Number:_____________

St Catherine's School
Waverley

Mathematics Extension 2

St. Catherine's School Waverley

2013

ASSESSMENT TASK 3 (15%)

Total marks – 36

- Attempt all questions
- Marks for each question are indicated on this page

General Instructions

- Working time – 55 minutes
Reading Time: 3 minutes

- Write using black or blue pen only.
- Board-approved calculators may be used.
- All necessary **working** must be shown.
- Marks may be deducted for careless or badly arranged work.
- Standard Integrals is attached as the last page,

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0 \qquad \text{Note: } \ln x = \log_e x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

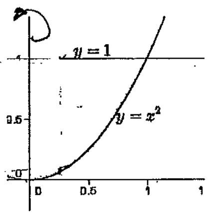
$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Circle the right answer.

1 The region bounded by $y = x^2$, the line y=1 and the y axis is rotated about the y axis. The volume of the solid is given by **1m**



A. $2\pi \int_0^1 xy\,dy$ B. $2\pi \int_0^1 xy\,dx$

C. $2\pi \int_0^1 x(1-x^2)\,dx$ D. $2\pi \int_0^1 x\sqrt{x}\,dx$

2 **1m**

$$\int \frac{sin^{-1}x}{\sqrt{1-x^2}}dx =$$

A. $\frac{sin^{-1}x^2}{2} + c$ B. $\frac{(sin^{-1}x)^2}{2} + c$

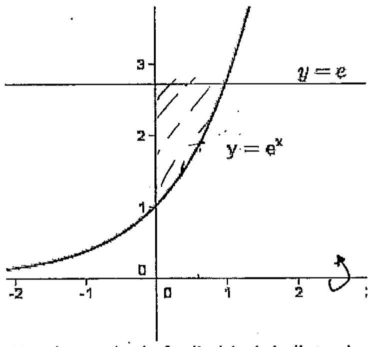
C. $sin^{-1}(\sqrt{1-x^2}+C$ D. $\sqrt{sin^{-1}x}$

3 a. $\int \frac{x}{x^2-4x+8}\,dx$ **4m**

b. $\int sin^3\theta\,d\theta$ **2m**

c. Use integration by parts twice to show that **4m**
$\int_0^{\pi} e^x \sin x\,dx = \frac{e^{\pi}+1}{2}$

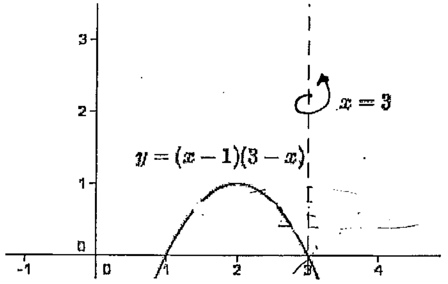
4 The area bound by $y = e^x$, the y axis and the line $y = e$ is rotated about the x axis



(i) Use the method of cylindrical shells to show that the volume of the solid generated is given by the integral **2m**
$V = 2\pi \int_1^e y\,log_e\,y\ dy$

(ii) Hence find the volume of the solid **3m**

5 The region bounded by the curve $y = (x-1)(3-x)$ and the $x - axis$ is rotated about the line $x = 3$.



By considering slices perpendicular to the axis of rotation, show that the volume of the solid is given by

i) $V = 4\pi \int_0^1 \sqrt{1-y}\, dy$ **4m**

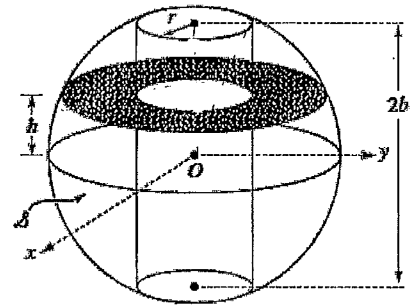
(ii) Hence find the volume of the solid. **2m**

6 A sequence u_n is defined by $u_0 = u_1 = 2$ and $u_n = 2u_{n-1} + u_{n-2}$, **4m**

Use mathematical Induction to show that

$u_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$, for all $n \geq 0$

7 A cylindrical hole of radius r is bored through a sphere of radius R. The hole is perpendicular to the xy-plane and its axis passes through the origin O., which is at the centre of the sphere. The resulting solid is denoted by S.
The cross section of the solid S at a height h from the xy plane is shown in the diagram.



(i) Show that the area of cross section shown above is given by $\pi(R^2 - h^2 - r^2)$ **2m**

(ii) If the length of the hole is 2b, show that the volume of the solid S is $\frac{4\pi b^3}{3}$ **3m**

8 (i) Show that $x^{n-1}(1+x)^{\frac{3}{2}} = (1+x)^{\frac{1}{2}}(x^{n-1} + x^n)$ **1m**

(ii) If $I_n = \int_{-1}^{0} x^n (1+x)^{\frac{1}{2}}\, dx$, show that **3m**

$$I_n = \frac{-2n}{2n+3} I_{n-1}$$

END of TASK.

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4 page writing booklet

3 a) $\int \frac{x}{x^2-4x+8}\,dx$ $\quad \frac{x-2}{2x-4}$ $\quad \frac{1}{2}\left(\frac{x-2}{x-2}\right)$

$= \int \frac{x-2}{x^2-4x+8} + \frac{2}{(x^2-4x+4)+4}\,dx$

$= \frac{1}{2}\ln(x^2-4x+8) + 2\int \frac{1}{(x-2)^2+4}\,dx$ ✓ 4

~~$= \frac{1}{2}\ln(x^2-4x+8) + \tan^{-1}\left(\frac{x-2}{4}\right) + C$~~

$= \frac{1}{2}\ln(x^2-4x+8) + \tan^{-1}\left(\frac{x-2}{2}\right) + C$

b) $\int \sin^3\theta\, d\theta$

$= \int \sin\theta(1-\cos^2\theta)\,d\theta$ 2

$= \int \sin\theta - \cos^2\theta\sin\theta\, d\theta$ ✓

$= -\cos\theta + \frac{\cos^3\theta}{3} + C$

$= \frac{\cos^3\theta}{3} - \cos\theta + C$

c) $\int_0^\pi e^x \sin x\,dx$ $\quad$ let $u = \cancel{e^x}\sin x \quad v' = e^x$; $u' = \cancel{e^x}\cos x \quad v = e^x$

$= [e^x \sin x]_0^\pi - \int_0^\pi e^x \cos x\,dx$ $\quad \int uv' = uv - \int u'v$

$= -[(e^x\cos x)_0^\pi + \int_0^\pi e^x \sin x\,dx]$ $\quad$ let $u = \cos x \quad v' = e^x$; $u' = -\sin x \quad v = e^x$

$= -(e^\pi \times -1 - 1) - I$

$2I = e^\pi + 1$

$I = \frac{e^\pi+1}{2}$ ✓ 4

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4. i $\Delta V = 2\pi\, x y\, dy$ $\quad y = e^x$

$= 2\pi\, y\ln y\, dy$ $\quad x = \ln y$

$\therefore V = 2\pi\int_1^e y\ln y\, dy$ ✓

ii $V = 2\pi\int_1^e y\ln y\, dy$ $\quad u = \ln y \quad v' = y$; $u' = \frac{1}{y} \quad v = \frac{y^2}{2}$

$V = 2\pi\left[\left(\ln y \cdot \frac{y^2}{2}\right)_1^e - \int_1^e \frac{y}{2}\,dy\right]$ $\quad \int uv' = uv - \int u'v$

$= 2\pi\left[\frac{e^2}{2} - \left(\frac{y^2}{4}\right)_1^e\right]$

$= 2\pi\left[\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}\right]$

$= 2\pi\left(\frac{e^2}{4} + \frac{1}{4}\right)$ ✓ 5

~~$= \frac{\pi}{2}(e^2+1)$~~

$= \pi\left(\frac{e^2}{2} + \frac{1}{2}\right)$

$= \frac{\pi}{2}(e^2+1)\ u^3$

(5)(i) $\Delta V = \pi((3-x_1)^2 - (3-x_2)^2)\,\Delta y$

$= \pi(9 - 6x_1 + x_1^2 - 9 + 6x_2 - x_2^2)\,\Delta y$

$= \pi(6(x_2-x_1) + (x_1^2 - x_2^2))\,\Delta y$

~~$= \pi(6(x_2-x_1) + ($~~ ✓

$= \pi(6(x_2-x_1) - (x_2-x_1)(x_1+x_2))\,\Delta y$

$x^2 - 4x + 3 + y = 0$ $\quad x_1$ & x_2 are roots

$x_1 + x_2 = 4$

$x_1 x_2 = 3 + y$

$(x_2-x_1)^2 = (x_1+x_2)^2 - 4x_1x_2$

$= 4 - 4y$

$x_2 - x_1 = 2\sqrt{1-y}$ ✓

~~$\Delta V = \pi(8\sqrt{1-y} - 4\sqrt{1-y})$~~ $\quad \Delta V = (12\sqrt{1-y} - 8\sqrt{1-y})\,\Delta y$

~~$= 4\pi\sqrt{1-y}$~~ $\quad = 4\pi\sqrt{1-y}\ \Delta y$

when $x = 2$ $\quad y = 1$ ✓

$\therefore V = 4\pi\int_0^1 \sqrt{1-y}\,dy$

5 ii $V = 4\pi \int_0^1 \sqrt{1-y}\,dy$

$= 4\pi\left[-\frac{2}{3}(1-y)^{\frac{3}{2}}\right]_0^1$

$= 4\pi\left(-\frac{2}{3} \times -1\right)$

$= \frac{8\pi}{3}\ u^3$ 2

6. $u_0 = u_1 = 2$ $u_n = 2u_{n-1} + u_{n-2}$

Show $u_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$ $n \geq 0$

~~Consider P(0):~~ let $P(n) = u_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$

Consider $P(0)$: $u_0 = 1+1$

$= 2$ (given) ∴ true for $n=0$

Consider $P(1)$: $u_1 = 1+\sqrt{2}+1-\sqrt{2}$

$= 2$ (given) ∴ true for $n=1$

Assume true for $n \leq k$

∴ $P(k) = (1+\sqrt{2})^k + (1-\sqrt{2})^k$

$P(k-1) = (1+\sqrt{2})^{k-1} + (1-\sqrt{2})^{k-1}$ $\frac{4}{4}$

Consider $P(k+1)$

~~$P(k+1) = (1+\sqrt{2})^{k+1} + (1-\sqrt{2})^k$~~

$P(k+1) = 2u_k + u_{k-1}$

(from the assumption) $= 2\left[(1+\sqrt{2})^k + (1-\sqrt{2})^k\right] + (1+\sqrt{2})^{k-1} + (1-\sqrt{2})^{k-1}$

$= 2(1+\sqrt{2})^k + 2(1-\sqrt{2})^k + \frac{(1+\sqrt{2})^k}{1+\sqrt{2}} + \frac{(1-\sqrt{2})^k}{1-\sqrt{2}}$

$\frac{(1-\sqrt{2})(1+\sqrt{2})}{1-\sqrt{2}+\sqrt{2}-2}$

$= (1+\sqrt{2})^k\left(2+\frac{1}{1+\sqrt{2}}\right) + (1-\sqrt{2})^k\left(2+\frac{1}{1-\sqrt{2}}\right)$

$= (1+\sqrt{2})^k\left(2+\frac{1-\sqrt{2}}{1-2}\right) + (1-\sqrt{2})^k\left(2+\frac{1+\sqrt{2}}{-1}\right)$

$= (1+\sqrt{2})^k(2-1+\sqrt{2}) + (1-\sqrt{2})^k(2-1-\sqrt{2})$

$= (1+\sqrt{2})^k(1+\sqrt{2}) + (1-\sqrt{2})^k(1-\sqrt{2})$

$= (1+\sqrt{2})^{k+1} + (1-\sqrt{2})^{k+1}$ ✓

∴ true for $P(k+1)$ if true for $P(k)$, since true for $n=0$, $n=1$, by PMI true for all $n \geq 0$

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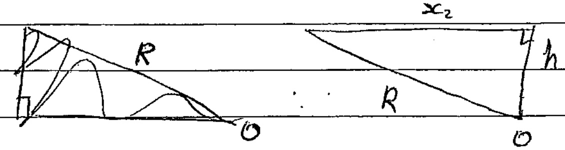
7 i ~~Area of CS =~~

let the larger radius $= x_2$, the smaller radius $= x_1$



$R^2 = (x_2)^2 + h^2$

$(x_2)^2 = R^2 - h^2$

$x_1 = r$ (given)

$(x_1)^2 = r^2$

Area $= \pi(x_2^2 - x_1^2)$

$= \pi(R^2 - h^2 - r^2)$ ✓ 2.

ii $V = \pi \int_0^{2b} R^2 - h^2 - r^2\,dh$

~~$= \pi\left[(R^2-r^2)h - \frac{h^3}{3}\right]_0^{2b}$~~ ✓

~~$= \pi\left[(R^2-r^2)2b - \frac{8b^3}{3}\right]$~~

~~$= 2\pi\left[(R^2-r^2)b - \frac{4b^3}{3}\right]$~~ ✓

~~$= 2\pi\left[b^3 - \frac{4b^3}{3}\right]$~~ ✓

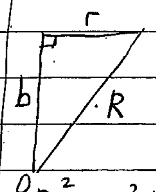
$= \pi\left[(R^2-r^2)h - \frac{h^3}{3}\right]_{-b}^{b}$ ✓

$= \pi\left[(R^2-r^2)b - \frac{b^3}{3} + (R^2-r^2)b - \frac{b^3}{3}\right]$

$= 2\pi\left[(R^2-r^2)b - \frac{b^3}{3}\right]$

$= 2\pi\left[b^3 - \frac{b^3}{3}\right]$

$= \frac{4\pi b^3}{3}$ ✓



$R^2 = r^2 + b^2$

$b^2 = R^2 - r^2$ ✓

good 3.

8 i $\cancel{x^{n-1}}$ $LHS = x^{n-1}(1+x)^{\frac{3}{2}}$

$\cancel{= x^{n-1}(1}$

$= x^{n-1}(1+x)\sqrt{1+x}$

$= (x^{n-1}+x^{n})(1+x^{\frac{1}{2}})$

$= RHS$

ii $I_n = \int_{-1}^{0} x^n (1+x)^{\frac{1}{2}}\,dx$ let $u = x^n$ $v' = (1+x)^{\frac{1}{2}}$

$u' = nx^{n-1}$ $v = \frac{2}{3}(1+x)^{\frac{3}{2}}$

$I_n = \left[\frac{2}{3}x^n(1+x)^{\frac{3}{2}}\right]_{-1}^{0} - \int_{-1}^{0} \frac{2}{3} n x^{n-1}(1+x)^{\frac{3}{2}}\,dx$ $\int uv' = uv - \int u'v$

$= -\int_{-1}^{0} \frac{2}{3} n (x^n + x^{n-1})(1+x)^{\frac{1}{2}}\,dx$

$= -\frac{2}{3}n\int_{-1}^{0} x^n(1+x)^{\frac{1}{2}} + x^{n-1}(1+x)^{\frac{1}{2}}\,dx$

$= -\frac{2}{3}n\,I_n - \frac{2}{3}n\,I_{n-1}$

$\left(\frac{2}{3}n+1\right) I_n = -\frac{2n}{3} I_{n-1}$

$\cancel{\frac{2n+3}{3} I_n =}$

$\frac{2n+3}{3} I_n = -\frac{2n}{3} I_{n-1}$

$I_n = -\frac{2n}{2n+3} I_{n-1}$

2.

good