NORTH SYDNEY GIRLS HIGH SCHOOL

2006
TRIAL HSC EXAMINATION

Mathematics Extension 1

General Instructions
- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 84
Attempt Questions 1 – 7
All questions are of equal value

NAME:________________ **TEACHER:**________________

NUMBER:________________

QUESTION	MARK
1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
TOTAL	/84

QUESTION 1. (12 marks) Use a *separate* writing booklet **Marks**

a) Evaluate $\lim_{x \to 0} \frac{x}{3\tan 3x}$. **2**

b) When the polynomial $P(x)$ is divided by $x^2 - 1$ the remainder is $3x - 1$. What is the remainder when $P(x)$ is divided by $x - 1$? **2**

c) Solve the inequality $\frac{x^2 - 4}{x} > 0$. **2**

d) Using the substitution $u = 2 + x^2$, find $\int x\sqrt{2 + x^2}\, dx$. **2**

e) Divide the interval PQ internally in the ratio 4 : 9, where P is the point (2 , 3) and Q is (5 , –7). **2**

f) Differentiate $e^{2x} \cos x$. **2**

QUESTION 2. (12 marks) Use a *separate* writing booklet

a) Differentiate $\sin^{-1}(5x)$. **2**

b) Find:

i) $\int \frac{2}{1 + 9x^2}\, dx$ **2**

ii) $\int 5\cos^2 x\, dx$ **2**

c) If α, β, γ are the roots of the equation $x^3 - x^2 + 4x - 1 = 0$ find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$. **2**

d) Consider the function $f(x) = \frac{1}{2}\cos^{-1}(1 - 3x)$.

i) State the domain and range of $f(x)$. **2**

ii) Hence, or otherwise, sketch the graph of $y = f(x)$. **2**

QUESTION 3. (12 marks) Use a *separate* writing booklet

Marks

a) The only information given about a certain graph is that **2**
$f(2)=3$, $f'(2)=1$ and $f''(2)=-2$
Describe in as much detail as possible, the graph of $f(x)$ near $x=2$

b) A formula for the rate of change in population of a colony of bacteria is given by $P=3200+400e^{kt}$. **4**

If the population doubles after 20 hours, how long would it take to triple the original population?

c) i) Show that the equation $5x^4-4x^5-0\cdot9=0$ has a root between $x=0$ and $x=1$. **1**

ii) Starting with the approximation $x=1$ attempt to find an improved value for this root using Newton's Method. Explain why this attempt fails. **2**

d) i) Express $\sqrt{3}\cos x-\sin x$ in the form $R\cos(x+\alpha)$ where $0<\alpha<\dfrac{\pi}{2}$ and $R>0$. **2**

ii) Hence, or otherwise, solve $\sqrt{3}\cos x-\sin x=1$ for $0\le x\le\dfrac{\pi}{2}$. **1**

QUESTION 4. (12 marks) Use a *separate* writing booklet

a) Prove $\tan^{-1}\dfrac{2}{3}+\cos^{-1}\dfrac{2}{\sqrt{5}}=\tan^{-1}\dfrac{7}{4}$. **2**

b) Evaluate $\displaystyle\int_{\frac{1}{2}}^{\frac{e}{2}}\frac{\ln 2x}{x}\,dx$, using the substitution $u=\ln 2x$. **3**

c) i) Sketch the curve $y=x+\dfrac{4}{x}$ showing clearly all the stationary points and asymptotes. **3**

ii) Hence, or otherwise, find the values of k such that $x+\dfrac{4}{x}=k$ has no real roots. **1**

d) Use the method of mathematical induction to prove that, for all positive integers n, **3**

$$1+2+4+\ldots+2^{n-1}=2^n-1$$

QUESTION 5. (12 marks) Use a *separate* writing booklet

Marks

a) Given that $0<x<\dfrac{\pi}{4}$, prove that **2**

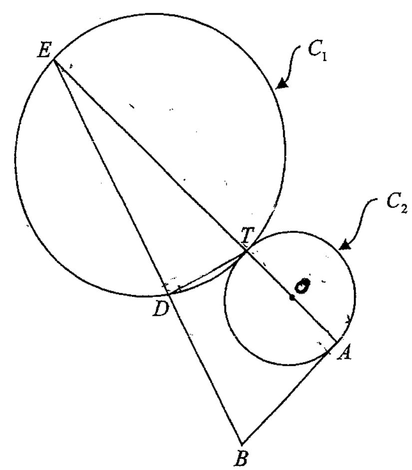
$$\tan\left(\frac{\pi}{4}+x\right)=\frac{\cos x+\sin x}{\cos x-\sin x}$$

b) i) Show that the graphs of $y=2x-1$ and $y=x^3$ intersect at $x=1$. **1**

ii) Find the size of the acute angle between the graphs at $x=1$. **2**

c) A polynomial is given by $p(x)=x^3+ax^2+bx-18$. **3**
Find values for a and b if $(x+2)$ is a factor of $p(x)$ and if -24 is the remainder when $p(x)$ is divided by $(x-1)$.

d)



NOT TO SCALE

Two circles C_1 and C_2 touch at T.
The line AE passes through O, the centre of C_2, and through T.
The point A lies on C_2 and E lies on C_1.
The line AB is a tangent to C_2 at A, D lies on C_1 and BE passes through D.
The radius of C_1 is R and the radius of C_2 is r.

i) Explain why $\angle EDT=90°$. **2**

ii) If $DE=2r$, show that $EB=\dfrac{2R(R+r)}{r}$. **2**

QUESTION 6. (12 marks) Use a *separate* writing booklet **Marks**

a) The volume, V, of a sphere of radius r mm is increasing at a constant rate of 200 mm^3 per second.

$\left(V = \frac{4}{3}\pi r^3 \quad ; \quad S = 4\pi r^2\right)$

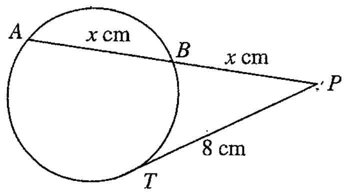
i) Find $\frac{dr}{dt}$ in terms of r. **2**

ii) Determine the rate of increase of the surface area, S, of the sphere when the radius is 50 mm **2**

b) In the diagram below, A, B and T are points on the circumference of the circle. P is an external point. The tangent PT is drawn 8 centimetres long, and B is the midpoint of secant AP.

Let AB be x centimetres. **3**
Find the value of x giving reasons.

NOT TO SCALE



c) i) Show that the equation of the normal at $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ is given by $x + py = ap^3 + 2ap$. **2**

ii) The normal intersects with the y-axis at point Q. Find the co-ordinates of Q and hence find the co-ordinates of R where R is the midpoint of PQ. **2**

iii) Hence find the Cartesian equation of the locus of R. **1**

QUESTION 7. (12 marks) Use a *separate* writing booklet **Marks**

a) Given that $x = \cos t + t\sin t$
$y = \sin t - t\cos t$

i) Show that $\frac{dx}{dt} = t\cos t$ **1**

ii) Hence, or otherwise, find $\frac{dy}{dx}$ in terms of t **2**

b) At the North Sydney Tennis Competition, Jemma served a ball from a height of 1·8 metres above the ground.
The ball was hit in a horizontal direction with an initial velocity of 35 m/s.
Assume that the equations of motion for the ball in flight are $y = -5t^2 + 1\cdot 8$ and $x = 35t$ where the acceleration due to gravity is taken at 10 m/s^2

i) How long does it take for the ball to hit the ground? **2**

ii) How far will the ball travel horizontally before bouncing? **1**

iii) The net is 0·95 metres high and is 14 metres away from where Jemma hit the ball. Will the ball clear the net? Explain your answer. **2**

c) A is the top of a vertical radio mast AB standing on level ground. **4**
C and D are points on the ground level such that C is due east of B and D is 500 metres due north of C.
The angles of elevation of A from C and D are respectively, 10°13′ and 7°18′
Calculate the height of the mast to the nearest metre.
Include a diagram with your answer.

END OF EXAMINATION

Year 12 2006 Trial HSC
Extension 1

Question 1

a) $\lim_{x \to 0} \frac{1}{3} \times \frac{3x}{\tan 3x} \times \frac{1}{3} = \frac{1}{9}$

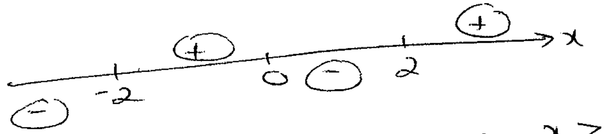
b) $P(x) = (x^2-1)Q(x) + (3x-1)$

$P(1) = 2$

$\therefore$ remainder when $P(x)$ is divided by $(x-1)$ is 2

c) $\frac{x^2(x^2-4)}{x} > 0 \times x^2$

$x(x-2)(x+2) > 0$



$\therefore \; -2 < x < 0$ OR $x > 2$

d) $u = 2 + x^2$

$\frac{du}{dx} = 2x$

$\int x\sqrt{2+x^2}\,dx = \frac{1}{2}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{2} u^{3/2} \times \frac{2}{3} + C$

$= \frac{1}{3} u^{3/2} + C$

$= \frac{1}{3}(2+x^2)^{3/2} + C$

$= \frac{1}{3}(2+x^2)\sqrt{2+x^2} + C$

Question 1 cont

e) $x = \frac{4 \times 5 + 9 \times 2}{4+9}$

$= \frac{38}{13}$

$y = \frac{4 \times -7 + 9 \times 3}{4+9}$

$= -\frac{1}{13}$

$\therefore$ point is $\left(\frac{38}{13}, -\frac{1}{13}\right)$

f) $\frac{d}{dx}(e^{2x}\cos x) = 2e^{2x}\cos x - e^{2x}\sin x$

$= e^{2x}(2\cos x - \sin x)$

Question 2

a) $\frac{d}{dx}(\sin^{-1} 5x) = \frac{1}{\sqrt{1-(5x)^2}} \times 5$

$= \frac{5}{\sqrt{1-25x^2}}$

b) i) $\int \frac{2}{1+9x^2}\,dx = 2\tan^{-1} 3x \times \frac{1}{3} + C$

$= \frac{2}{3}\tan^{-1} 3x + C$

ii) $\int 5\cos^2 dx = \frac{5}{2}\int (1+\cos 2x)\,dx$

$= \frac{5}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

$= \frac{5}{4}(2x + \sin 2x) + C$

Question 2 cont

c) $(\alpha+1)(\beta+1)(\gamma+1) = (\alpha+1)(\beta\gamma+\beta+\gamma+1)$

$= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1$

$= 1 + 4 + 1 + 1$

$= 7$

d) $f(x) = \frac{1}{2}\cos^{-1}(1-3x)$

i) Domain: $-1 \le 1-3x \le 1$

$-2 \le -3x \le 0$

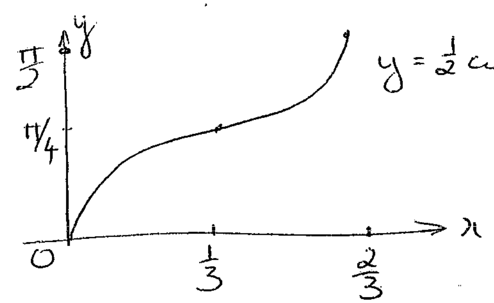
$2 \geq 3x \geq 0$

$0 \le x \le \frac{2}{3}$

Range: $0 \le \cos^{-1}(1-3x) \le \pi$

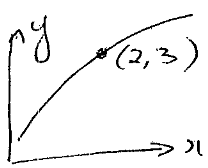
$0 \le f(x) \le \frac{\pi}{2}$

ii)



Question 3

a) at $x=2$; the value of $f(x)$ is 3
gradient of curve is 1
∴ curve is increasing
second derivative is 2
∴ curve is concave downwards



at (2,3) the curve is increasing & concave downwards

Question 3 cont

b) $P = 3200 + 400e^{kt}$

when $t=0$, $P = 3200 + 400$
$= 3600$

when $t = 20$, $P = 7200$

$7200 = 3200 + 400e^{20k}$

$4000 = 400e^{20k}$

$e^{20k} = 10$

$k = \frac{\ln 10}{20}$

$10800 = 3200 + 400e^{\frac{t\ln 10}{20}}$

$7600 = 400e^{\frac{t\ln 10}{20}}$

$e^{\frac{t\ln 10}{20}} = 19$

$t = \ln 19 \times \frac{20}{\ln 10}$

$\doteqdot 25.575\ldots$

Population triples after approximately 25.6 hrs

c) i) $f(x) = 5x^4 - 4x^5 - 0.9$ $\qquad f'(x) = 20x^3 - 20x^4$

$f(0) = -0.9 < 0$ $\qquad f'(1) = 0$

$f(1) = 0.1 > 0$

sign change ($f(x)$ is continuous)
∴ root exists between $x=0$ & $x=1$

ii) $x = 1 - \frac{f(1)}{f'(1)}$

$= 1 - \frac{0.1}{0}$

This attempt fails since $f'(1) = 0$
ie stationary point at $x=1$

Question 3 cont

d) i) $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\sin\alpha = 1$

$R\cos\alpha = \sqrt{3}$

$\tan\alpha = \frac{1}{\sqrt{3}}$ $\quad R^2 = 1+3$

$\alpha = \frac{\pi}{6}$ $\quad = 4$

$R = 2$

$\therefore \sqrt{3}\cos x - \sin x = 2\cos(x+\frac{\pi}{6})$

ii) $2\cos(x+\frac{\pi}{6}) = 1$

$\cos(x+\frac{\pi}{6}) = \frac{1}{2}$

$x+\frac{\pi}{6} = \frac{\pi}{3}$

$x = \frac{\pi}{6}$

Question 4

a) Let $\tan^{-1}\frac{2}{3} = x \Rightarrow \tan x = \frac{2}{3}$

$\cos^{-1}\frac{2}{\sqrt{5}} = y \Rightarrow \cos y = \frac{2}{\sqrt{5}}$

$\tan(\tan^{-1}\frac{2}{3} + \cos^{-1}\frac{2}{\sqrt{5}}) = \tan(x+y)$

$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$= \frac{\frac{2}{3}+\frac{1}{2}}{1-\frac{2}{3}\times\frac{1}{2}}$

$= \frac{7}{6} \div \frac{2}{3}$

$= \frac{7}{4}$

$\therefore \tan^{-1}\frac{2}{3} + \cos^{-1}\frac{2}{\sqrt{5}} = \tan^{-1}(\frac{7}{4})$

Question 4 cont

b) $\int_{\frac{1}{2}}^{e/2} \frac{\ln 2x}{x}\,dx$ $\quad u = \ln 2x$

$\frac{du}{dx} = \frac{1}{x}$

when $x = \frac{1}{2}$, $u = \ln 1 = 0$

when $x = e/2$, $u = \ln e = 1$

$= \int_0^1 u\,du$

$= \left[\frac{u^2}{2}\right]_0^1$

$= \frac{1}{2} - 0$

$= \frac{1}{2}$

c) $y = x + \frac{4}{x}$ $\quad x \neq 0$

vertical asymptote at $x = 0$

as $x \to \infty$, $y \to x$

$\therefore y = x$ is an asymptote

$\frac{dy}{dx} = 1 - 4x^{-2}$

$1 - \frac{4}{x^2} = 0$

$x^2 = 4$

$x = \pm 2$

stationary points at $(2,4)$ & $(-2,-4)$

$\frac{d^2y}{dx^2} = 8x^{-3}$

$= \frac{8}{x^3}$

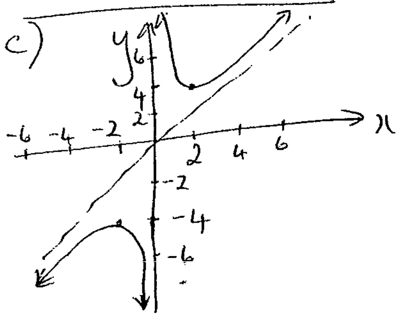
at $x = 2$, $\frac{d^2y}{dx^2} = 1 > 0$ minimum turning point

at $x = -2$ $\frac{d^2y}{dx^2} = -1 < 0$ maximum turning point

$\frac{8}{x^3} \neq 0 \therefore$ no points of inflexion

Question 4 cont

c)



ii) $4 + \frac{4}{x} = k$ has no real roots

if $-4 < k < 4$

d) $1 + 2 + 4 + \cdots + 2^{n-1} = 2^{n-1}$

let $n = 1$ LHS $= 1$

RHS $= 2^1 - 1$

$= 1$

∴ result is true for $n = 1$

Assume true for $n = k$

ie $1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1$

consider $n = k + 1$

LHS $= 1 + 2 + 4 + \cdots + 2^{k-1} + 2^k$

$= (2^k - 1) + 2^k$

$= 2 \times 2^k - 1$

$= 2^{k+1} - 1$

∴ true for $n = k+1$ if true for $n = k$

Since the result is true for $n = 1$, it is true for $n = 1+1 = 2$ & hence for all positive integers n.

Question 5

a) $\tan\left(\frac{\pi}{4} + x\right) = \dfrac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4} \times \tan x}$

$= \dfrac{1 + \tan x}{1 - \tan x}$

$= \dfrac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}}$

$= \dfrac{\cos x + \sin x}{\cos x - \sin x}$

b) i) $y = 2x - 1$ when $x = 1$, $y = 1$

$y = x^3$ when $x = 1$, $y = 1$

∴ graphs intersect at $x = 1$

ii) $y = 2x - 1$ $y = x^3$

$\frac{dy}{dx} = 2$ $\frac{dy}{dx} = 3x^2$

$= 3$ when $x = 1$

$m_1 = 2$ $m_2 = 3$

$\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_2 m_1}\right|$

$= \left|\dfrac{3 - 2}{1 + 3 \times 2}\right|$

$= \frac{1}{7}$

$\theta = 8°8'$ (to nearest minute)

the acute angle between the curves is $8°8'$.

Question 5 cont

c) $p(x) = x^3 + ax^2 + bx - 18$

$p(-2) = -8 + 4a - 2b - 18 = 0$

$p(1) = 1 + a + b - 18 = -24$

$4a - 2b = 26$

$2a - b = 13$

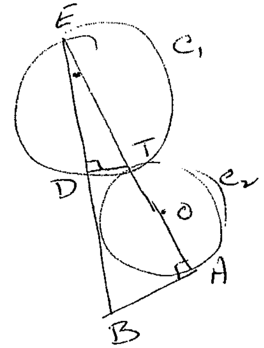
$a + b = -7$

$\therefore 3a = 6$

$a = 2$

$b = -9$

d)



i) AE passes thru centre of C_2 + point of contact of C_1 + C_2 (given)

$\therefore$ AE passes thru centre of C_1

$\therefore E\hat{D}T = 90°$ (angle in a semi circle is a right angle)

ii) $E\hat{A}B = 90°$ (angle between tangent + radius at point of contact)

$\hat{E}$ is common

$\therefore \Delta EAB \,|||\, \Delta EDT$ (equiangular)

$\therefore \frac{EB}{ET} = \frac{EA}{ED}$ (ratio of sides of similar triangles)

$\frac{EB}{2R} = \frac{2R+2r}{2r}$

$EB = \frac{2R(2R+2r)}{2r}$

$= \frac{2R(R+r)}{r}$

Question 6

a) i) $\frac{dV}{dt} = 200$ $\qquad \frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= 200 \times \frac{1}{4\pi r^2}$

$= \frac{50}{\pi r^2}$

ii) $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$= 8\pi r \times \frac{50}{\pi r^2}$

$= \frac{400}{r}$

when $r = 50$, $\frac{dS}{dt} = 8$ mm² per second

b) $AP \cdot PB = PT^2$

$2x \times x = 8^2$

$2x^2 = 64$

$x = \sqrt{32}$

$= 4\sqrt{2} \quad (x > 0)$

c) i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

when $x = 2ap$, $\frac{dy}{dx} = p$ $\quad \therefore$ slope of normal $= -\frac{1}{p}$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$

Question 6 cont

c) ii) $py = ap^3 + 2ap$

$y = ap^2 + 2a$

$Q(0, ap^2+2a)$

$P(2ap, ap^2)$

R: $x = \frac{0+2ap}{2} = ap$ $\qquad$ $y = \frac{ap^2+ap^2+2a}{2} = \frac{2ap^2+2a}{2} = ap^2+a$

$R(ap, ap^2+a)$

iii) $x = ap \Rightarrow p = \frac{x}{a}$

$y = ap^2 + a$

$= a \times \frac{x^2}{a^2} + a$

$= \frac{x^2}{a} + a$

$ay = x^2 + a^2$

$x^2 = a(y-a)$

Question 7

a) i) $x = \cos t + t\sin t$

$\frac{dx}{dt} = -\sin t + \sin t + t\cos t$

$= t\cos t$

ii) $y = \sin t - t\cos t$

$\frac{dy}{dt} = \cos t - \cos t + t\sin t$

$= t\sin t$

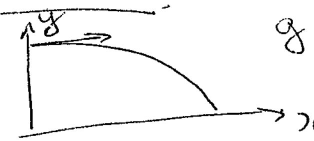
$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= t\sin t \times \frac{1}{t\cos t}$

$= \tan t$

Question 7 cont

b)



$g = 10$

$x = 35t$ $\qquad$ $y = -5t^2 + 1.8$

i) $y = 0 \Rightarrow -5t^2 + 1.8 = 0$

$t^2 = 0.36$

$t = 0.6 \quad (t > 0)$

∴ ball hits the ground after 0.6 seconds

ii) when $t = 0.6 \Rightarrow x = 35 \times 0.6$

$= 21$

∴ ball travels 21 metres before bouncing

iii) $x = 14 \Rightarrow 35t = 14$

$t = \frac{14}{35}$

$= 0.4$

$y = -5 \times 0.4^2 + 1.8$

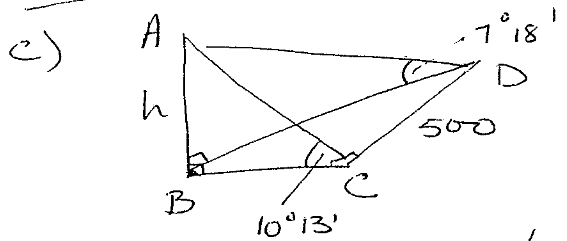
$= 1$

$1 > 0.95$

since ball is at a height of 1m when $x = 14$, the ball will clear the net.

Question 7 cont

c)



let height of AB be h

$\frac{h}{BD} = \tan 7°18' \Rightarrow BD = \frac{h}{\tan 7°18'}$

$\frac{h}{BC} = \tan 10°13' \Rightarrow BC = \frac{h}{\tan 10°13'}$

$BC^2 + 500^2 = BD^2$

$\frac{h^2}{\tan^2 10°13'} + 500^2 = \frac{h^2}{\tan^2 7°18'}$

$\frac{h^2}{\tan^2 7°18'} - \frac{h^2}{\tan^2 10°13'} = 500^2$

$h^2\left(\frac{1}{\tan^2 7°18'} - \frac{1}{\tan^2 10°13'}\right) = 500^2$

$h^2 = 500^2 \div \left(\frac{1}{\tan^2 7°18'} - \frac{1}{\tan^2 10°13'}\right)$

$h = 91.057\ldots$

height of tower is 91 metres.

Extension 1 Mathematics **2006 Trial HSC**

	Algebra, co-ord geom, parameters	Calculus	Trig	Circle geom	Polynomials	Inverse functions	Applications of calculus	Total
1a			2					
1b					2			
1c	2							
1d		2						
1e	2							
1f		2						12
2a						2		
2b	4							
2c					2			
2d						4		12
3a	2							
3b		4						
3c					3			
3d			3					12
4a						3		
4b							3	
4c		3						
4d	3							12
5a			2					
5b	3							
5c					3			
5d				4				12
6a							4	
6b				3				
6c	5							12
7a	3							
7b							5	
7c			4					12
Totals	24	11	11	7	10	9	12	84