# NSW INDEPENDENT SCHOOLS

**TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

**1997**

# MATHEMATICS

## 3 UNIT (ADDITIONAL)
## AND
## 3/4 UNIT (COMMON)

*Time Allowed - Two hours*
*(Plus 5 minutes reading time)*

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Write your student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the seven questions are to be handed in separately clearly marked Question 1, Question 2, etc..
- *The question paper must be handed to the supervisor at the end of the examination.*

STUDENT NUMBER / NAME........................................

**Question 1 (Start a new page)** **Marks**

a. Find the exact value of $\displaystyle\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{2}{\sqrt{9 - x^2}}\,dx$ 3

b. Use the substitution $u = 3 - x^2$ to find $\displaystyle\int \frac{x}{\sqrt{3 - x^2}}\,dx$ 3

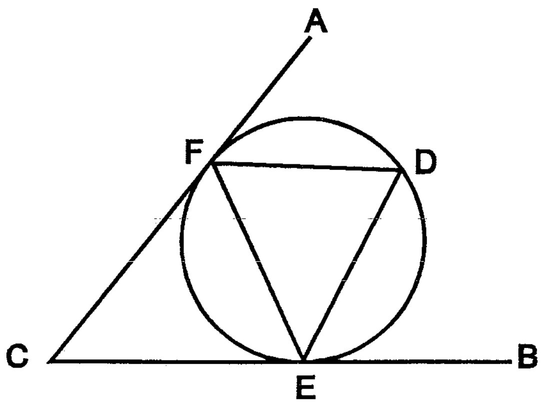
c. For the expansion of $\left(x - \dfrac{2}{x}\right)^8$, find the term independent of $x$. 3

d. Solve the inequality $\dfrac{2x - 3}{x} > 1$ 3

**Question 2 (Start a new page)**

a. In the diagram, AC and BC are tangents to the circle, touching the circle at F and E respectively. $\angle ACB$ equals $50°$. Copy the diagram into your workbook. 3

Show that $\angle CEF$ is $65°$ and hence find $\angle EDF$.




b. i. In how many ways can the letters of the word **MOUSE** be arranged? 3

ii. How many of these arrangements

1. start with the letter **M** and end with the letter **E**?
2. have the *vowels* together?

[A *vowel* is one of the letters **A, E, I, O, U**]

c. A curve is defined by the parametric equations $x = t - 3,\ y = t^2 - 9$ 3

i. Find $\dfrac{dy}{dx}$ in terms of $t$.

ii. Find the equation of the tangent to the curve at the point where $t = -3$

d. The polynomial equation $8x^3 - 36x^2 + 22x + 21 = 0$ has roots which form an arithmetic progression. Find the roots. 3

**Question 3 (Start a new page)** **Marks**

a. The arc of the curve $y = \cos 3x$ between the lines $x = 0$ and $x = \frac{\pi}{6}$ is rotated about the $x$-axis. **4**

Find the volume of the solid formed.

b. Consider the function $y = x \ln x - 1, \quad (x > 0)$ **6**

i. Find the stationary point and determine its nature.

ii. With an initial approximation of $x = 2$, use Newton's Method once to find the $x$-intercept.

iii. Show that the curve is always concave upwards.

iv. Sketch the curve, showing all of its main features.

c. Sketch the graph of the function $f(x) = 3 \sin^{-1} \frac{x}{2}$ **2**

**Question 4 (Start a new page)**

a. Prove by Mathematical Induction that $3^{2n} - 1$ is divisible by 8 when $n$ is an integer greater than 0. **4**

b. From a balloon 500 metres above a road junction, the angle of depression to a point, $P$, due south of the junction is 42°. To another point, $Q$, bearing 080° from the junction, the angle of depression is 32°. How far apart are $P$ and $Q$? **4**

c. It is known that 5% of all gear boxes made in Factory A are faulty whereas 7% of gear boxes made in factory B are faulty. If 10 gear boxes from each factory are bought, find the probability that exactly two are faulty. **4**

**Question 5 (Start a new page)** **Marks**

a. A golf ball is lying on a horizontal fairway when a golfer hits it. It just passes over a 2.25 metre high tree 1½ seconds later. The tree is 60 metres away from the point from which the ball was hit. Taking $g = 10 \text{ m s}^{-1}$, **5**

i. calculate the initial velocity and angle of projection.

ii. How far from where the golfer hits it does the ball land?

b. The shape of a hollow vessel is that of the parabola $9x^2 = 4y$ when rotated about the $y$-axis. The distance across the open end of the vessel is 1.5 metres. **7**

i. Show that, when the depth of water in the vessel is $h$ metres, the volume of water is $\dfrac{2\pi h^2}{9}$ $\text{m}^3$

ii. Find the depth of the water in the vessel when it is half full.

iii. Water is being poured into the vessel at a rate of 0.1 $\text{m}^3$ per second. At what rate is the level rising when the depth of water is 20 centimetres?

**Question 6 (Start a new page)** **Marks**

a. A particle at the origin starts from rest and moves in a straight line so that its acceleration $a$ is given by **3**

$$a = \frac{1}{\sqrt{x^2 + 16}}$$

where $x$ is its displacement.

Use the table of standard integrals to find its velocity when $x$ is 3.

b. A particle is moving in simple harmonic motion with a period of $\pi$ seconds. When it is at a point 5 cm from the centre of the motion, its speed is 2 cm per second. Find the maximum speed and acceleration of the particle. **4**

c. Newton's Law of Cooling states that when an object at temperature $T$°C is placed in an environment at temperature $T_0$°C, the rate of temperature loss is given by the equation **5**

$$\frac{dT}{dt} = k(T - T_0)$$

where $t$ is the time in seconds and $k$ is a constant.

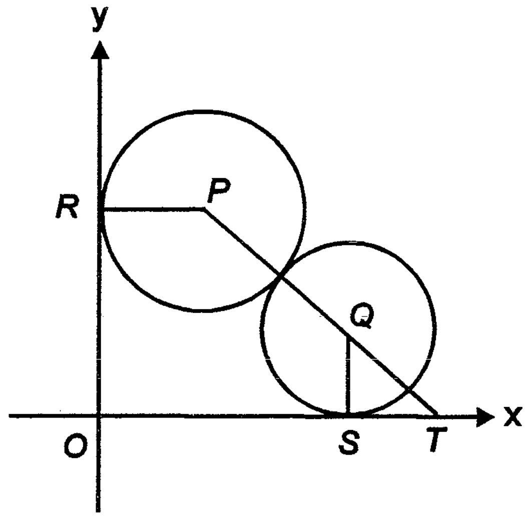
i. Show that $T = T_0 + Ae^{kt}$ is a solution to the equation.

ii. A packet of peas, initially at 24°C is placed in a snap-freeze refrigerator in which the internal temperature is maintained at –40°C. After 5 seconds, the temperature of the packet is 19°C. How long will it take for the packet's temperature to reduce to 0°C?

**Question 7 (Start a new page)** **Marks**

a. Nathan, who will soon turn 21, wants to invest some money on his birthday each year so that he will have \$500,000 when he retires on his 65th birthday. He can open an account which will give him 6.2% p.a. compounded yearly over that time period. How much should he invest each year to achieve his goal? **4**

b. The diagram shows two touching circles, with centres $P$ and $Q$. The circle with centre $P$ has a radius of 4 units and touches the $y$-axis at $R$. The circle with centre $Q$ has a radius of 3 units and touches the $x$-axis at $S$. $PQ$ produced meets the $x$-axis at $T$ and $\angle QTS = \theta$. **8**




i. Show that $OR = 3 + 7\sin\theta$ and $OS = 4 + 7\cos\theta$

ii. Show that $RS^2 = 42\sin\theta + 56\cos\theta + 74$

iii. Hence express $RS^2$ in the form $74 + r\cos(\theta - \alpha)$, clearly stating the values of $r$ and $\alpha$

iv. Find the maximum length of $RS$ and the value of $\theta$ for which this occurs.

# STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

# 3u MATHS SOLUTIONS -1997

## NSW INDEPENDENT TRIAL EXAMS

QUESTION 1

a) $\displaystyle\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{2}{\sqrt{9-x^2}}\,dx = 2\left[\sin^{-1}\frac{x}{3}\right]_{\frac{3\sqrt{3}}{2}}^{3}$

$= 2\left(\sin^{-1}1 - \sin^{-1}\frac{\sqrt{3}}{2}\right)$

$= 2\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= \pi/3$

b) If $u = 3 - x^2$; $du = -2x\,dx$

$\displaystyle\int \frac{x}{\sqrt{3-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$

$\displaystyle = -\frac{1}{2}\int u^{-1/2}\,du$

$\displaystyle = -\frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right] + C$

$= -\sqrt{3-x^2} + C$

c) $\displaystyle\left(x - \frac{2}{x}\right)^8 = \sum_{0}^{8} {}^8C_r\, x^{8-r}.\left(-\frac{2}{x}\right)^r$

Term $= {}^8C_r . x^{8-r} . x^{-r} . (-2)^r$

$= {}^8C_r (-2)^r . x^{8-2r}$

Put $8-2r = 0 \Rightarrow r = 4$

$\therefore$ Term $= {}^8C_4 (-2)^4$

$= 1120$

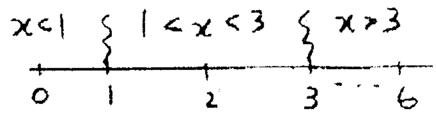
(d) $\displaystyle\frac{2x-3}{x} > 1$

A critical value exists at $x=0$

Also, at $\displaystyle\frac{2x-3}{x} = 1$

$2x-3 = x$

$\therefore\ x = 3$



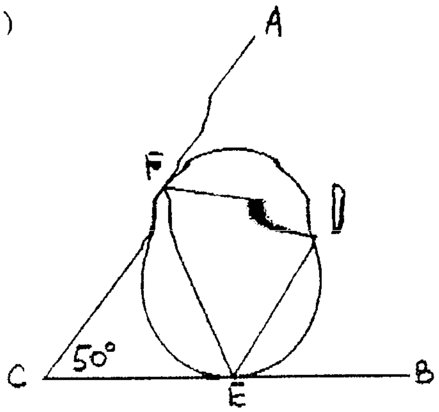

Testing: if $x=6$, $\displaystyle\frac{2\times6-3}{6} > 1$ TRUE

$\therefore$ Solution set is

$x<1$, $x>3$

QUESTION 2

(a)




AC and BC are tangents (given)

$\therefore$ CF = CE (tangents from an external point equal)

$\therefore \triangle ECF$ is isosceles (two equal sides)

$\therefore \angle EFC = \angle CEF$ (base angles equal)

But $\angle EFC + \angle CEF + \angle FCE$

$= 180°$ (angle sum of triangle)

$\therefore 2 \times \angle EFC + 50° = 180°$

$\therefore \angle EFC = 65° = \angle CEF$

Hence, $\angle EDF = 65°$ (alternate segment theorem)

(b) i. $5! = 120$

ii. 1: $3! = 6$

2: $3!3! = 36$

(c) i. $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 2t \times 1$

$= 2t$

ii. At $t = -3$, $x = -6$, $y = 0$; $m = -6$

$\therefore y - 0 = -6(x+6)$

$y = -6x - 36$

(d) $\alpha + \beta + \gamma = 36/8 = 9/2$ .... (1)

$\alpha\beta + \alpha\gamma + \beta\gamma = 22/8 = 11/4$ .... (2)

$\alpha\beta\gamma = -21/8$ .... (3)

Also $\beta = \frac{\alpha + \gamma}{2}$

or $\alpha + \gamma = 2\beta$ ..... (4)

(1) & (4) $\Rightarrow \beta + 2\beta = 9/2$

$\therefore \beta = 3/2$

(3) $\Rightarrow \alpha\gamma \cdot 3/2 = -21/8$

$\therefore \alpha\gamma = -7/4$ ... (5)

(1) $\Rightarrow \alpha + \gamma + 3/2 = 9/2$

$\therefore \alpha + \gamma = 3$ ... (6)

Solving (5) & (6):

$\alpha(3 - \alpha) = -7/4$

$\Rightarrow 4\alpha^2 - 12\alpha - 7 = 0$

$(2\alpha + 1)(2\alpha - 7) = 0$

$\alpha = -\frac{1}{2}$ or $\frac{7}{2}$

The roots are $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}$

QUESTION 3

a) $V = \pi \int y^2 \, dx$

$= \pi \int_0^{\pi/6} \cos^2 3x \, dx$

But $\cos 2\theta = 2\cos^2\theta - 1$

$\therefore \cos 6\theta = 2\cos^2 3\theta - 1$

$+ \quad \cos^2 3\theta = \frac{1}{2}(1 + \cos 6\theta)$

So $V = \pi \int_0^{\pi/6} \frac{1}{2}(1 + \cos 6x)\, dx$

$= \frac{\pi}{2}\left[x + \frac{1}{6}\sin 6x\right]_0^{\pi/6}$

$= \frac{\pi}{2} \times \frac{\pi}{6}$

$= \frac{\pi^2}{12}$

b) $y = x \ln x - 1$

$y' = x \cdot \frac{1}{x} + 1 \cdot \ln x$

$= 1 + \ln x$

$y'' = \frac{1}{x}$

(i) Stationary point if $1 + \ln x = 0$

$\Rightarrow x = \frac{1}{e}, \; y = \frac{-1}{e} - 1$

At $x = \frac{1}{e}$, $y'' > 0 \therefore$ minimum

(ii) By Newton's method,

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= x_1 - \frac{x_1 \ln x_1 - 1}{1 + \ln x_1}$

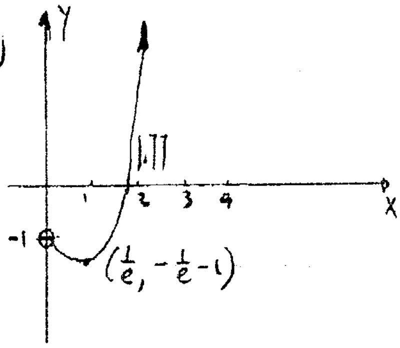
If $x_1 = 2$, $x_2 = 2 - \frac{2\ln 2 - 1}{1 + \ln 2}$

$= 1.77$

(iii) As $x > 0$, $y'' = \frac{1}{x} > 0$

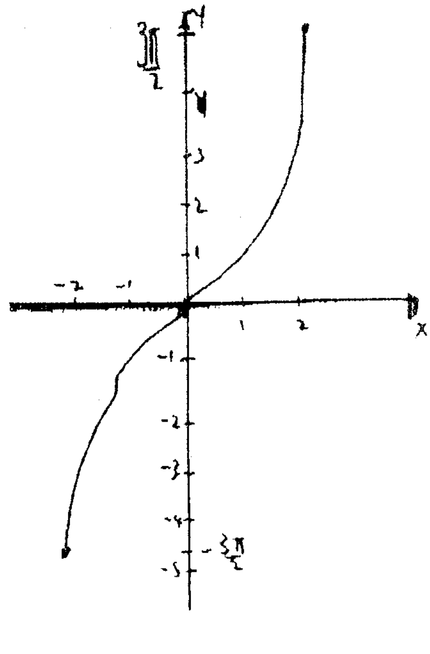
$\therefore$ curve always concave up

(iv)




(c) Domain: $-2 \le x \le 2$

Range: $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$



QUESTION 4

a) $S(n)$: $3^{2n} - 1 = 8I$, where $I$ is an integer

Test. $S(1)$: $3^2 - 1 = 8 = 8 \times 1$

$\therefore S(1)$ is true

Assume $S(k)$: $3^{2k} - 1 = 8I$, $I$ an integer

Prove $S(k+1)$.

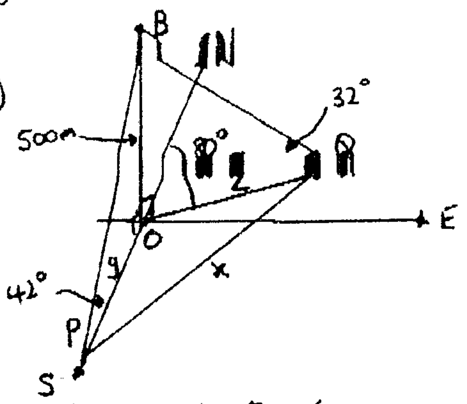
$$\begin{aligned} LHS &= 3^{2(k+1)} - 1 \\ &= 3^{2k+2} - 1 \\ &= 3^{2k} . 3^2 - 1 \\ &= (3^{2k} - 1) . 9 + 8 \\ &= 8I . 9 + 8 \\ &= 8(9I + 1) \end{aligned}$$

But $I$ is an integer; so $9I + 1$ is an integer

$\therefore$ If $S(k)$ is true, $S(k+1)$ is true.

But $S(1)$ is true so $S(2)$ is true and so on for all integers, $n$, greater than 0.

b)




$\tan 42 = 500/y$

$\therefore y = 500/\tan 42$

Similarly, $z = 500/\tan 32$

$\angle POQ = 100°$

By the same rule:

$$x^2 = \left(\frac{500}{\tan 42}\right)^2 + \left(\frac{500}{\tan 32}\right)^2 - 2 \frac{500}{\tan 42} \frac{500}{\tan 32} \cos 100$$

$x = 1050 \cdot 21$

$\doteqdot 1050$ metres

(c) For A + B: let $p$ = faulty gearbox

$q$ = good gearbox

$X$ = no. of faulty gearboxes.

For A: $P(X=r) = {}^{10}C_r (\cdot 05)^r (\cdot 95)^{10-r}$

B: $P(X=k) = {}^{10}C_k (\cdot 07)^k (\cdot 93)^{10-k}$

To get 2 faulty gearboxes:

$r = 2$; $k = 0$

$\Rightarrow {}^{10}C_2 (\cdot 05)^2 (\cdot 95)^8 \times {}^{10}C_0 (\cdot 07)^0 (\cdot 93)^{10}$

$r = 1$; $k = 1$

$\Rightarrow {}^{10}C_1 (\cdot 05)^1 (\cdot 95)^9 \times {}^{10}C_1 (\cdot 07)^1 (\cdot 93)^9$

$r = 0$; $k = 2$

$\Rightarrow {}^{10}C_0 (\cdot 05)^0 (\cdot 95)^{10} \times {}^{10}C_2 (\cdot 07)^2 (\cdot 93)^8$

Adding gives

$P(X = 2) = 0 \cdot 2248$

QUESTION 5

(a) $\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$

$\dot{x} = V\cos\alpha$ $\qquad$ $\dot{y} = -10t + V\sin\alpha$

$x = Vt\cos\alpha$ $\qquad$ $y = -5t^2 + Vt\sin\alpha$

(assuming (0,0) at point of impact)

(i) At $t = 1\cdot5$, $x = 60$; $y = 2\cdot25$

$\Rightarrow 60 = V\cos\alpha \times 1\cdot5 \Rightarrow V\cos\alpha = 40$

$\Rightarrow -5\times(1\cdot5)^2 + V\sin\alpha \times 1\cdot5 = 2\cdot25$

$\Rightarrow V\sin\alpha = 9$

whence $V = 41$ m/s

& $\tan\alpha = 9/40$

so $\alpha = 12\cdot68°$

(ii) When $y = 0$: $-5t^2 + Vt\sin\alpha = 0$

$t(-5t + V\sin\alpha) = 0$

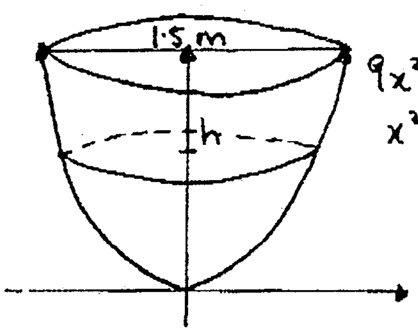
$\Rightarrow t = 0$ or $t = \frac{V\sin\alpha}{5} = \frac{9}{5}$

so $x = Vt\cos\alpha$

$= 40 \times \frac{9}{5}$

$= 72$ metres.

(b)




$9x^2 = 4y$

$x^2 = \frac{4}{9}y$

$V = \pi\int x^2\,dy$

$V = \pi\int_0^h \frac{4}{9}y\,dy$

$= \frac{4\pi}{9}\left[\frac{y^2}{2}\right]_0^h$

$= \frac{2\pi h^2}{9}$

(ii) If $x = 0\cdot75$, $y = 1\cdot2656$ so the volume when full is

$V = \frac{2\pi}{9} \times (1\cdot2656)^2 = 1\cdot1183\ m^3$

$\therefore$ when half full, $V = 0\cdot5591$

i.e. $\frac{2\pi}{9} \cdot h^2 = 0\cdot5591$

$h^2 = 0\cdot8009$

$h = 0\cdot8949$

$\doteqdot 0\cdot9$ m.

(iii) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$= \frac{4\pi}{9}\cdot h \times \frac{dh}{dt}$

$\frac{dV}{dt} = 0\cdot1\ m^3/s$; $h = 0\cdot2$ m

so $0\cdot1 = \frac{4\pi}{9}\cdot 0\cdot2 \times \frac{dh}{dt}$

$\frac{dh}{dt} = 0\cdot0143$ m/s

i.e about $1\cdot4$ cm/s.

QUESTION 6

(a) $a = \dfrac{1}{\sqrt{x^2+16}}$

$\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \dfrac{1}{\sqrt{x^2+16}}$

$\tfrac{1}{2}v^2 = \ln\left(x+\sqrt{x^2+16}\right) + C$

At $t=0, v=0, x=0$

$\Rightarrow C = -\ln 4$

At $x=3$,

$\tfrac{1}{2}v^2 = \ln\left(3+\sqrt{3^2+16}\right) - \ln 4$

$= \ln 8 - \ln 4$

$\therefore v^2 = 2\ln 2$

$v = 1.1774 \doteqdot 1.18$

(b) $T = \dfrac{2\pi}{n} = \pi \Rightarrow n = 2$

For Simple Harmonic Motion:

$v^2 = n^2(a^2 - x^2)$

and $\ddot{x} = -n^2 x$

When $x = 5$, $v = 2$

$\therefore 4 = 4(a^2 - 25)$

$\therefore a = \pm\sqrt{26}$

Maximum velocity when $x = 0$:

$v^2 = 4 \times 26 \Rightarrow v = 2\sqrt{26}$ cm/s

Maximum acceleration when $x = \sqrt{26}$

$\ddot{x} = -4 \times \sqrt{26} \Rightarrow \ddot{x} = -4\sqrt{26}$ cm/s²

(c) i. $T = T_0 + Ae^{kt}$

$\dfrac{dT}{dt} = 0 + A.e^{kt} \times k$

$= (T - T_0) \times k$

$= k(T - T_0)$

ii. $T = 24°$ when $t = 0$

$T_0 = -40°$

$T = 19°$ when $t = 5$

So: $24 = -40 + Ae^0$

$\therefore A = 64$

and $19 = -40 + 64e^{5k}$

$59 = 64e^{5k}$

$e^{5k} = 59/64$

$5k = \log_e(59/64)$

$\therefore k = \tfrac{1}{5}\log_e(59/64) = -0.01$

So for $T = 0°$:

$0 = -40 + 64e^{-0.0163t}$

$\dfrac{40}{64} = e^{-0.0163t}$

$-0.0163t = \log_e\left(\dfrac{40}{64}\right)$

$t = \dfrac{\log_e(40/64)}{-0.0163}$

$= 28.8893$

$\doteqdot 29$ seconds.

QUESTION 7

(a) Let the amount invested be \$P.

$R = 1 + \frac{6.2}{100} = 1.062$

Let $A_i$ = accumulated value of the $i$ th investment:

$A_1 = P(1.062)^{44}$

$A_2 = P(1.062)^{43}$

$\vdots$

$A_{44} = P(1.062)^1$

$\text{Total} = P(1.062) + P(1.062)^2 + \ldots + P(1.062)^{44}$

$= P.(1.062).\frac{(1.062^{44} - 1)}{(1.062 - 1)}$

$= P.\ 1.062.\frac{(1.062^{44} - 1)}{0.062}$

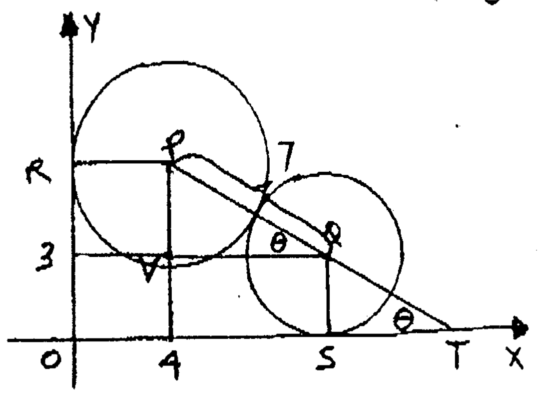
This must equal 500 000

$\therefore P = \frac{500000 \times 0.062}{1.062[(1.062)^{44} - 1]}$

$= \$2226.83$

He must invest \$2226.83 per year

(b)




(i) $OR = 3 + PV = 3 + 7\sin\theta$

$OS = 4 + VQ = 4 + 7\cos\theta$

(ii) $RS^2 = OR^2 + OS^2$

$= (3 + 7\sin\theta)^2 + (4 + 7\cos\theta)^2$

$= 9 + 42\sin\theta + 49\sin^2\theta$

$+ 16 + 56\cos\theta + 49\cos^2\theta$

$= 74 + 42\sin\theta + 56\cos\theta$

(iii) $r\cos(\theta - \alpha) = 42\sin\theta + 56\cos\theta$

$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = 42\sin\theta + 56\cos\theta$

$\therefore r\cos\alpha = 56\cos\theta$

$r\sin\alpha = 42\sin\theta$

so $\tan\alpha = \frac{42}{56} \Rightarrow \alpha = 36°52'$

d $r = 70$

$RS^2 = 74 + 70\cos(\theta - 36°52')$

iv. The maximum value of RS occurs when $\cos(\theta - 36°52') = 1$

i.e. when $\theta = 36°52'$

$RS^2 = 74 + 70 = 144$

$+ RS = 12$