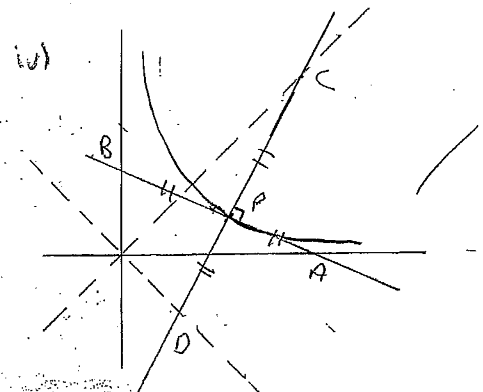
$$= \left[\frac{ct^2 - ct^2 + c}{t}\right]^2 + \left[\frac{c - ct^2 + c}{t}\right]^2$$

3 $$= \frac{c^2}{t^2} + c^2t^2$$

$$PA = \frac{c}{t^2} + c^2t^2$$

$\therefore$ $PC = PD = PA$

iv)



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v) The diagonals AB, CD are equal and bisect each other at right angles

1 Hence ABCD is a square

c) $\displaystyle\int \frac{\tan^{-1}x}{1+x^2}\,dx = \frac{[\tan^{-1}x]^2}{2}$

1

**Question 8**

a) i) $m_T = -\frac{1}{p^2}$ (derived earlier)

$m_N = p^2$

Reqd eqn: $y - y_1 = m(x - x_1)$

$$y - \frac{c}{p} = p^2(x - cp)$$

$$py - c = p^3(x - cp)$$

ii) The $x$ coordinates of P and Q are the solution of simultaneous eqns

$$py - c = p^3(x - cp) \quad \text{— (1)}$$

$$xy = c^2 \quad \text{— (2)}$$

$$py - c = p^3x - cp^4$$

$$py - c - p^3x + cp^4 = 0$$

$$p\left(\frac{c^2}{x}\right) - c - p^3x + cp^4 = 0 \quad \text{since } y = \frac{c^2}{x}$$

$$pc^2 - cx - p^3x^2 + cp^4x = 0$$

$$-pc^2 + cx + p^3x^2 - cp^4x = 0$$

$$p^3x^2 + c(1-p^4)x - pc^2 = 0$$

$$x^2 + c\left(\frac{1-p^4}{p^3}\right)x - \frac{pc^2}{p^3} = 0$$

$$x^2 + c\left(\frac{1}{p^3} - p\right)x - \frac{c^2}{p^2} = 0$$

$$x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0$$

iii) One of the roots of above eqn is $cp$

Prod of roots: $cp \times \beta = \frac{c}{a}$

$$cp\,\beta = -\frac{c^2}{p^2}$$

$$\beta = -\frac{c^2}{p^2} \times \frac{1}{cp}$$

$$= -\frac{c}{p^3}$$

when $x = -\frac{c}{p^3}$ $\longrightarrow$ $-\frac{c}{p^3}y = c^2$

$$-cy = p^3c^2$$

$$y = -p^3c$$

$\therefore$ Coords of Q are $\left(-\frac{c}{p^3}, -cp^3\right)$

$$m_{RP} = \frac{c/p}{cp} = \frac{1}{p^2}$$

Coords of R: $\left[-cp, -\frac{c}{p}\right]$

$$\therefore m_{QR} = \frac{-cp^3 - \left(-\frac{c}{p}\right)}{-\frac{c}{p^3} - (-cp)}$$

$$= \frac{-cp^6 + cp^2}{-c + cp^4}$$

3 $$= \frac{cp^2(1-p^4)}{-c(1-p^4)}$$

$$= -p^2$$

$\therefore$ $m_{QR} \times m_{RP} = -1$

$\therefore$ $\angle QRP$ is right angled.

b) i) $\displaystyle\int_0^a f(x)\,dx = \int_a^0 f(a-u)(-du)$ let $u = a - x$, $du = -dx$

$$= -\int_a^0 f(a-u)\,du \quad \text{when } x = a,\ u = 0;\ x = 0,\ u = a$$

2 $$= \int_0^a f(a-u)\,du$$

$$= \int_0^a f(a-x)\,dx$$

ii) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx = \int_0^{\pi/2} \frac{\cos(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x) + \sin(\frac{\pi}{2}-x)}\,dx$

$$= \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$$

2 $\therefore$ $\displaystyle\int_0^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x}\,dx = \int_0^{\pi/2} dx$

$$= [x]_0^{\pi/2} = \frac{\pi}{2}$$

But this is $2\displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$

$\therefore$ $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx = \frac{\pi}{4}$

c) i) $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$

Now $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_{-a}^0 f(x)\,dx$

Consider $\displaystyle\int_{-a}^0 f(x)\,dx = \int_a^0 f(-u)\cdot -du$ $\quad u = -x$, $du = -dx$

$$= -\int_a^0 f(-u)\,du \quad \text{when } x = 0,\ u = 0;\ x = -a,\ u = a$$

$$= \int_0^a f(-u)\,du$$

$$= \int_0^a f(-x)\,dx$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$$

ii) $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\sin x} = \int_0^{\pi/4} \left[\frac{1}{1+\sin x} + \frac{1}{1+\sin(-x)}\right] dx$

$$= \int_0^{\pi/4} \left[\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right] dx$$

$$= \int_0^{\pi/4} \frac{1-\sin x+1+\sin x}{(1+\sin x)(1-\sin x)}\,dx$$

$$= 2\int_0^{\pi/4} \frac{dx}{1-\sin^2 x}$$

$$= 2\int_0^{\pi/4} \frac{dx}{\cos^2 x}$$

$$= 2\int_0^{\pi/4} \sec^2 x\,dx$$

$$= 2[\tan x]_0^{\pi/4} = 2$$