WESTERN REGION

2010
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time - 5 minutes.
- Working Time - 2 hours.
- Write using a blue or black pen.
- Board Approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (**84**)

- Attempt Questions 1-7.
- All questions are of equal value.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) The point $P(3, 2)$ divides the interval MN internally in the ratio 3 : 2. If M is the point (6, -1), find the coordinates of N. **2**

b) Find $\int \frac{2}{\sqrt{x^2+25}}\,dx$. **2**

c) Solve $\frac{2x}{x-3} \le 2$. **3**

d) Evaluate $\sum_{r=1}^{5} r^2 + 2r$. **1**

e) Show that $(x+3)$ is a factor of $P(x) = x^3 - 19x - 30$ and determine whether $(x-5)$ or $(x+5)$ is also a factor of $P(x)$. **2**

f) Using the substitution $u = x^2 - 2$, or otherwise, find $\int \frac{x}{\sqrt{x^2-2}}\,dx$. **2**

End of Question 1

Question 2 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The equation of the tangents at P and Q respectively are $y = px - ap^2$ and $y = qx - aq^2$.

(i) The tangents at P and Q meet at the point R. Show that the coordinates of R are $(a(p+q), apq)$. **2**

(ii) The equation of the chord PQ is $y = \frac{p+q}{2}x - apq$ (Do NOT show this.) **1**

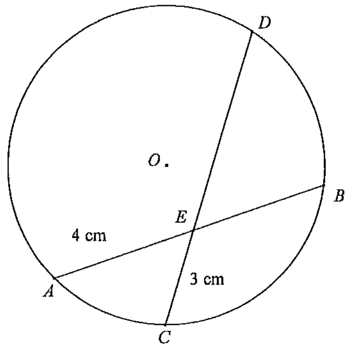
If the chord PQ passes through $(0, a)$, show that $pq = -1$.

(iii) Find the equation of the locus of R if the chord PQ passes through $(0, a)$ **2**

b) Find $\int \sin^2 6x\,dx$. **2**

c) There are eight parking spaces at the front of a motel which are all vacant at 2 pm. Two utilities and six cars arrive in the next hour and park randomly in the eight spaces. What is the probability that the two utilities park side by side? **2**

d) In the circle centred at O, the chords AB and CD intersect at E. The length of AB is x cm and of CD is y cm. $AE = 4$ cm and $CE = 3$ cm. **3**

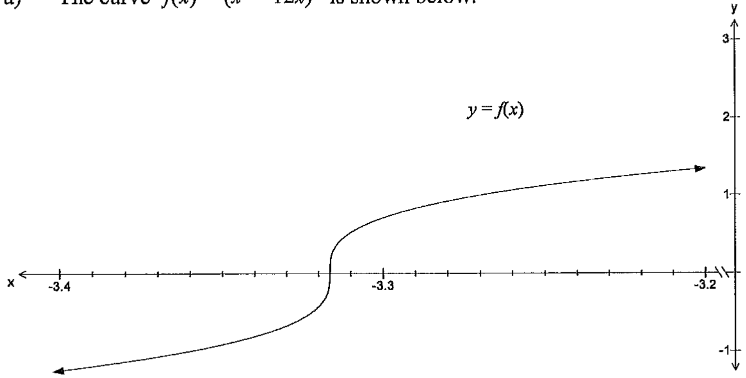


Show that $4x = 3y + 7$

End of Question 2

Question 3 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) The curve $f(x) = (x^3 - 12x)^{\frac{1}{3}}$ is shown below.



(i) Find $f'(x)$. (No need to simplify your answer.) **1**

(ii) Taking an initial estimate of $x_1 = -3.3$, use one application of Newtons' Method to obtain another approximation to the root of $f(x) = 0$. **2**

(iii) Explain why using $x = -3.3$ does not produce a better approximation to the root than the original estimate. **1**

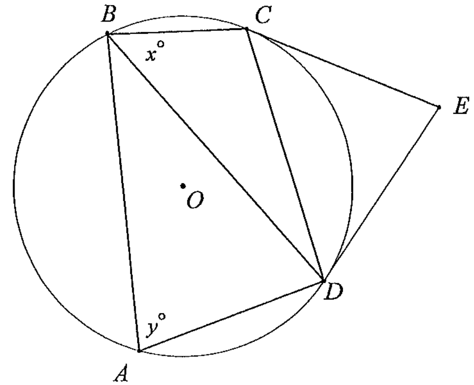
b) (i) Use the expansion for $\sin(A + B)$ and the exact values for $\cos\frac{\pi}{4}$ and $\sin\frac{\pi}{4}$ to show that $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sin x + \cos x}{\sqrt{2}}$. **2**

(ii) Hence, or otherwise, solve $\frac{\sin x + \cos x}{\sqrt{2}} = \frac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$. **2**

Question 3 continues

Question 3 continued **Marks**

c) The circle $ABCD$ has centre O. Tangents are drawn from an external point E to contact the circle at C and D. $\angle CBD = x^\circ$ and $\angle BAD = y^\circ$.



(i) Show that $\angle CED = (180 - 2x)^\circ$. **2**

(ii) Show that $\angle BDC = (y - x)^\circ$. **2**

End of Question 3

Question 4 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) Find the term independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^9$. **3**

b) The polynomial $P(x)$ is defined as $P(x) = x^3 + mx^2 + 2mx + n$ (where m and n are constants). The zeros of $P(x)$ are -3, 2 and α. Find the values of m and of α. **3**

c) Given $f(x) = \log_e\left(\sqrt{9 - x^2}\right)$, state the domain of $f(x)$. **2**

d) On a busy highway, 5% of vehicles passing a checking point are heavy vehicles.

(i) What is the probability that two vehicles chosen at random passing the checking point will be heavy vehicles? **1**

(ii) A random sample of 25 vehicles, passing the checking point, is photographed each hour. What is the probability (correct to 3 significant figures) that three of the 25 vehicles will be heavy vehicles? **1**

(iii) What is the probability (correct to 2 significant figures) that at least four of the vehicles from the sample of 25 will be heavy vehicles? **2**

End of Question 4

Question 5 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) (i) Show, using sketches on separate sets of axes : **2**

the area enclosed between $y = \sin^{-1}x$, the x axis, and the line $x = 1$ **and**

the area enclosed between $y = \sin x$, the x axis, and the line $x = \frac{\pi}{2}$.

(ii) Using the graphs, explain why $\int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x \, dx$. **2**

b) (i) Using the expansion of $(1 + x)^{n-1}$ show that: **2**

$$\binom{n-1}{1} + \binom{n-1}{2} + \ldots \binom{n-1}{n-2} = 2^{n-1} - 2.$$

(ii) Find the least positive integer n, such that: **2**

$$\binom{n-1}{1} + \binom{n-1}{2} + \ldots \binom{n-1}{n-2} > 1\,000$$

c) The velocity of a particle moving along the x axis, in simple harmonic motion is given by:

$$v^2 = 24 + 2x - x^2.$$

(i) What are the endpoints of the motion? **1**

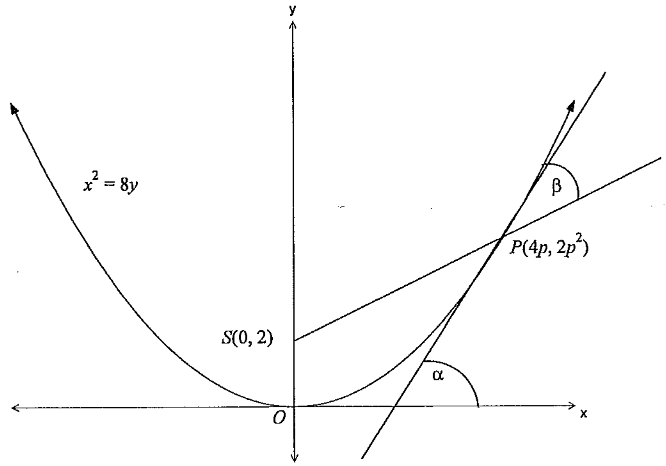
(ii) Write an equation for the acceleration of the particle in terms of x. **2**

(iii) Find the period of the motion. **1**

End of Question 5

Question 6 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) The point $P(4p, 2p^2)$ is a point on the parabola $x^2 = 8y$, with focus $S(0, 2)$.



The tangent at P makes an angle of α with the x axis and an angle of β with the line SP.

(i) Show that the gradient of the tangent is p. **1**

(ii) Show that the gradient of SP is $\frac{p^2 - 1}{2p}$. **1**

(iii) Show that $\tan\beta = \frac{1}{p}$ **2**

(iv) Show that $\alpha + \beta = \frac{\pi}{2}$ **1**

Question 6 continues

Question 6 continued **Marks**

b) Prove, using the method of mathematical induction, that $4^n + 14$ is divisible by 6 for all $n \geq 1$. **3**

c) Use the definition of the derivative, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find $f'(x)$ where $f(x) = 4x^2 - 5x$. **2**

d) Differentiate $2x^2 \cos^{-1} 2x$. **2**

End of Question 6

Question 7 **(12 Marks)** Use a Separate Sheet of paper **Marks**

a) Making use of an appropriate graph or trigonometric diagram, show that: **2**

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

b) A particle is moving in a straight line such that its acceleration is given by $\ddot{x} = -x$.

(i) Given that, when $x = 0$, $\dot{x} = 1$, show that $|\dot{x}| = \sqrt{1-x^2}$. **2**

(ii) Given that, when $x = 0$, $t = 0$, find an expression for x in terms of t. **2**

c) A spherical weather balloon is being inflated from empty using a container of helium, so that its' volume is increasing at a constant rate of 0.5 m^3/s. (Assume the balloon maintains a spherical shape throughout inflation.)

(i) Show that the radius at a time t is given by $r = \sqrt[3]{\frac{3t}{8\pi}}$, **2**

(ii) Show that the rate of increase of its surface area after 8 seconds is **3**
$\sqrt[3]{\frac{\pi}{3}}$ m^2/s.

(iii) If the maximum safe surface area before there is a risk that the balloon will burst is 200 m^2, what is the maximum time that the inflation should be allowed to proceed? **1**

End of Examination

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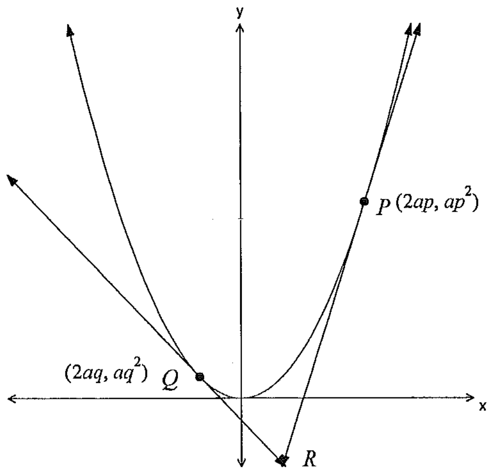
Mathematics Extension 1

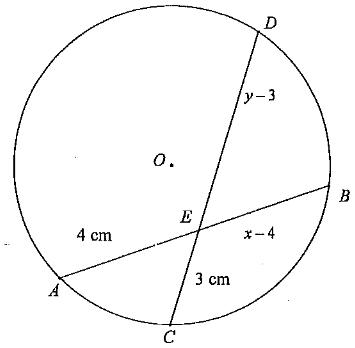
SOLUTIONS

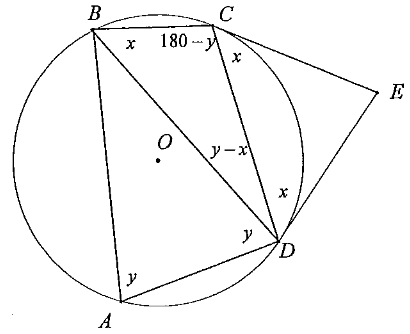
Question 1	Trial HSC Examination - Mathematics Extension 1		2010
Part	**Solution**	**Marks**	**Comment**
a)	$x=\frac{kx_2+lx_1}{k+l}$ $y=\frac{ky_2+ly_1}{k+l}$ $3=\frac{3x_2+2\times 6}{3+2}$ $2=\frac{3y_2+2\times -1}{3+2}$ $15=3x_2+12$ $10=3y_2-2$ $3=3x_2$ $12=3y_2$ $x_2=1$ $y_2=4$ The point M is $(1,4)$	2	1 for correct substitution in formula 1 for point.
b)	$\int \frac{2}{\sqrt{x^2+25}}\,dx = 2\int \frac{1}{\sqrt{x^2+25}}\,dx$ $= \int \frac{1}{\sqrt{x^2+5^2}}\,dx$ $= 2\ln\left(x+\sqrt{x^2+25}\right)+C$ Using Standard Integrals.	2	1 for changing to standard form 1 for integral.
c)	$\frac{2x}{x-3} \le 2$ $x \ne 3$ from denominator. $\frac{2x}{x-3}(x-3)^2 \le 2(x-3)^2$ $2x(x-3) \le 2x^2-12x+18$ $2x^2-6x \le 2x^2-12x+18$ $6x \le 18$ $x \le 3$ But $x \ne 3$ So $x < 3$	3	1 for eliminating the denominator 1 for working solution to a linear equation 1 for solution
d)	$\sum_{r=1}^{5} r^2+2r = 1^2+2\times 1+2^2+2\times 2+3^2+2\times 3+4^2+2\times 4+5^2+2\times 5$ $= 3+8+15+24+35$ $= 85$	1	1 for answer

Question 1	Trial HSC Examination - Mathematics Extension 1		2010
Part	**Solution**	**Marks**	**Comment**
e)	$P(x) = x^3 - 19x - 30$ $P(-3) = (-3)^3 - 19(-3) - 30$ $= -27 + 57 - 30$ $= 0$ $\therefore (x - -3) = (x + 3)$ is a factor of $P(x)$ Test $P(-5) = (-5)^3 - 19(-5) - 30$ $= -125 + 95 - 30 = -60$ $P(5) = (5)^3 - 19(5) - 30$ $= 125 - 95 - 30 = 0$ $\therefore (x - 5)$ is a factor of $P(x)$	2	1 for showing (x+3) is a factor 1 for finding either of other factors
f)	$u = x^2 - 2$ $du = 2x\,dx$ $\int \frac{x}{\sqrt{x^2-2}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-2}}dx$ $= \frac{1}{2}\int \frac{du}{\sqrt{u}}$ $= \frac{1}{2}\int u^{-\frac{1}{2}}du$ $= u^{\frac{1}{2}} + C$ $= \sqrt{x^2-2} + C$	2	1 For correct method of substitution including du 1 for integral
		/12	

Question 2	Trial HSC Examination - Mathematics Extension 1		2010
Part	**Solution**	**Marks**	**Comment**
a) (i)	 Equations of tangents: $y = px - ap^2$ and $y = qx - aq^2$ To find R, solve simultaneously $px - ap^2 = qx - aq^2$ $px - qx = ap^2 - aq^2$ $x(p-q) = a(p+q)(p-q)$ $x = a(p+q)$ $y = p \times a(p+q) - ap^2$ $= ap^2 + apq - ap^2$ $= apq$ R is the point $(a(p+q), apq)$	2	NB Graph not needed for marks. 1 for reasonable attempt to solve simultaneously 1 for correct result.
a) (ii)	Substitute $(0, a)$ into $y = \frac{p+q}{2}x - apq$ $a = \frac{p+q}{2} \times 0 - apq$ $-apq = a$ $pq = -1$	1 mark	

Question 2	Trial HSC Examination - Mathematics Extension 1		2010
Part	**Solution**	**Marks**	**Comment**
a) (iii)	If chord passes through $(0, a)$ then $pq = -1$ and $p = -\frac{1}{q}$ R is the point $(a(p+q), apq)$. Which becomes $\left(a\left(-\frac{1}{q}+q\right), a\times(-1)\right)$ $x = a\left(-\frac{1}{q}+q\right)$ and $y = -a$ $x = -y\left(\frac{q^2-1}{q}\right)$ $qx = y(1-q^2)$ $y = \frac{qx}{1-q^2}$ OR SIMILARLY $y = \frac{px}{1-p^2}$	2	1 for introducing $pq = -1$ to eliminate p or q 1 for relating x and y and obtaining the equation of the locus.
b)	$\cos 2A = 1 - 2\sin^2 A$ $2\sin^2 A = 1 - \cos 2A$ $\sin^2 A = \frac{1}{2}(1-\cos 2A)$ $\int \sin^2 6x\, dx = \frac{1}{2}\int 1 - \cos 12x\, dx$ $= \frac{1}{2}\left(x - \frac{1}{12}\sin 12x\right) + C$ $= \frac{x}{2} - \frac{\sin 12x}{24} + C$	2	1 for transformation of the integral or recalling a formula 1 for integration
c)	There are ${}^8\mathbf{P}_8$ (8!) ways the 8 vehicles can park. If the two utes are together, treat them as one, so there are 7 vehicles. These can park in ${}^7\mathbf{P}_7$ (7!) ways with ${}^2\mathbf{P}_2$ (2!) ways of arranging the utes among themselves. So the 7 are arranged in $\frac{7!}{2!}$ ways. Probability $= \frac{7!}{2!} \div 8!$ $= \frac{7!}{8!\,2!}$ $= \frac{1}{8\times 2}$ $= \frac{1}{16}$	2	1 for arrangement of vehicles with utes together. 1 for probability

Question 2	Trial HSC Examination - Mathematics Extension 1		2010
Part	**Solution**	**Marks**	**Comment**
d)	 Since $AB = x$, $EB = x - 4$ and since $CD = y$, $ED = y - 3$ $AE.EB = CE.ED$ (Ratio of intercepts on chords) $4(x-4) = 3(y-3)$ $4x - 16 = 3y - 9$ $4x = 3y + 7$	3	1 1 1
		/12	

Question 3	Trial HSC Examination - Mathematics Extension 1		2010
Part	**Solution**	**Marks**	**Comment**
a) (i)	$f(x) = (x^3 - 12x)^{\frac{1}{3}}$ $f(x) = \frac{1}{3}(x^3 - 12x)^{-\frac{2}{3}}.(3x^2 - 12)$	1	No need to simplify further
a) (ii)	$x_1 = -3.3$ $f(x_1) = ((-3.3)^3 - 12(-3.3))^{\frac{1}{3}}$ ≈ 1.54 $f'(x_1) = \frac{1}{3}((-3.3)^3 - 12(-3.3))^{-\frac{2}{3}} \times (3(-3.3)^2 - 12)$ ≈ 2.90 $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $= -3.3 - \frac{1.54}{2.90}$ ≈ -3.83 (2 dec places)	2	1 for evaluating function and derivative. 1 for substitution into Newtons Method formula.
a) (iii)	As Newtons Method uses the intercept that the tangent makes, from the graph, the tangent at -3.3 is quite flat compared to the sudden drop in the curve to meet the axis. Hence the tangent would meet the axis much further along than the graph, so the second approximation is not as good as the first.	1	Mark for mention of the tangent meeting the axis or similar
b) (i)	$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}$ $= (\sin x) \times \frac{1}{\sqrt{2}} + (\cos x) \times \frac{1}{\sqrt{2}}$ $= \frac{\sin x + \cos x}{\sqrt{2}}$	2	1 correct definition 1 correct evaluation
b) (ii)	$\frac{\sin x + \cos x}{\sqrt{2}} = \frac{\sqrt{3}}{2}$ $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$ $x + \frac{\pi}{4} = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ $\left(\frac{\pi}{4} \le x + \frac{\pi}{4} \le \frac{9\pi}{4}\right)$ $x = \frac{\pi}{12}$ or $\frac{5\pi}{12}$ $(0 \le x \le 2\pi)$	2	1 for initial solution of $\frac{\pi}{3}$ and set. 1 for final solution for x
c) (i)	 $\angle EDC = \angle CBD = x$ (Angle between a tangent and a chord is equal to the angle in the alternate segment) Similarly $\angle ECD = \angle CBD = x$ Hence $\angle ECD = \angle EDC = x$ Or $EC = ED$ (Tangents from an external point are equal) Hence $\angle ECD = \angle EDC = x$ $\angle CED = 180 - \angle ECD - \angle EDC$ (Angle sum of triangle) $\angle CED = (180 - 2x)^\circ$	2	1 for partially completed proof with some of the required points or with single error 2 for completely correct proof Or any other valid proof
c) (ii)	$\angle BCD = 180 - y^\circ$ (Opposite angles of cyclic quadrilateral are supplementary) $\angle BDC = 180 - \angle CBD - \angle BCD$ $= 180 - x - (180 - y)$ $= 180 - x - 180 + y$ $= (y - x)^\circ$	2	1 for cyclic quad or similar partial proof. 2 for full proof.
		/12	

Question 4	Trial HSC Examination - Mathematics Extension 1		2010
Part	**Solution**	**Marks**	**Comment**
a)	In expansion of $(a+b)^n$ $T_{r+1} = {}^n\mathbf{C}_r a^{n-r} b^r$ $T_{r+1} = {}^9\mathbf{C}_r (x^2)^{9-r}\left(-\frac{2}{x}\right)^r$ $= {}^9\mathbf{C}_r x^{18-2r}(-2)^r x^{-r}$ In the expansion of $\left(x^2 - \frac{2}{x}\right)^9$ $= {}^9\mathbf{C}_r (-2)^r x^{18-3r}$ For the term independent of x, $18 - 3r = 0$ $3r = 18$ $r = 6$ $T_7 = {}^9\mathbf{C}_6(-2)^6 x^0$ $= 84 \times 64$ $= 5\,376$ Accept ${}^9\mathbf{C}_6(-2)^6$ for full marks.	3	1 for writing the general term or starting to write out the expansion. 1 for simplifying the term in x and setting to zero. 1 for term either as a single number or the unexpanded expression given.

b)	$P(x) = x^3 + mx^2 + 2mx + n$ $P(-3) = 0$ $(-3)^3 + m(-3)^2 + 2m(-3) + n = 0$ $3m + n = 27$ ① $P(2) = 0$ $(2)^3 + m(2)^2 + 2m(2) + n = 0$ $8m + n = -8$ ② $5m = -35$ ② − ① $m = -7$ $3(-7) + n = 27$ $n = 48$ $P(\alpha) = 0$ $\alpha^3 - 7\alpha^2 - 14\alpha + 48 = 0$ Three factors $(x-\alpha)(x-2)(x+3)$ give a constant term of 48 $-6(-\alpha) = 48$ $\alpha = 8$ OR Test factors, Try $P(8)$ $(8)^3 - 7(8)^2 - 14(8) + 48 = 0$ OR use division transformation ANSWER $m = -7$, $n = 48$, $\alpha = 8$	3	1 for setting up simultaneous equations 1 for solving for m and n 1 for evaluating α
c)	$f(x) = \log_e\left(\sqrt{9-x^2}\right)$, The log function has its argument greater than zero $\sqrt{9-x^2} > 0$ $9 - x^2 > 0$ $(3-x)(3+x) > 0$ Domain $-3 < x < 3$	2	1 for argument >0 1 for solving for the domain.
d) (i)	$P(Heavy, Heavy) = 0.05 \times 0.05$ $= 0.0025$	1	
d) (ii)	$P(3\ are\ heavy) = {}^{25}\mathbf{C}_3(0.95)^{22} \times (0.05)^3$ $= 2300 \times (0.95)^{22} \times (0.05)^3$ $= 0.0930$ (3 sig fig)	1	