SYDNEY TECHNICAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 3

JUNE 2013

Mathematics

General Instuctions

- Working time - 70 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in questions 6 to 13
- Start each question on a new page
- A table of standard integrals is provided at the back of the paper

Total marks - 55

Section 1 - 5 marks

Attempt Questions 1 – 5.
Allow about 7 minutes for this section.

Section 2 - 50 marks

Attempt Questions 6 – 11.
Allow about 63 minutes for this section.

Name : ______________________

Teacher : ______________________

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section 1

5 marks
Attempt Questions 1 – 5
Allow about 7 minutes for this section

Use the multiple-choice answer sheet in your answer booklet for Questions 1 – 5.
Do not remove the multiple-choice answer sheet from your answer booklet.

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1. What is the period of the function $y = 5 - 3\cos 2x$?

A) 3

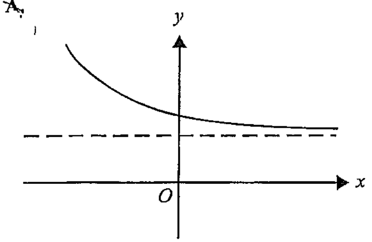
B) 5

C) 4π

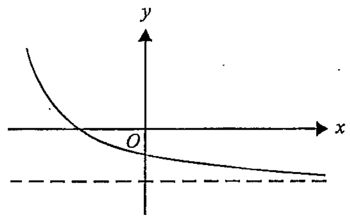
D) π

2. If k is a negative real number and P is a positive real number, which one of the following is most likely to be the graph of the function with equation $y = e^{kx} + P$?

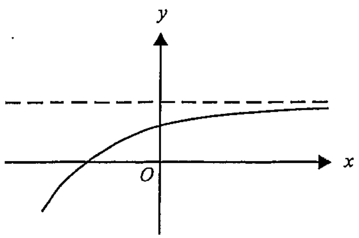
A.



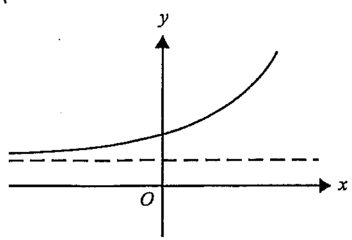
B.



C.



D.



3. If $\int_0^a \sec^2 2x \, dx = \frac{1}{2}$, then a is equal to

A. $\frac{\pi}{4}$

B. $\frac{\pi}{8}$

C. $\frac{\pi}{12}$

D. $\frac{\pi}{2}$

4. If $g(t) = e^{-t} - 1$ then $g'(0)$ equals

A. $-e$

B. -2

C. -1

D. 0

5. If $\int_1^3 f(x)\, dx = 5$ then $\int_1^3 (2f(x) - 3)\, dx$ is equal to

A. 4

B. 5

C. 7

D. 10

Section 2

50 marks
Attempt Questions 6 – 10
Allow about 63 minutes for this section
Start each question on a new page

Question 6 (10 marks)

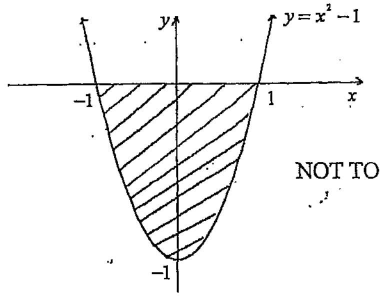
a) Evaluate $e^{-2.8}$ giving your answer correct to 2 significant figures. 2

b) Find the exact value of $\cos\frac{5\pi}{6}$. 1

c) Sketch the graph of $y = 1 - \cos x$ for $0 \le x \le 2\pi$. 2

d) Evaluate $\int_0^{\frac{\pi}{6}} \sin 2x \; dx$ 2

e) The area bounded by $y = x^2 - 1$ and the x - axis is rotated about the y - axis . Find the volume of the solid of revolution formed. 3



NOT TO SCALE

Question 7 (10 marks) (Start a new page)

a) Solve $\sqrt{3}\,\tan\theta - 1 = 0$ for $0 \le \theta \le 2\pi$. 2

b) Differentiate with respect to x.

i) $\tan 3x + \sin x$ 2

ii) $(x+2)e^{2x}$ 2

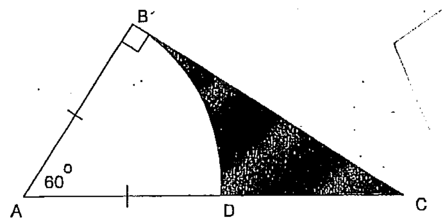
c) Consider the function $f(x) = \frac{x^2}{x+4}$

i) Use the trapezoidal rule with 4 function values to approximate $\int_1^7 f(x)\,dx$, giving your answer correct to 1 decimal place. 3

ii) For all values of x , between 1 and 7 , $f(x) > 0$, $f'(x) > 0$ and $f''(x) > 0$.
Use this information to decide whether the approximation found in part i) is an over-estimate or an under-estimate of the true value of the integral. Give a brief reason. 1

Question 8 (10 marks) (Start a new page)

a)



In the diagram above, angle B = 90°, angle A = 60° and $AB = AD = 10\ m$.

BD is an arc of the circle with centre A.

i) Calculate the exact length of the arc BD. 1

ii) Calculate the shaded area in exact form. 3

b) The area bounded by $y = x^2$ and $y = 4$ is rotated about the $x-axis$ to form a solid. Find the volume of the solid. 3

c) Find the equation of the tangent to $y = e^{4x} + x$ at the point where $x = 0$. 3

Question 9 (10 marks) (Start a new page)

a) i) Draw a neat sketch of the curve 2

$y = 3\sin\frac{x}{2}$ for $-2\pi \le x \le 2\pi$,

showing clearly all the important features.

ii) Draw on your diagram a line, clearly labelled, which can be used to solve the following equation: 1

$3\sin\frac{x}{2} - x - 1 = 0$

iii) Determine the number of solutions the equation 1

$3\sin\frac{x}{2} - x - 1 = 0$ has over the domain $-2\pi \le x \le 2\pi$.

b) i) Show that $\frac{d^2}{dx^2}(e^x \sin x) = 2e^x \cos x$ 2

ii) Hence find $\int e^x \cos x\ dx$ 2

c) For what values of k does $y = 3e^{kx}$ satisfy the equation $\frac{d^2y}{dx^2} - 9y = 0$? 2

Question 10 (10 marks) (Start a new page)

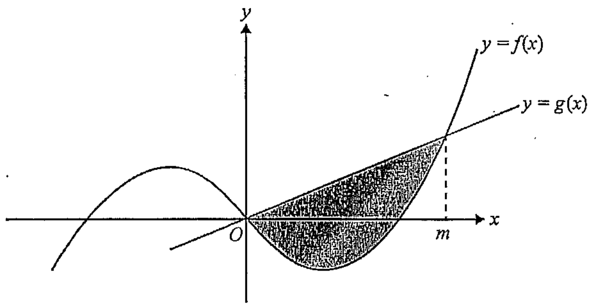
a) A function $f(x)$ is defined by $f(x) = \frac{e^{-x}}{x}$.

i) Differentiate $f(x)$ with respect to x. 2

ii) Find the coordinates of any stationary points of the graph of $y = f(x)$, and determine their nature. 2

iii) Sketch the graph of $y = f(x)$, showing all important features. (Not inflexion points) 2

b)



Parts of the graphs of the functions $f(x) = x^3 - ax$, $a > 0$ and

$$g(x) = ax, a > 0$$

are shown in the diagram above.

The graphs intersect when $x = 0$ and when $x = m$. $(m \neq 0)$

i) Show that $m^2 = 2a$. 2

ii) If the area of the shaded region is 64 square units, find the value of a and m. 2

End of Paper

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MULTIPLE CHOICE ANSWER SHEET

Name : SOLUTIONS:

Teacher:

Completely fill the response oval representing the most correct answer.

Do not remove this sheet from the answer booklet.

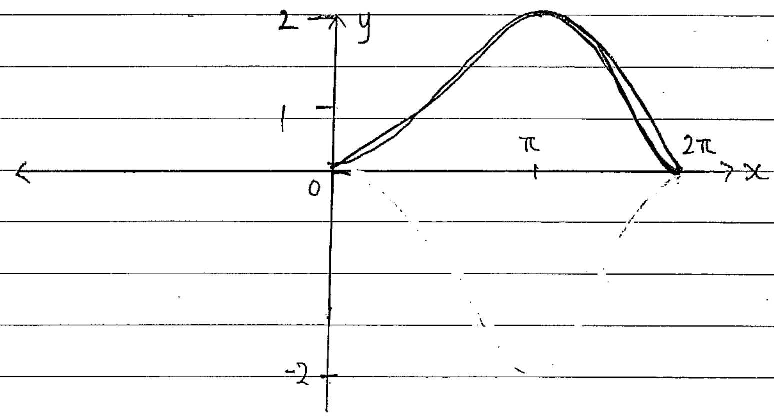
1. A ○ B ○ C ○ D ● ✓
2. A ● B ○ C ○ D ○ ✓
3. A ○ B ● C ○ D ○ ✓
4. A ○ B ○ C ● D ○ ✓
5. A ● B ○ C ○ D ○ ✓

Question 6

a) $e^{-2.8} = 0.061$ (2 sig. figs.) √ X

b) $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ ✓

c)



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d) $\int_0^{\frac{\pi}{6}} \sin 2x \, dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}}$

$= \left[\left(-\frac{1}{4}\right) - \left(-\frac{1}{2}\right)\right]$

$= \frac{1}{4}$ // ✓✓

e) $V_y = \pi \int_{-1}^{0} x^2 \, dy$

$= \pi \int_{-1}^{0} y+1 \, dy$

$= \pi \left[\frac{y^2}{2} + y\right]_{-1}^{0}$

$= \pi \left[0 - \left(\frac{1}{2} - 1\right)\right]$ ✓✓✓

$= \frac{\pi}{2}$ units^3 //

9

Question 7

a) $\sqrt{3}\tan\theta - 1 = 0$

$\tan\theta = \frac{1}{\sqrt{3}}$ S | A ✓ / ✓T | C

$\theta = \frac{\pi}{6}, \frac{7\pi}{6}$ // ✓ 2

b) i) $y = \tan 3x + \sin x$

$\frac{dy}{dx} = 3\sec^2 3x + \cos x$ // ✓ 2

ii) $y = (x+2)e^{2x}$ $u = x+2$ $v = e^{2x}$

$u' = 1$ $v' = 2e^{2x}$

$\frac{dy}{dx} = e^{2x} + 2e^{2x}(x+2)$

$= e^{2x}(1 + 2(x+2))$

$= e^{2x}(1 + 2x + 4)$

$= e^{2x}(2x+5)$ // ✓ 2

c) i)

x	1	3	5	7
$f(x)$ ~~y~~	1/5	9/7	25/9	49/11

$A_{\text{Trapezoidal}} = \frac{2}{2}\left[\frac{1}{5} + \frac{49}{11} + 2\left(\frac{9}{7} + \frac{25}{9}\right)\right]$

$= 12.8$ units^2 // ✓ 3

ii) It is an ~~under~~ over estimation because $f'(x) > 0$ i.e gradient is more than zero. $f(x)$ is an increasing function $+ f''(x) > 0$ ~~1~~

Question 8

a) i) $l = \theta^c r$

$60^\circ \times \frac{\pi}{180} = \frac{\pi}{3}$

$l = \frac{\pi}{3} \times 10$

$= \frac{10}{3}\pi$ m

1

ii) $BC = \frac{10}{\tan 30}$

$= 10\sqrt{3}$

~~Area of triangle~~ $= \frac{1}{2} \times 10\sqrt{3} \times 10$

$= 50\sqrt{3}$

~~Area of sector~~ $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2}(10)^2 \frac{\pi}{3}$

$= \frac{50}{3}\pi$

~~Shaded area~~ $= \left(50\sqrt{3} - \frac{50}{3}\pi\right) m^2$

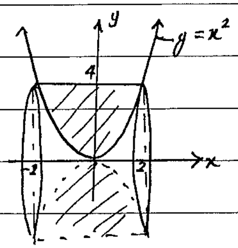
3

b) $y = x^2$

$4 = x^2$

$x = \pm 2$

$A_x = \cancel{2\pi \int_0^2} \; \pi \int_{-2}^{2} (x^2)^2 \, dx$



Shaded volume = Vol. of cylinder $- 2\int_0^2 \pi y^2 \, dx$

$= \pi(4)^2 \times 4 - 2\pi \int_0^2 x^4 \, dx$

$= 64\pi - 2\pi \left[\frac{x^5}{5}\right]_0^2$

$= 64\pi - 2\pi\left[\frac{32}{5}\right]$

$= \frac{4 \times 64\pi}{5}$ cu units

$= \frac{256}{5}\pi$ unit^3

$= 4$?

1

c) $y = e^{4x} + x$ sub $x = 0$

$y = e^0 + 0$

$= 1$

$(0, 1)$

$\frac{dy}{dx} = 4e^{4x} + 1$ ~~sub $x=0$~~

$= 4 + 1$

$= 5$

$y - 1 = 5(x - 0)$

$\therefore y = 5x + 1$

3

8

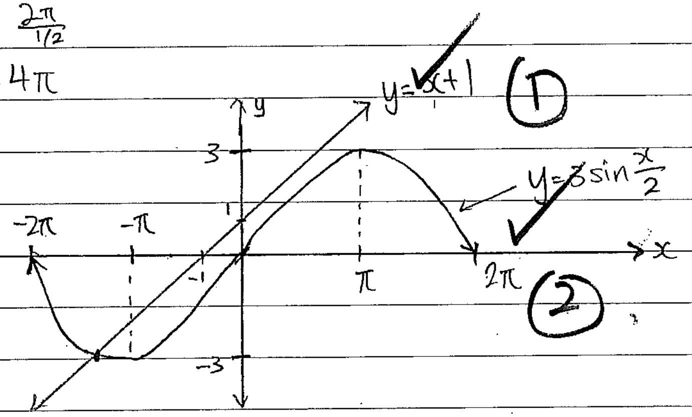
Question 9

a) i) Amp = 3

Period $= \frac{2\pi}{1/2}$

$= 4\pi$

ii)



(1)

(2)

iii) It has one solution (1)

$\frac{10}{10}$

b) i) $\frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x$ $\quad U = e^x \quad V = \sin x$ $\quad U' = e^x \quad V' = \cos x$

$= e^x(\sin x + \cos x)$

$\frac{d}{dx}(e^x(\sin x + \cos x)) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$ $\quad U = e^x \quad V = \sin x + \cos x$ $\quad U' = e^x \quad V' = \cos x - \sin x$

$= \cancel{e^x \sin x} + e^x \cos x + e^x \cos x - \cancel{e^x \sin x}$

$= 2e^x \cos x$ (2)

$\therefore \frac{d^2}{dx^2}(e^x \sin x) = 2e^x \cos x$ //

ii) ~~$e^x \cos x = \frac{d}{dx}$ ... $\frac{d}{dx}(e^x(\sin x + \cos x$~~

$\int e^x \cos x = \frac{1}{2} \times \int \frac{d}{dx}(e^x(\sin x + \cos x))$

$\therefore \int e^x \cos x = \frac{1}{2} e^x(\sin x + \cos x) + C$ // (2)

c) $y = 3e^{kx}$

$\frac{dy}{dx} = k3e^{kx}$ $\quad U = k \quad V = 3e^{kx}$ $\quad U' = 0 \quad V' = k3e^{kx}$

$\frac{d^2y}{dx^2} = 3k^2 e^{kx}$

$\frac{d^2y}{dx^2} - 9y = 3k^2 e^{kx} - 9(3e^{kx})$

$3k^2e^{kx} - 27e^{kx} = 0$

$\cancel{3(k^2e}\ 3e^{kx}(k^2 - 9) = 0$

$3e^{kx} = 0 \qquad k^2 - 9 = 0$

$e^{kx} = 0 \qquad k^2 = 9$

$k = \pm 3$

$\therefore\ k = \pm 3$ //

2

Question 10

a) i) $f(x) = \dfrac{e^{-x}}{x}$ $D: x \neq 0$ $\qquad u = e^{-x} \qquad v = x$

$u' = -e^{-x} \qquad v' = 1$

$$f'(x) = \frac{-xe^{-x} - e^{-x}}{x^2}$$

$$= \frac{e^{-x}(-x-1)}{x^2}$$ //

ii) St points $f'(x) = 0$

$$\frac{e^{-x}(-x-1)}{x^2} = 0$$

$e^{-x} = 0$	$-x - 1 = 0$
$e^{x} > 0$	$x = -1$
No solutions	

Test concavity

x	-2	-1	1
$f'(x)$	$+$	0	$-$

max

Sub $x = -1$ into $f(x)$

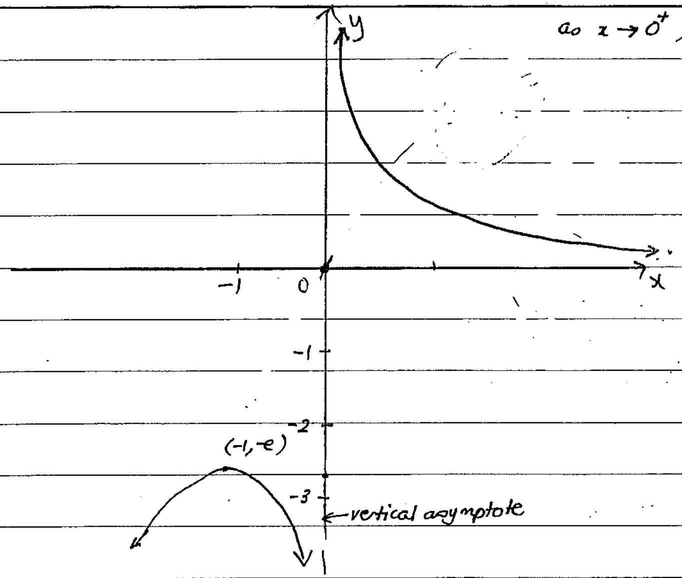
$\dfrac{e^{1}}{-1} = -e$

$\therefore (-1, -e)$ is a maximum st. point. //

$x>0$, as $x\to\infty$, $\frac{e^{-x}}{x}\to 0^+$

iii)

as $x\to 0^+$, $\frac{e^{-x}}{x}\to\infty$



b) i) Function intersects when $f(x)=g(x)$

$$x^3-ax=ax$$
$$x^3-2ax=0$$
$$x(x^2-2a)=0$$
$$x=0 \quad \Big| \quad x^2-2a$$
$$x^2=2a$$
(when they intersect, $x=m$, $m\neq 0$)

$$\therefore m^2=2a$$

ii)
$$A=\int_0^m \cancel{x^3} \; ax-(x^3-ax)$$
$$=\int_0^m -x^3$$
$$\cancel{=\left[-\frac{x^4}{4}\right]_0^m}$$
$$\cancel{=\left[\left(\frac{-m^4}{4}\right)-\left(\frac{-0^4}{4}\right)\right]}$$

(ii) Cont'd.

$$=\int_0^m -x^3+2ax$$
$$=\left[-\frac{x^4}{4}+\frac{2ax^2}{2}\right]_0^m$$
$$=-\frac{m^4}{4}+am^2$$
$$-\frac{m^4}{4}+am^2=64 \qquad \text{sub } m^2=2a$$
$$-\frac{(2a)^2}{4}+a(2a)=64$$
$$-\frac{4a^2}{4}+2a^2=64$$
$$-a^2+2a^2=64$$
$$a^2=64$$
$$a=\pm 8 \quad \text{but } a>0$$
$$a=8$$

Sub $a=8$ into $m^2=2a$
$$m^2=16$$
$$m=\pm 4 \quad \text{but } m>0$$
$$\therefore a=8,\ m=4$$

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