NEWINGTON COLLEGE

Trial Higher School Certificate 2003

MATHEMATICS

Extension 2

Time allowed - 3 hours

(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- All questions are of equal value.
- All questions may be attempted.
- In every question, show all necessary working.
- Each question must be started in a new booklet.
- Marks may not be awarded for careless or badly arranged work.
- Approved non-progammable calculators may be used.
- A table of standard integrals is provided for your convenience.
- The answers to the questions in this paper are to be returned in separate bundles clearly marked Question 1and Question 2, etc.
- Each booklet must show the candidate's computer number.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Unless otherwise stated candidates should leave their answers in simplest exact form.

Question 1 (15 Marks) **Use a SEPARATE writing booklet.** **Marks**

a) Find $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x = \sec\theta$. **3**

b) Find $\int \frac{dx}{(x+1)(x^2+2)}$. **3**

c) Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$. **3**

d) Given that $I_n = \int \sec^n x\,dx$, $n \geq 2$, show that $(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$.

Hence evaluate $\int_0^{\frac{\pi}{4}} \sec^6 x\,dx$. **6**

/Q2 ...Page 2

Question 2 (15 Marks) **Use a SEPARATE writing booklet.**

a) Let $z=-1+i\sqrt{3}$. 5

i) Express z in modulus-argument form.

ii) Hence or otherwise, find in real-imaginary form:

(α) z^5

(β) $z\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$.

b) Sketch the region represented by

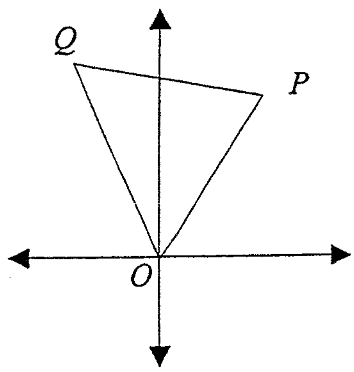
$|z|\le 4$ and $\frac{\pi}{3}<\arg z\le\frac{2\pi}{3}$. 2

c) Let ω be a complex root of $z^3=1$ 3

i) Show that $\omega^2+\omega+1=0$.

ii) Hence simplify $(1+\omega)^8$.

d) The points P, Q and O represent the complex numbers z_1, z_2 and z_3 respectively. 5



i) If P, Q and O are vertices of an equilateral triangle, show that

$\frac{z_2}{z_1}=\frac{1+i\sqrt{3}}{2}$ and deduce that $z_1^2+z_2^2=z_1z_2$.

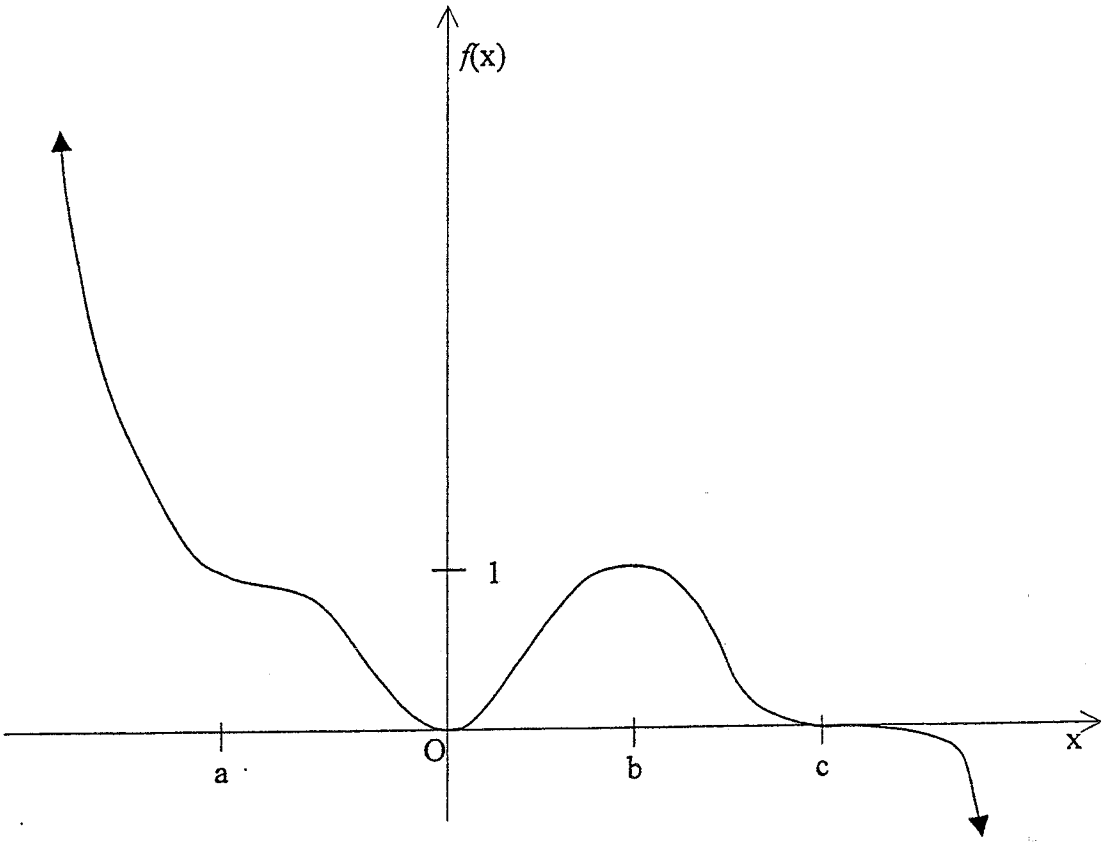
ii) Deduce that if z_1, z_2 and z_3 are ANY three complex numbers at the vertices of an equilateral triangle then

$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$.

/Q3 ...Page 3

Question 3 (15 Marks) **Use a SEPARATE writing booklet.** **Marks**

a)



Using the above function $f(x)$, sketch

i) $f'(x)$ **2**

ii) $f''(x)$ **2**

iii) $\dfrac{1}{f(x)}$ **2**

iv) $f(|x|)$ **3**

v) $\ln f(x)$ **3**

b) Sketch showing all asymptotes the graph of $y^2 = \dfrac{x^2}{x^2 - 1}$. **3**

/Q4 …Page 4

Question 4 (15 Marks) **Use a SEPARATE writing booklet.**

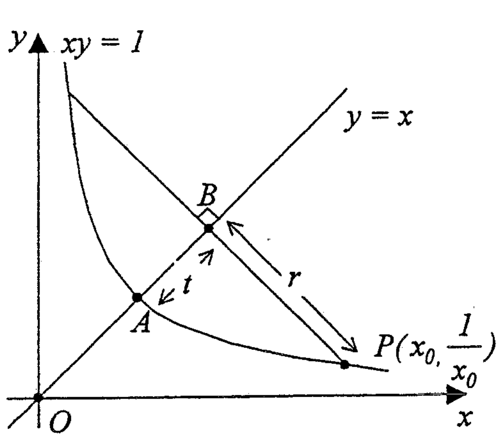
a) $x^3 + 3px + q = 0$ has a double root of $x = k$. **4**

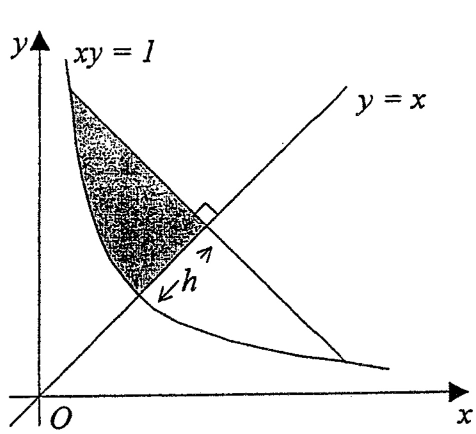
i) Show that $p = -k^2$.

ii) Show that $4p^3 + q^2 = 0$.

iii) Hence factorise $x^3 - 6ix + 4 - 4i$ into linear factors, given that it has a repeated factor.

b)





i) From the diagram on the left above, show that $2r^2 = \left(x_0 - \dfrac{1}{x_0}\right)^2$. **2**

ii) Hence show that $OP^2 = 2(1 + r^2)$. **2**

iii) Hence or otherwise show that $r^2 = \left(t + \sqrt{2}\right)^2 - 2$ **2**

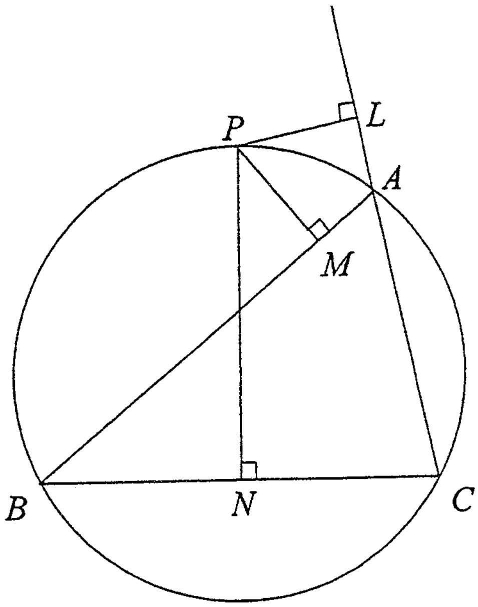
iv) Hence find the volume of the solid formed by rotating the shaded area above through 2π radians about the line $y = x$. **5**

Question 5 (15 Marks) **Use a SEPARATE writing booklet.**

a) i) Show that the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (where $a > b$) at the point $P(a\sec\theta, b\tan\theta)$ has the equation $bx\sec\theta - ay\tan\theta = ab$ **4**

ii) If this tangent passes through a focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a > b$) show that it is parallel to one of the lines $y = x$, $y = -x$ and that its point of contact with the hyperbola lies on the directrix of the ellipse. **5**

b)



In the diagram above, ABC is a triangle inscribed in a circle. P is a point on the minor arc AB. L, M and N are the feet of the perpendiculars from P to CA (produced), AB, and BC respectively. **6**

i) Copy the diagram.
ii) State a reason why P, M, A and L are concyclic points.
iii) State a reason why P, B, N and M are concyclic points.
iv) Show that L, M and N are collinear.

/Q6 …Page 6

Question 6 (15 Marks) **Use a SEPARATE writing booklet.**

A body of unit mass falls under gravity through a resisting medium. The body falls from rest from a height of *50* metres above the ground. The resistance to its motion is $\frac{v^2}{100}$ where v metres per second is the speed of the body when it has fallen a distance x metres.

a) Draw a diagram to show the forces acting on the body. Show that the equation of motion of the body is $\ddot{x} = g - \frac{v^2}{100}$. **3**

b) Show that the terminal velocity V of the body is given by: $V = \sqrt{100g}$. **1**

c) Show that $v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$. **3**

d) Find the distance fallen in metres until the body reaches a velocity equal to *50%* that of the terminal velocity. **3**

e) Find the velocity reached as a percentage of the terminal velocity when the body hits the ground. **2**

f) If $v = v_1$ when $x = d$ and $v = v_2$ when $x = 2d$ show that $v_2^2 = v_1^2\left(2 - \frac{v_1^2}{V^2}\right)$. **3**

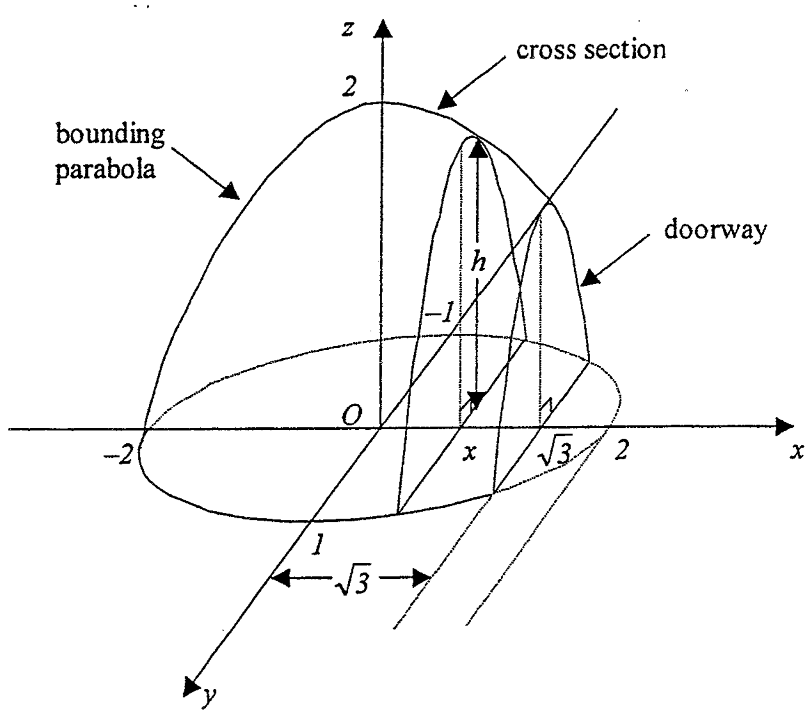
/Q7 …Page 7

Question 7 (15 Marks) **Use a SEPARATE writing booklet.**

a) Use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to show that $\cos^4\theta = \frac{1}{8}(3+4\cos 2\theta+\cos 4\theta)$. **2**

b) A parabola passes through the points $(-a,0)$, $(0,h)$ and $(a,0)$ where $a>0,\ h>0$. Show that the area enclosed by this parabola and the x axis is $\frac{4}{3}ah$. **2**

c) A mathematically inclined Eskimo decides to build himself an igloo based on conic sections. The base of the interior of the igloo is an ellipse with semi-axes *2* metres and *1* metre. Vertical cross sections taken at right angles to the major axis are bounded by parabolic arcs, the axis of the parabola being vertical and passing through the major axis of the ellipse. The cross section taken vertically through the centre of the ellipse along the major axis is also bounded by a parabolic arc with axis vertical through the centre of the ellipse. The igloo is shown in the diagram, with coordinate axes taken through the centre of the elliptical base. The maximum height of the interior is to be *2* metres as shown, and the internal dimensions are indicated on the diagram. The Eskimo forms the entrance to the igloo by slicing the structure vertically at right angles to the major axis of the ellipse at a distance $\sqrt{3}$ metres from the centre and removing the material from this point outwards to the end of the major axis, as shown in the diagram.



/Q7*cont*...Page 8

Question 7 (continued)

i) Find the equation of the ellipse which bounds the floor of the interior of the igloo. **1**

ii) By finding the equation of the bounding parabola indicated in the diagram, show that the height h of the vertical cross section, drawn at a distance x from the centre of the ellipse is given by: $h=\frac{1}{2}\left(4-x^2\right)$. **2**

iii) Find the maximum height and width of the doorway. **2**

iv) By finding the area of the indicated cross section of height h, show that the volume of air in the igloo is given by $V=\frac{1}{3}\int_{-2}^{\sqrt{3}}\left(4-x^2\right)^{\frac{3}{2}}\,dx$. **3**

v) Calculate the volume of air in the igloo by using the substitution $x = 2\sin\theta$ in the above integral for V. **3**

/Q8…Page 9

Question 8 (15 Marks) **Use a SEPARATE writing booklet.**

a) A polynomial $P(x)$ is divided by x^2-a^2 where $a\neq 0$ and the remainder is $px+q$.

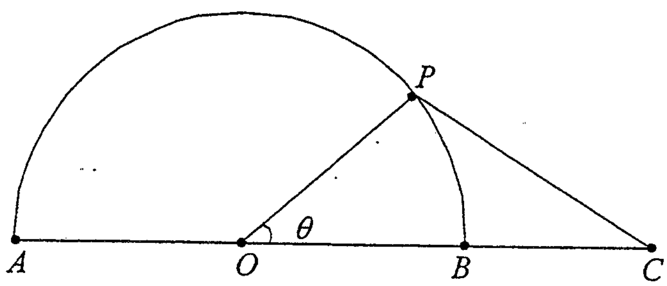
i) Show that $p=\frac{1}{2a}[P(a)-P(-a)]$ and $q=\frac{1}{2}[P(a)+P(-a)]$.. 2

ii) Find the remainder when the polynomial $P(x)=x^n-a^n$ is divided by x^2-a^2 for the cases

α) n even 2

β) n odd 2

b) In the diagram above the fixed points *A, O, B* and *C* are on a straight line such that *AO =OB=BC=1* unit. The points *A* and *B* are also joined by a semicircle and *P* is a variable point on this semicircle such that the angle *POC* is *θ*. *R* is the region bounded by the arc *AP* of the semicircle and the straight lines *AC* and *PC*.



i) Show that the area *S* of *R* is given by $S=\frac{\pi}{2}-\frac{\theta}{2}+\sin\theta$ and find the value of θ for which *S* is a maximum. 3

ii) Show that the perimeter *L* of *R* is given by $L=3+\pi-\theta+\sqrt{5-4\cos\theta}$. 2

iii) Show that *L* has just one stationary point and that occurs at the same value of θ for which *S* is a maximum. Find the least value of *L* and the greatest value of *L* in the interval $0\leq\theta\leq\pi$ 4

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

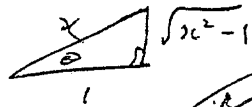
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Q1

a) $\int \frac{dx}{x\sqrt{x^2-1}}$ let $x = \sec\theta$



$= \int \frac{\sec\theta . \tan\theta \, d\theta}{\sec\theta \sqrt{\sec^2\theta - 1}}$ ✓

$= \int 1 \, d\theta$ ✓

$= \theta + c$

$= \tan^{-1}(\sqrt{x^2-1}) + c$ ✓

b) $\int \frac{dx}{(x+1)(x^2+2)} = \frac{1}{3}\int \left[\frac{1}{x+1} - \frac{x-1}{x^2+2}\right] dx$ ✓

$= \frac{1}{3}\int \left\{\frac{1}{x+1} - \frac{x}{x^2+2} + \frac{1}{2+x^2}\right\} dx$ ✓

$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2+2| + \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$ ✓

c) $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$ let $t = \tan\frac{x}{2}$

where $\cos x = \frac{1-t^2}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$ ✓ – t form

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$= \int_0^1 \frac{\frac{2}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)} dt$

$= 2\int_0^1 \frac{1}{9+t^2} dt$ ✓

$= \frac{2}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^1$

$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$ ✓

Q1 (d) $\int \sec^n x \, dx \quad , n \geq 2$

$= \int \sec^{n-2} x \frac{d[\tan x]}{dx} . dx$

$\therefore \int \sec^n x \, dx = \tan x \sec^{n-2} x - (n-2) \int \tan x . \sec^{n-3} x . \sec x \tan x \, dx$ ✓

$= \tan x \sec^{n-2} x - (n-2) \int \tan^2 x . \sec^{n-2} x \, dx$

$= \tan x \sec^{n-2} x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x \, dx$ ✓

$I_n = \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$ ✓

$\therefore (n-1) I_n = \tan x \sec^{n-2} x + (n-2) I_{n-2}$ # ✓

$\int_0^{\frac{\pi}{4}} \sec^6 x \, dx = \frac{1}{5} [\tan x \sec^4 x]_0^{\frac{\pi}{4}} + \frac{4}{5} I_4$

$= \frac{4}{5} + \frac{4}{5} \times \left[\frac{1}{3} [\tan x \sec^2 x]_0^{\frac{\pi}{4}} + \frac{2}{3} I_2 \right] \quad \sec\frac{\pi}{4} = \sqrt{2}$

$= \frac{4}{5} + \frac{4}{5} \times \frac{1}{3} \times 2 + \frac{8}{15} \times \frac{I}{2}$

$= \frac{4}{5} + \frac{8}{15} \times \int_0^{\frac{\pi}{4}} \sec^2 x \, dx$ ✓

$= \frac{4}{5} + \frac{8}{15} [\tan x]_0^{\frac{\pi}{4}}$

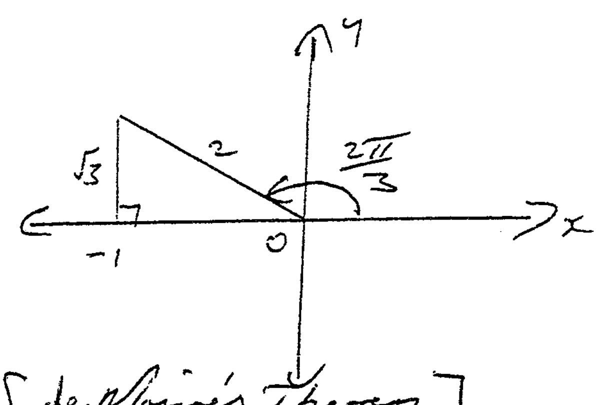
=

$= \# \quad 1 \frac{13}{15}$ # ✓

Q 2

(a) (i) $z = -1 + i\sqrt{3}$

$z = 2\left(\text{cis } \frac{2\pi}{3}\right)$ ✓



(ii) (α) $Z^5 = \left(2 \text{ cis } \frac{2\pi}{3}\right)^5$

$= 32 \text{ cis } \frac{10\pi}{3}$ [de Moivre's Theorem]

$= 32 \text{ cis } \left(-\frac{2\pi}{3}\right)$

$= 32\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$

$= -16 - 16\sqrt{3}\, i$ ✓✓

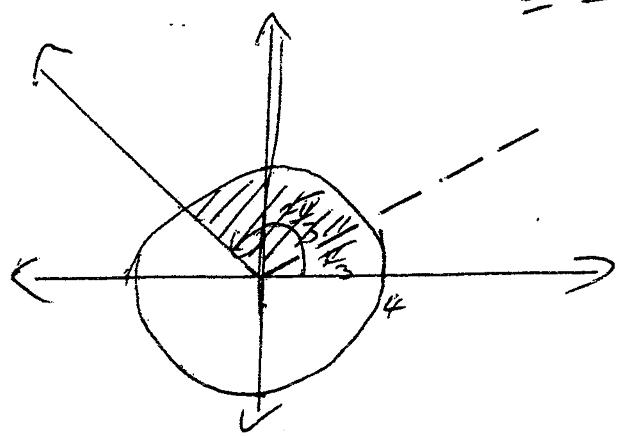
(β) $Z\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2 \text{ cis } \frac{2\pi}{3} \times \text{cis } \frac{\pi}{6}$

$= 2 \text{ cis } \frac{5\pi}{6}$

$= 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$

$= -\sqrt{3} + i$ ✓✓

(b)



✓ – $|z| < 4$, $\frac{\pi}{3} < \arg z \le \frac{2\pi}{3}$

✓ – correct shading.

(c) (i) The roots of $z^3 - 1 = 0$ are $1, \omega, \omega^2$

or Find roots $1, \text{cis } \frac{2\pi}{3}, \text{cis}\left(-\frac{2\pi}{3}\right)$

as Sum of roots $= -\frac{\text{coeff of } z^2}{\text{coeff of } z^3}$

$= 0$ ✓

$\therefore 1 + \omega + \omega^2 = 0$

(ii) $(1+\omega)^8 = (-\omega^2)^8$ ✓

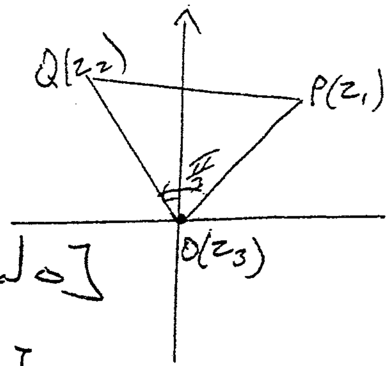
$= \omega^{16}$

$= (\omega^3)^5 \times \omega$ ✓

$= \omega$ ... $\omega^3 = 1$

(d)

(i) Now $\arg z_2 - \arg z_1 = \frac{\pi}{3}$



$\therefore \arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{3}$ [equilateral △]

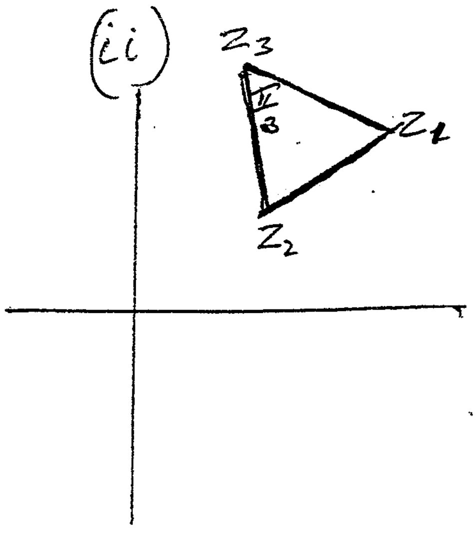
and $\frac{|z_2|}{|z_1|} = 1$ [equil. △]

$$\therefore \frac{z_2}{z_1} = 1 \operatorname{cis}\frac{\pi}{3}$$
$$= \frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$= \frac{1 + i\sqrt{3}}{2} \#$$

Now as $z_2 = z_1\left(\frac{1+i\sqrt{3}}{2}\right)$

$$z_1^2 + z_2^2 = z_1^2 + z_1^2\left(\frac{1+i\sqrt{3}}{2}\right)^2$$
$$= z_1^2\left[1 + \left(\frac{1+i\sqrt{3}}{2}\right)^2\right]$$
$$= z_1^2\left[1 + \frac{1}{4}\left[1 + 2\sqrt{3}i - 3\right]\right]$$
$$= z_1^2\left[1 + \frac{\sqrt{3}i}{2} - \frac{1}{2}\right]$$
$$= z_1^2\left(\frac{1+i\sqrt{3}}{2}\right)$$
$$= z_1 \times z_1\left(\frac{1+i\sqrt{3}}{2}\right)$$
$$= z_1 z_2 \#$$

(ii)



Now as $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right)$

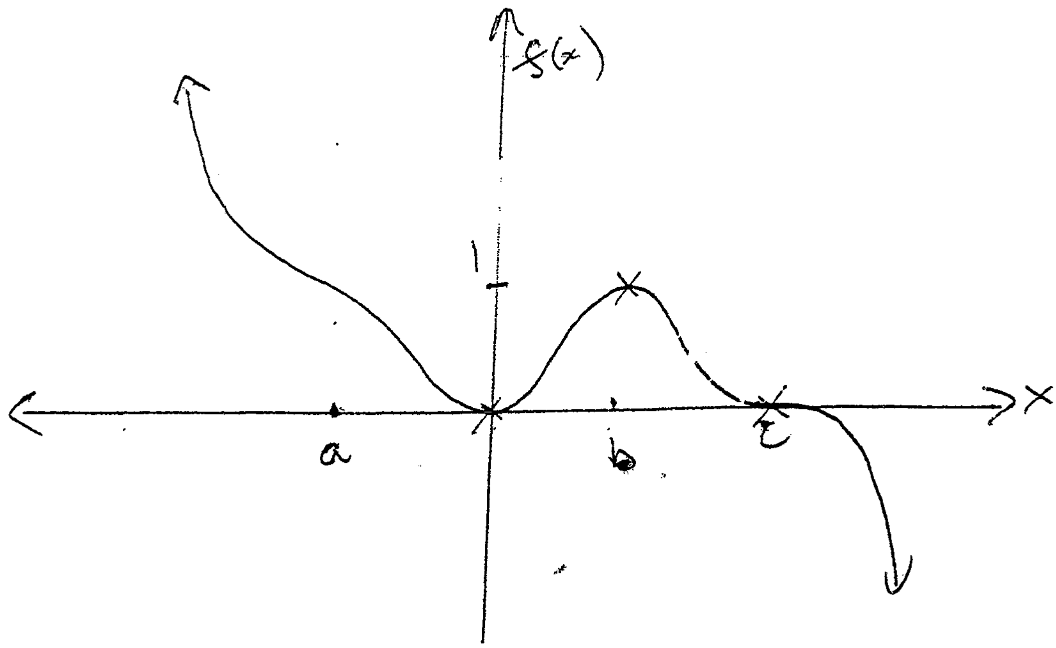
$$\therefore \frac{z_2 - z_1}{z_3 - z_1} = \frac{z_1 - z_3}{z_2 - z_3}$$

$$\therefore z_2^2 - z_1 z_3 - z_2 z_3 + z_1 z_3 = z_1 z_3 - z_3^2 - z_1^2 + z_1 z_3$$

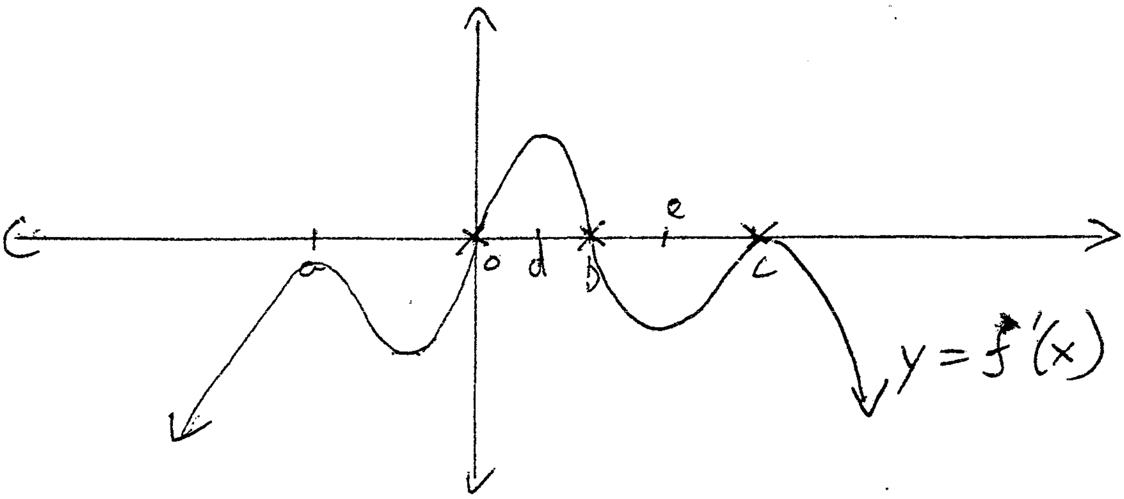
$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_1 z_3 + z_2 z$$

Q3

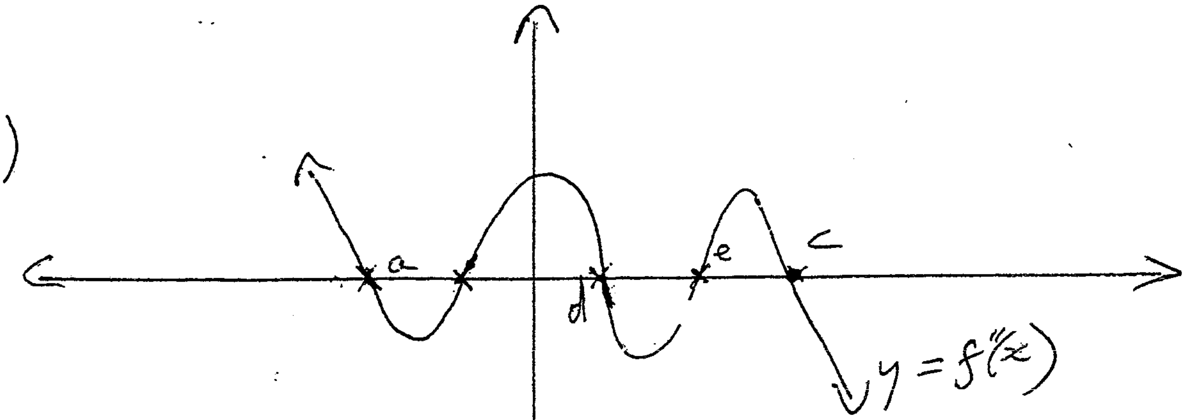
(a)



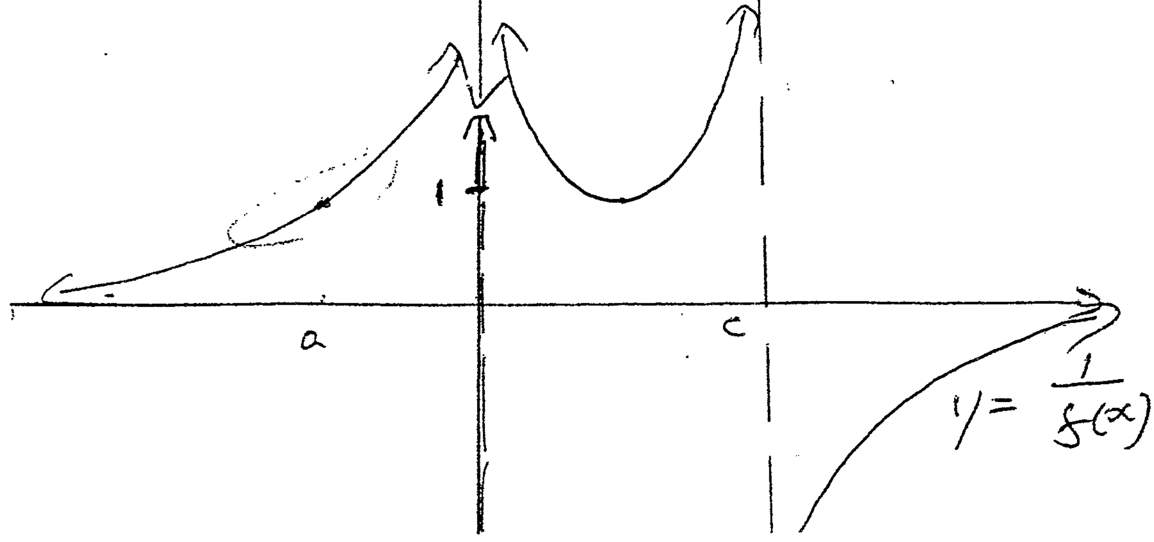
i)



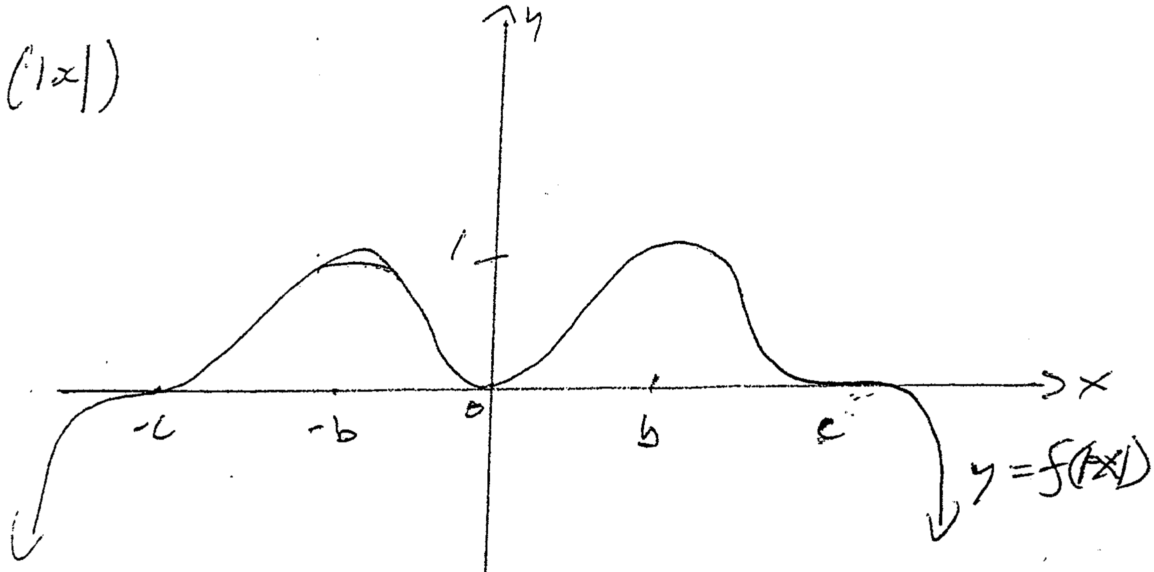
ii)



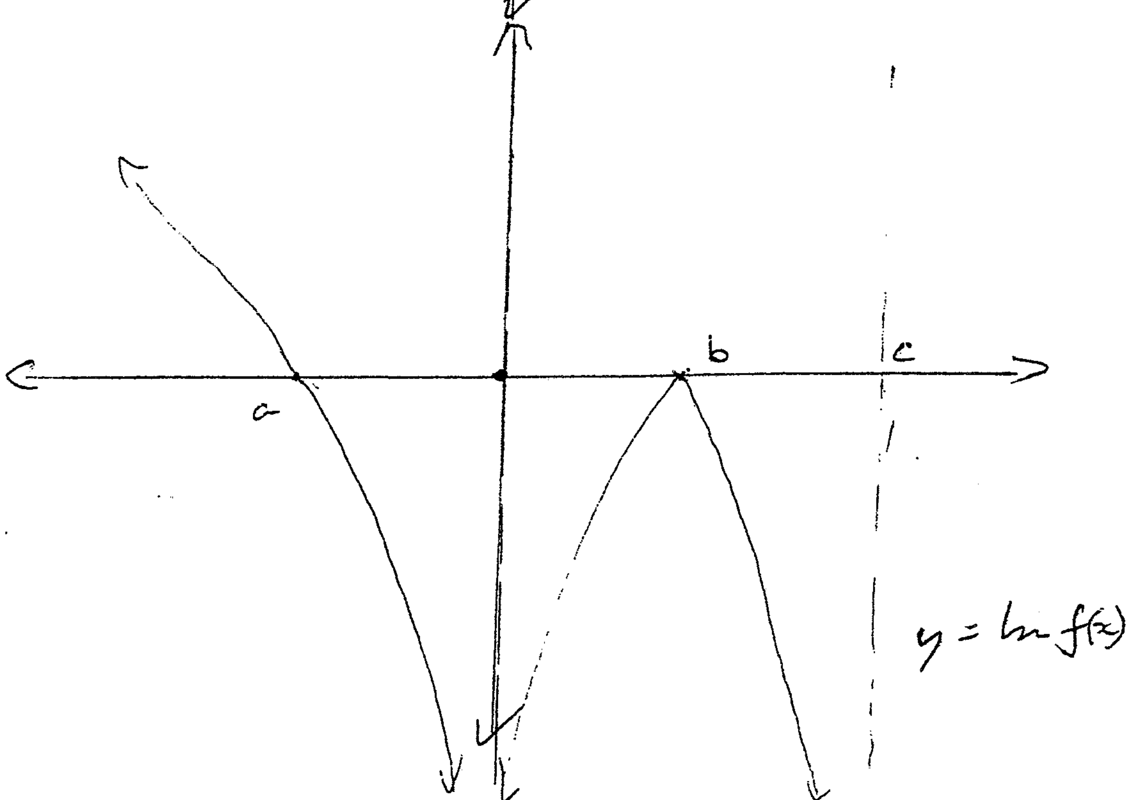
iii)



(iv) $y = f(|x|)$



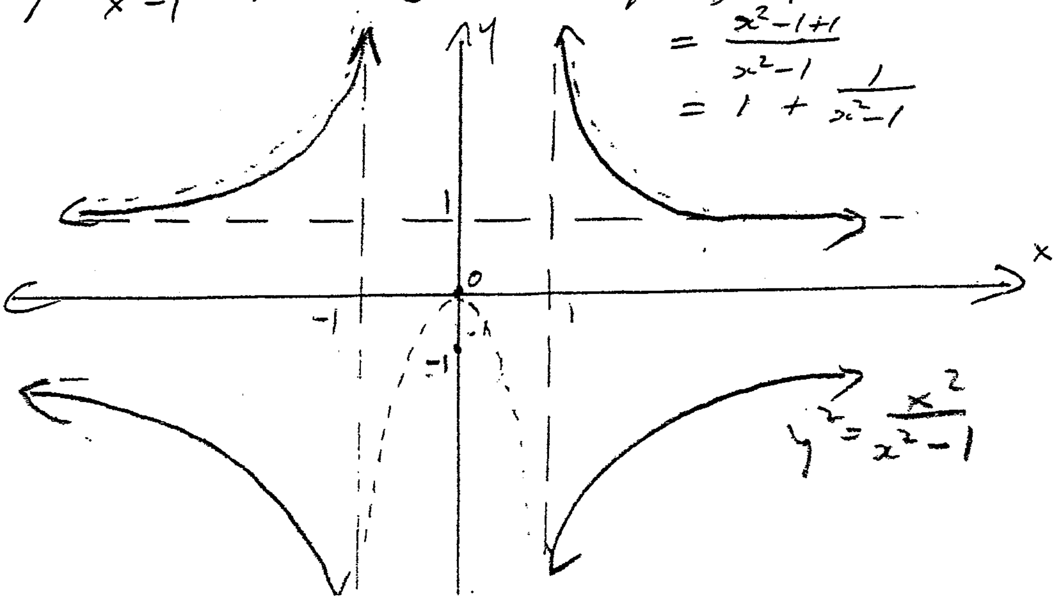
v)



b) $y^2 = \frac{x^2}{x^2-1}$, Firstly sketch $y = \frac{x^2}{x^2-1}$

$= \frac{x^2-1+1}{x^2-1}$

$= 1 + \frac{1}{x^2-1}$



Q4 a) $x^3 + 3px + q = 0$ $x = k$ is a double root.

$\therefore k^3 + 3pk + q = 0$ ... ①

and $3k^2 + 3p = 0$

i) $\therefore p = -k^2$ ... ②

ii) from ① $k(k^2 + 3p) = -q$

$\therefore k^2(k^2+3p)^2 = q^2$

$\therefore -p \times (2p)^2 = q^2$

$\therefore -4p^3 = q^2$

$4p^2 + q^2 = 0$ #

iii) let $k = a + ib$

now $3p = -6i$

$p = -2i$

$\therefore$ as $k^2 = -p$

$\therefore (a+ib)^2 = 2i$

$\therefore k = \pm(1+i)$, take $k = (1+i)$

$\therefore x^3 - 6ix + 4 - 4i \equiv \{x + (1+i)\}^2\{x + (c+id)\}$

$= \{x + (1+i)\}^2\{x - 2 - 2i\}$

7(c)

i) $\frac{x^2}{4} + y^2 = 1$ ✓

ii) $y = \frac{2}{4}(4 - x^2)$ ✓

$h = \frac{1}{2}(4 - x^2)$ ✓

iii) $x = \sqrt{3}$

$\therefore h = \frac{1}{2}(4 - 3)$

$= 0.5$

$x = \sqrt{3}$

$y = \sqrt{1 - \frac{3}{4}}$

$= 0.5$

$\therefore$ Max Height is 0.5 m ✓

Max Width is 1 m. ✓

iv). Slice is parabolic segment as in 7(b), where

$a = \sqrt{1 - \frac{x^2}{4}}$

$\therefore h = \frac{1}{2}(4 - x^2)$ ✓

$\therefore A_{slice} = \frac{4}{3} \cdot \frac{1}{2}\sqrt{4-x^2} \cdot \frac{1}{2}(4 - x^2)$

$= \frac{1}{3}(4 - x^2)^{3/2}$ ✓

$\therefore V_{slice} = \frac{1}{3}(4-x^2)^{3/2}\delta x$

$\therefore V = \lim_{\delta x \to 0} \sum \frac{1}{3}(4-x^2)^{3/2}\delta x$ ✓

$= \frac{1}{3}\int_{-2}^{\sqrt{3}}(4-x^2)^{3/2}dx$

Q7(v) $x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\frac{dx}{d\theta} = 2\cos\theta$

$x = -2$, $\theta = \frac{\pi}{2}$

$x = \sqrt{3}$, $\theta = \frac{\pi}{3}$

$\therefore \quad V = \frac{1}{3}\int_{-2}^{\sqrt{3}} (4-x^2)^{3/2}\,dx$

$= \frac{16}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \cos^3\theta \cdot \cos\theta \, d\theta$

$= \frac{16}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \cos^4\theta \, d\theta$

$= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} (3 + 4\cos 2\theta + \cos 4\theta)\,d\theta$ from 7(i)

$= \frac{2}{3}\left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{3}}$

$= \frac{2}{3}\left[\frac{5\pi}{2} + \sqrt{3} - \frac{\sqrt{3}}{8}\right]$

$= \frac{1}{12}\left[20\pi + 7\sqrt{3}\right]$

Q8 (a) i) $P(x) = (x^2 - a^2)Q(x) + (px + q)$
$= (x-a)(x+a)Q(x) + (px+q)$ (✓)

$P(a) = pa + q$
$P(-a) = -pa + q$ (✓)

so $P(a) - P(-a) = 2pa$

$\therefore p = \frac{1}{2a}[P(a) - P(-a)]$ $q = \frac{1}{2}[P(a) + P(-a)]$

ii) When $P(x) = x^n$ then

α) when n is even, $P(a) = 0$ and $P(-a) = 0$ (✓✓)
$\therefore$ remainder is 0.

β) when n is odd, $P(a) = 0$ and $P(-a) = -2a^n$ (✓✓)
so remainder is $a^{n-1}x - a^n$

b) i) S = area of sector AOP + area of triangle OCP

$= \frac{1}{2}(1)^2 . A\hat{O}P + \frac{1}{2} OC . OP . \sin P\hat{O}C$

$= \frac{1}{2}(\pi - \theta) + \sin\theta$ (✓✓)

$= \frac{\pi}{2} - \frac{\theta}{2} + \sin\theta$

$S' = -\frac{1}{2} + \cos\theta$

$\therefore S' = 0$ when $\theta = \frac{\pi}{3}$ sign of S': $/ \frac{\pi}{3} \backslash$ (✓)

$\therefore$ Max when $\theta = \frac{\pi}{3}$

(ii) L = length AC + arc length AP + length PC (✓✓)

$= 3 + (\pi - \theta) + \sqrt{5 - 4\cos\theta}$

(iii) $L' = -1 + \frac{2\sin\theta}{\sqrt{5 - 4\cos\theta}}$ (✓)

let $L' = 0$ $\therefore 2\sin\theta = \sqrt{5 - 4\cos\theta}$ (✓)

$\therefore 4\sin^2\theta = 5 - 4\cos\theta$

$4(1 - \cos^2\theta) = 5 - 4\cos\theta$

$4\cos^2\theta - 4\cos\theta + 1 = 0$

$(2\cos\theta - 1)^2 = 0$ (✓)

$\therefore \cos\theta = \frac{1}{2}$

L is stationary when $\theta = \frac{\pi}{3}$ $[0 \leq \theta \leq \pi]$ (✓)

$\therefore$ when $\theta = \pi$, $L_{min} = 6$ and when $\theta = 0$,
$L_{max} = 4 + \pi$