NEWINGTON COLLEGE

Trial Examination

12 MATHEMATICS

2003

Extension 1

Time allowed: 2 hours (plus five minutes reading time)

DIRECTIONS TO CANDIDATES

- All questions may be attempted.
- In every question, show all necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Approved silent calculators may be used.
- A table of standard integrals is provided for your convenience.
- The answers to the questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2, etc.
- Each bundle must show the candidate's computer number.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Unless otherwise stated, candidates should leave their answers in simplest exact form.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 12 marks marks

a) Differentiate $\tan^{-1}\frac{x}{3}$. 2

b) Evaluate: 6

(i) $\int_1^{\sqrt{2}} \frac{x}{\sqrt{4-x^2}}\,dx$ using the substitution $u = 4 - x^2$.

(ii) $\int_0^1 \sqrt{1-x^2}\,dx$ using the substitution $x = \sin\theta$.

c) Solve the equation $3\sin\theta + 4\cos\theta = 2.5$ for values of θ between 0° and 360°. Give your answer correct to the nearest minute. 4

Question 2 12 marks **Start a New Booklet**

a) (i) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dt}$. 5

(ii) The acceleration of a particle moving in a straight line is given by $\ddot{x} = -2e^{-x}$ where x metres is the displacement from the origin. Initially, the particle is at the origin with velocity 2 ms^{-1}.

Prove that $v = 2e^{\frac{-x}{2}}$.

(iii) What happens to v as x increases without bound?

b) (i) By considering the graph of $y = e^x$, show that the equation $e^x + x + 1 = 0$ has only one real root and that this root is negative. 4

(ii) Taking $x = -1\cdot5$ as a first approximation to this root, use one application of Newton's method to find a better approximation.

c) In how many ways can the letters of the word *GEOMETRY* be arranged in a straight line if the vowels must occupy the 2nd, 4th and 6th places. (NOTE: The vowels in the English alphabet are the letters *A, E, I, O, U*). 3

Q3 ... Page 2

Question 3 12 marks **Start a New Booklet** marks

a) Find the general solution for $\sqrt{3}\sin 2\theta = \cos 2\theta$. 3

b) The region bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{12}$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution about the x-axis. Find the volume of the solid so formed. 3

c) Two points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x^2 = 4y$. 6

(i) Show that the equation of the tangent to the parabola at P is $y = px - p^2$.

(ii) The tangent at P and the line through Q parallel to the y axis intersect at T. Find the coordinates of T.

(iii) Write down the coordinates of M, the midpoint of PT.

(iv) Determine the locus of M when $pq = -1$.

Question 4 12 marks **Start a New Booklet**

a) If $\tan A$ and $\tan B$ are the roots of the equation $3x^2 - 5x - 1 = 0$, find the value of $\tan(A + B)$. 3

b) A particle is moving with simple harmonic motion. When it is at a distance d from the centre of motion, its speed is V. If its speed is $\frac{V}{2}$ when the distance from the centre is $2d$, show that the period of the motion is $\frac{4\pi d}{V}$ and the amplitude is $d\sqrt{5}$. 5

c) The rate at which a body cools in air is assumed to be proportional to the difference between its temperature T and the constant temperature S of the surrounding air. This can be expressed by the differential equation $\frac{dT}{dt} = k(T - S)$ where t is the time in hours and k is a constant. 4

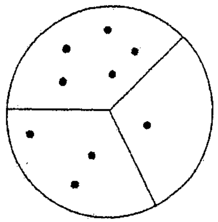
(i) Show that $T = S + Be^{kt}$, where B is a constant, is a solution of the differential equation.

(ii) A heated body cools from 80 °C to 40 °C in 2 hours. The air temperature S around the body is 20 °C. Find the temperature of the body after one further hour has elapsed. Give your answer correct to the nearest degree.

Q5 ... Page 3

<u>Question 5 12 marks</u> **<u>Start a New Booklet</u>** <u>marks</u>

a) Nine points lie inside a circle. No three of the points are collinear. Five of the points lie in sector *1*, three lie in sector *2*, and the other point lies in sector *3*. 5



(i) Show that *84* triangles can be made using these points as vertices.

(ii) One triangle is chosen at random from all the possible triangles. Find the probability that the vertices of the triangle chosen lie one in each sector.

(iii) Find the probability that the vertices of the triangle chosen lie all in the same sector.

b) Find the roots of the equation $x^3 - 12x^2 + 12x + 80 = 0$ given that they are three consecutive terms in an Arithmetic Series. 3

c) Consider the binomial expansion $1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n=(1+x)^n$. 4

(i) Show that $1-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^n\binom{n}{n}=0$.

(ii) Show that $1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\ldots+(-1)^n\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}$.

<u>Question 6 12 marks</u> **<u>Start a New Booklet</u>**

a) Colour-blindness affects *5%* of all men. What is the probability that any random sample of *20* men should contain: 5

(i) no colour-blind men.

(ii) only one colour-blind man.

(iii) two or more colour-blind men.

Q6 cont. ... Page 4

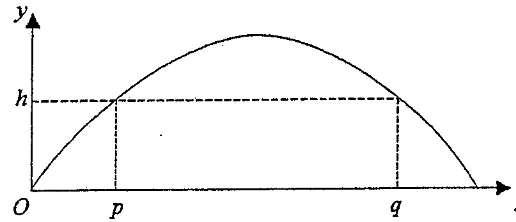
<u>marks</u>

b) When $(3 + 2x)^n$ is expanded in increasing powers of x, it is found that the coefficients of x^5 and x^6 have the same value. Find the value of n and show that the two coefficients mentioned are greater than all other coefficients in the expansion. 7

<u>Question 7 12 marks</u> **<u>Start a New Booklet</u>**

a) Prove by induction that $2^3+4^3+6^3+\ldots+(2n)^3=2n^2(n+1)^2$. 5

b) 7



A particle is projected with velocity V ms^{-1} from a point O at an angle of elevation α. Axes Ox and Oy are taken horizontally and vertically through O. The particle just clears two vertical chimneys of height h meters at horizontal distances of p metres and q metres from O. The acceleration due to gravity is taken as *10* ms^{-2} and air resistance if ignored.

(i) Write down expressions for the horizontal displacement x and the vertical displacement y of the particle after time t seconds.

(ii) Show that $V^2=\dfrac{5p^2(1+\tan^2\alpha)}{p\tan\alpha-h}$.

(iii) Show that $\tan\alpha=\dfrac{h(p+q)}{pq}$.

END OF PAPER

1/(a) $\dfrac{\frac{1}{3}}{1+\frac{x^2}{9}} = \dfrac{3}{x^2+9}$

(b)(i) $\displaystyle\int_1^{\sqrt{2}} \frac{x}{\sqrt{4-x^2}}\,dx$ $u = 4 - x^2$, $\dfrac{du}{dx} = -2x$, $x\,dx = -\dfrac{du}{2}$ $x=1,\ u=3$; $x=\sqrt{2},\ u=2$

$= -\displaystyle\int_3^2 \frac{du}{2\sqrt{u}}$

$= -\displaystyle\int_3^2 \tfrac{1}{2}u^{-\frac{1}{2}}\,du$

$= \left[u^{\frac{1}{2}}\right]_2^3$

$= \sqrt{3} - \sqrt{2}$

(ii) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ $x = \sin\theta$, $dx = \cos\theta\,d\theta$ $x=0,\ \theta=0$; $x=1,\ \theta=\frac{\pi}{2}$

$= \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,.\cos\theta\,d\theta$

$= \displaystyle\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$= \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$

$= \dfrac{1}{2}\left[\theta + \dfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

c) $5\left(\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right) = 2\cdot5$

$\sin(\theta+\alpha) = \frac{1}{2}$, $\alpha = \sin^{-1}\frac{4}{5}$

$\theta+\alpha = 30°, 150°, 390°$

$\theta = 30°-\alpha, 150°-\alpha, 390°-\alpha$

$= -23°8', 96°52', 336°52'$

$= 96°52', 336°52'$ (nearest minute)

2/ (a) (i) $\dfrac{dv}{dt} = \dfrac{dv}{dx}\cdot\dfrac{dx}{dt}$

$= v\dfrac{dv}{dx}$

$= \dfrac{d}{dv}\left(\tfrac{1}{2}v^2\right).\dfrac{dv}{dx}$

$= \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

(ii) $\ddot{x} = -2e^{-x}$

$\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -2e^{-x}$

$\tfrac{1}{2}v^2 = 2e^{-x} + c$

$x=0, v=2$

$2 = 2e^0 + c,\ c=0$

$\tfrac{1}{2}v^2 = 2e^{-x}$

$v^2 = 4e^{-x}$

$v = \pm 2e^{-\frac{x}{2}}$

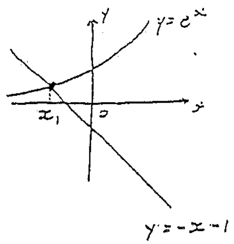
Initially $v > 0$ and $v^2 \neq 0$ ∴ reject −ve v

$v = 2e^{-\frac{x}{2}}$

(iii) as $x \to \infty$, $v \to 0$

(b) $y = e^x$

$y = -x-1$ (3)



$f(x) = e^x + x + 1$

$f'(x) = e^x + 1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= -1\cdot5 - \frac{e^{-1\cdot5} - 1\cdot5 + 1}{e^{-1\cdot5} + 1}$

$= -1\cdot27$ (correct to 2 dec. pl)

c) — V — V — V — —

No. of arrangements $= \frac{3!}{2!}\, 5!$

$= 360$

3(a)

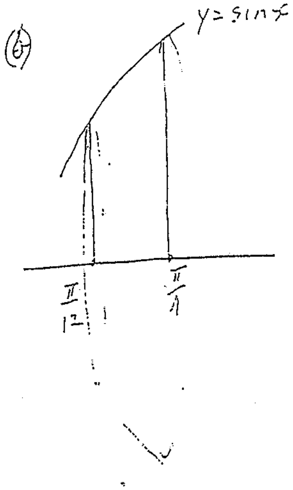
i) $\tan 2\theta = \frac{1}{\sqrt{3}}$

$2\theta = n\pi + \frac{\pi}{6}$

$\theta = n\frac{\pi}{2} + \frac{\pi}{12}$

Q 3. cont. (4)

(b)



$V = \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2 x \, dx$

$= \frac{\pi}{2} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1 - \cos 2x)\, dx$

$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$

$= \frac{\pi}{2} \left(\frac{\pi}{4} - \frac{1}{2} - \left(\frac{\pi}{12} - \frac{1}{2} \cdot \frac{1}{2} \right) \right)$

$= \frac{\pi}{2} \left(\frac{\pi}{6} - \frac{1}{4} \right)$

$= \frac{\pi(2\pi - 3)}{24}$ units^3

(c) (i) $\frac{dy}{dx} = \frac{x}{2y}$ at $P(2p, p^2)$

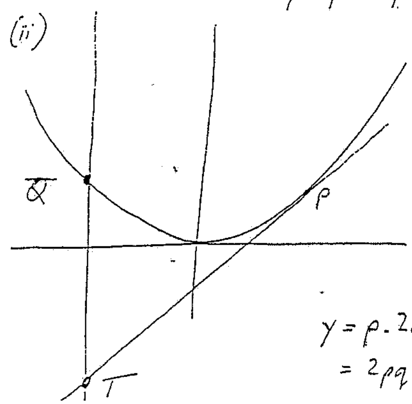
$= p$

required equation: $y - p^2 = p(x - 2p)$

$= px - 2p^2$

$y = px - p^2$

(ii)



$y = p \cdot 2q - p^2$

$= 2pq - p^2$

3/ cont.

(c)

iii) $M(p+q, pq)$

iv) $y = -1$

4/ (a) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan A + \tan B = \frac{5}{3}$

$\tan A \tan B = -\frac{1}{3}$

$\therefore \tan(A+B) = \dfrac{\frac{5}{3}}{1 - -\frac{1}{3}}$

$= \frac{5}{4}$

(b) $\ddot{x} = -n^2 x$

$\frac{1}{2}v^2 = -\frac{n^2 x^2}{2} + C$

$v = V, \quad x = d$

$\therefore C = \frac{1}{2}V^2 + \frac{n^2 d^2}{2}$

$\frac{1}{2}v^2 = -\frac{n^2 x^2}{2} + \frac{1}{2}V^2 + \frac{n^2 d^2}{2}$

$v^2 = V^2 + n^2(d^2 - x^2)$

$v = \frac{V}{2} \quad x = 2d$

$\left(\frac{V}{2}\right)^2 = V^2 + n^2(d^2 - 4d^2)$

$n^2 = \left(V^2 - \frac{V^2}{4}\right) \div 3d^2$

$= \frac{3V^2}{4} \times \frac{1}{3d^2}$

Q4 cont

(b)

Period $= \frac{2\pi}{n}$

$= 2\pi \div \frac{2d}{V}$

$= \frac{4\pi d}{V}$

When $v = 0$, $V^2 + n^2(d^2 - x^2) = 0$

$V^2 + \frac{V^2}{4d^2}(d^2 - x^2) = 0$

$x^2 - d^2 = V^2 \cdot \frac{4d^2}{V^2}$

$x^2 = 5d^2$

$\therefore$ amplitude $= \sqrt{5d^2}$

$= d\sqrt{5}$

(c)

(i) $\frac{dT}{dt} = kBe^{kt}$

$= k(S + Be^{kt} - S)$

$= k(T - S)$

(ii) $t = 0, \; T = 80°$

$80° = 20° + Be^0$

$B = 60$

$T = 20 + 60e^{kt}$

$40 = 20 + 60e^{2k}$

$e^{2k} = \frac{1}{3}$

$2k = \ln\left(\frac{1}{3}\right)$

$k = \frac{1}{2}\ln\left(\frac{1}{3}\right)$

$t = 3, \; T = 20 + 60e^{\frac{1}{2}\ln\left(\frac{1}{3}\right) \cdot 3}$

$= 20 + 60^{\ln\left(\frac{1}{3}\right)^{3/2}}$

$= 20 + 60$

$-3/_{2}$

6(b) cont

$$\frac{t_{r+1}}{t_r} = \frac{{}^{14}C_r\, 3^{14-r} (2x)^r}{{}^{14}C_{r-1}\, 3^{15-r} (2x)^{r-1}}$$ where t_r = rth term in the expansion

$$= \frac{14!}{(14-r)!\, r!} \cdot \frac{(15-r)!\,(r-1)!}{14!} \cdot \frac{2x}{3}$$

Let c_r = coefficient of the rth term

$$\frac{C_{r+1}}{C_r} = \frac{15-r}{r} \cdot \frac{2}{3}$$

$$= \frac{30-2r}{3r}$$

$\frac{C_{r+1}}{C_r} = 1$ when $r = 6$ $\therefore C_6 = C_7$

$\frac{C_{r+1}}{C_r} > 1$ when $r < 6$ $\therefore C_1 < C_2 < C_3 < C_4 < C_5 < C_6$

$\frac{C_{r+1}}{C_r} < 1$ when $r > 6$ $\therefore C_7 > C_8 > C_9 > C_{10} > C_{11} > C_{12} >$

C_6 & C_7 are greatest coefficients

7(a) Step 1 Let $n=1$

$LHS = (2\times1)^3$

$= 8$ $\therefore$ true for $n=1$

$RHS = 2\times1^2(1+1)^2$

$= 8$

Step 2 Assume result true for $n=k$, k is a positive integer

i.e. $2^3 + 4^3 + 6^3 + \dots + (2k)^3 = 2k^2(k+1)^2$

Step 3 Prove result true for $n=k+1$

$2^3 + 4^3 + 6^3 + \dots + (2k)^3 + (2(k+1))^3 = 2(k+1)^2((k+1)+1)^2$

7(a) cont.

$LHS = 2k^2(k+1)^2 + (2(k+1))^3$ from assumption

$= 2(k+1)^2(k^2 + 4(k+1))$

$= 2(k+1)^2(k+2)^2$

$= RHS$

Step 4 Result is true for $n=1$, Hence it is true for $n=1+1=2$, $n=2+1$ etc. $\therefore$ The result is true for all positive integers

(b) (i) $x = vt\cos\alpha$

$y = vt\sin\alpha - \frac{1}{2}gt^2$

(ii) $p = vt\cos\alpha$

$t = \frac{p}{v\cos\alpha}$

$$h = v \cdot \frac{p}{v\cos\alpha} \cdot \sin\alpha - \frac{1}{2} g \frac{p^2}{v^2\cos^2\alpha}$$

$$= p\tan\alpha - \frac{5p^2}{v^2\cos^2\alpha}$$

$$= p\tan\alpha - \frac{5p^2(\tan^2\alpha+1)}{v^2}$$

$$5p^2(\tan^2\alpha+1) = v^2(p\tan\alpha - h)$$

$$v^2 = \frac{5p^2(\tan^2\alpha+1)}{p\tan\alpha - h}$$

(iii) $$v^2 = \frac{5q^2(\tan^2\alpha+1)}{q\tan\alpha - h}$$

$$\therefore \frac{5p^2(\tan^2\alpha+1)}{p\tan\alpha - h} = \frac{5q^2(\tan^2\alpha+1)}{q\tan\alpha - h}$$

$$p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$$

$$\tan\alpha(p^2q - q^2p) = p^2h - q^2h$$

$$\tan\alpha = \frac{h(p-q)(p+q)}{\dots} = \frac{h(p+q)}{\dots}$$