NSW INDEPENDENT SCHOOLS

2014
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided at the back of the paper
- All necessary working should be shown in Question 11 – 14
- Write your student number and/or name at the top of every page

Total marks – 70

Section I - Pages 3 – 5

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 6 – 9

60 marks

Attempt Questions 11 – 14

Allow about 1 hour 45 minutes for this section

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

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STANDARD INTEGRALS

$\int x^n dx$	$= \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x} dx$	$= \ln x, \quad x > 0$
$\int e^{ax} dx$	$= \frac{1}{a}e^{ax}, \quad a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a}\sin ax, \quad a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a}\cos ax, \quad a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a}\tan ax, \quad a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a}\sec ax, \quad a \neq 0$
$\int \frac{1}{a^2+x^2} dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}} dx$	$= \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}} dx$	$= \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}} dx$	$= \ln\left(x + \sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

Student name / number ____________________

Marks

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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1 What is the value of $\lim_{x\to 0}\frac{\sin 3x}{2x}$? 1

(A) 0

(B) $\frac{2}{3}$

(C) 1

(D) $\frac{3}{2}$

2 Which of the following is a solution of the equation $2^x = 5$? 1

(A) $x=\sqrt{5}$

(B) $x=\log_e 5$

(C) $x=\frac{\log_e 5}{\log_e 2}$

(D) $x=\frac{\log_e 2}{\log_e 5}$

3 Which of the following is an expression for $\cos^4 x - \sin^4 x$? 1

(A) $\cos 2x$

(B) $\cos^2 2x$

(C) $\cos 4x$

(D) $\cos^2 4x$

4 Which of the following is an expression for $\frac{dy}{dx}$ if $x=\frac{1}{2}at$ and $y=at^2$? 1

(A) t

(B) $2t$

(C) $2at$

(D) $4t$

Student name / number ____________________

Marks

5 Which of the following is an expression for $1+\sec x$ in terms of $t=\tan\frac{x}{2}$? 1

(A) $\frac{2}{1+t^2}$

(B) $\frac{2}{1-t^2}$

(C) $\frac{2t^2}{1+t^2}$

(D) $\frac{2t^2}{1-t^2}$

6 Which of the following is the range of the function $y=2\sin^{-1}x+\frac{\pi}{2}$? 1

(A) $-\pi \le y \le \pi$

(B) $-\pi \le y \le \frac{3\pi}{2}$

(C) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(D) $-\frac{\pi}{2} \le y \le \frac{3\pi}{2}$

7 Which of the following is an asymptote of the curve $y=\frac{x^2-4}{x}$? 1

(A) $y=x$

(B) $x=2$

(C) $x=1$

(D) $y=0$

8 Which of the following is an expression for $\int \frac{1}{4x^2+1}\,dx$? 1

(A) $\frac{1}{2}\tan^{-1}\frac{x}{2}$

(B) $\tan^{-1}\frac{x}{2}$

(C) $\frac{1}{2}\tan^{-1}2x$

(D) $\tan^{-1}2x$

Student name / number ____________________

Marks

9 Which of the following is the coefficient of the term in x^n in the expansion of $(1+x)^n+(1-x)^n$? 1

(A) 0

(B) $\frac{1+(-1)^n}{2}$

(C) $1+(-1)^n$

(D) 2

10 $P(x\,,y)$ is a variable point which moves on the curve $y=x^3$ such that the x coordinate of P is increasing at a constant rate of $0\cdot05\text{ cm s}^{-1}$. What is the rate at which the y coordinate of P is increasing when $x=2$? 1

(A) $0\cdot4\text{ cm s}^{-1}$

(B) $0\cdot6\text{ cm s}^{-1}$

(C) 8 cm s^{-1}

(D) 12 cm s^{-1}

Student name / number ____________________

Marks

Section II

60 Marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer the questions on your own paper, or in writing booklets if provided.
Start each question on a new page.
All necessary working should be shown in every question.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) Find the number of ways in which the letters of the word CIRCLE can be arranged in a row so that the two vowels are in the two end positions. 2

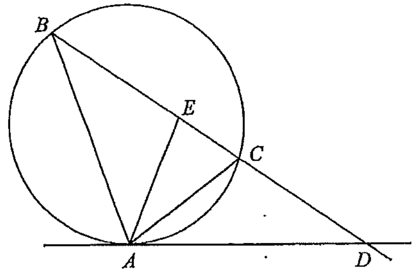
(b) $P(2ap\,,ap^2)$ and $Q(2ap^2\,,ap^4)$ are two points on the parabola $x^2=4ay$.

(i) Show that the chord PQ has gradient $m=\frac{1}{2}(p^2+p)$. 1

(ii) Find the minimum gradient of the chord PQ and the coordinates of the point P on the parabola at which this minimum value occurs. 2

(c) Solve the inequality $\frac{x^2+x-6}{x}\geq 0$. 3

(d)



In the diagram ABC is a triangle inscribed in a circle. The tangent to the circle at A meets BC produced at D. E is the point on BD such that $DA=DE$.
Show that EA bisects $\angle BAC$. 3

(e)(i) Show that the curves $y=\sin^{-1}x$ and $y=\cos^{-1}x$ intersect at the point where $x=\frac{1}{\sqrt{2}}$. 1

(ii) Find correct to the nearest degree the acute angle between the tangents to the curves at their point of intersection. 3

Student name / number ____________________

Marks

Question 12 (15 marks) **Use a separate writing booklet.**

(a) Use the method of Mathematical Induction to show that $5^n - 4n - 1$ is divisible by 4 for all positive integers $n \geq 2$. 3

(b) A curve $y = f(x)$ has gradient function $\frac{dy}{dx} = 2\cos^2 2x$ and passes through the point $\left(\frac{\pi}{4}, 0\right)$. Find the equation of the curve. 3

(c) Use the substitution $u = x - 1$ to evaluate $\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$. 3

(d)(i) Show that the x coordinates of the stationary points on the curve $y = x\cos x$ satisfy the equation $\tan x - \frac{1}{x} = 0$. 2

(ii) Show that the equation $\tan x - \frac{1}{x} = 0$ has a root α such that $\frac{\pi}{8} < \alpha < \frac{3\pi}{8}$. 2

(iii) Use one application of Newton's method with an initial approximation $\alpha_0 = \frac{\pi}{4}$ to find the next approximation, giving your answer correct to one decimal place. 2

Student name / number ____________________

Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) At time t years after observations began at the start of 2010 the number N of individuals in a population is given by $N = 100 + 500\,e^{-0.1t}$.

(i) Show that $\frac{dN}{dt} = -0{\cdot}1\left(N - 100\right)$. 1

(ii) Sketch the graph of N as a function of t showing clearly the initial population size and the limiting population size. 2

(iii) Find the year and month during which the rate of decrease of the population is expected to fall to half its initial value. 2

(b) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} and acceleration $\ddot{x}$ ms^{-2} given by $\ddot{x} = -2(x-1)$. Initially the particle is 2m to the right of O and moving away from O with speed $\sqrt{6}$ ms^{-1} .

(i) Show by integration that $v^2 = 2\left(3 + 2x - x^2\right)$. 2

(ii) Find the values of a and b if $x = b + a\cos(nt + \alpha)$ and $a > 0$. 2

(iii) Find the values of n and α in this expression for x if $n > 0$ and $0 < \alpha < 2\pi$. 2

(c) Four different-coloured, fair dice are rolled together.

(i) Find the probability that all four dice show different scores. 2

(ii) Find the probability that all four dice show the same score. 2

Student name / number ______________________

Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity $v\text{ ms}^{-1}$ given by $v=\frac{1}{2}\left(1-x^2\right)$ and acceleration $a\text{ ms}^{-2}$. Initially the particle is at O.

(i) Find an expression for a in terms of x. 1

(ii) Show that $\dfrac{2}{1-x^2}=\dfrac{1}{1+x}+\dfrac{1}{1-x}$. 1

(iii) Show that $x=\dfrac{e^t-1}{e^t+1}$. 3

(iv) Find the limiting position of the particle. 1

(b) A vertical tower of height 20 metres stands on horizontal ground. A particle is projected from a point O at the top of the tower with speed 35 ms^{-1} at an angle α above the horizontal. It moves in a vertical plane under gravity, where the acceleration due to gravity is $g=10\text{ ms}^{-2}$, and hits the ground after T seconds at a distance 140 metres from the foot of the tower. At time t seconds its horizontal and vertical displacements from O are x metres and y metres respectively.

(i) Use integration to show that $x=35t\cos\alpha$ and $y=35t\sin\alpha-5t^2$. 2

(ii) Show that $T^4-57T^2+800=0$. 2

(iii) Find the two possible times of flight until the particle hits the ground. 2

(c) Use the Binomial expansion of $(1+x)^n$ and differentiation to show that 3

$\sum_{r=2}^{n} {}^rC_2\,{}^nC_r={}^nC_2\,2^{n-2}$ for all integers $n\geq 2$.

Independent Trial HSC 2014 Mathematics Extension 1 Marking Guidelines

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1.	D	$\lim_{x\to 0}\frac{\sin 3x}{2x}=\frac{3}{2}\left\{\lim_{x\to 0}\frac{\sin 3x}{3x}\right\}=\frac{3}{2}\times 1=\frac{3}{2}$	H5
2.	C	$2^x=5$, $x\log_e 2=\log_e 5$ $\therefore x=\frac{\log_e 5}{\log_e 2}$	H3
3.	A	$\cos^4 x-\sin^4 x=(\cos^2 x+\sin^2 x)(\cos^2 x-\sin^2 x)=\cos 2x$	H5
4.	D	$x=\frac{1}{2}at$, $\frac{dx}{dt}=\frac{1}{2}a$; $y=at^2$, $\frac{dy}{dt}=2at$ $\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=4t$	PE4
5.	B	$t=\tan\frac{x}{2}\ \Rightarrow\ 1+\sec x=1+\frac{1+t^2}{1-t^2}=\frac{(1-t^2)+(1+t^2)}{1-t^2}=\frac{2}{1-t^2}$	H5
6.	D	$-2\times\frac{\pi}{2}\le 2\sin^{-1}x\le 2\times\frac{\pi}{2}$ $\therefore -\frac{\pi}{2}\le 2\sin^{-1}x+\frac{\pi}{2}\le\frac{3\pi}{2}$	HE4
7.	A	$y=\frac{x^2-4}{x}=x-\frac{4}{x}$ $\therefore \lvert y-x\rvert=\left\lvert\frac{4}{x}\right\rvert\to 0$ as $x\to\infty$	P5
8.	C	$\int\frac{1}{4x^2+1}\,dx=\frac{1}{2}\int\frac{2}{(2x)^2+1}\,dx=\frac{1}{2}\tan^{-1}2x$	HE4
9.	C	Term in x^n is ${}^nC_n x^n+{}^nC_n(-x)^n={}^nC_n x^n\{1+(-1)^n\}=x^n\{1+(-1)^n\}$	HE3
10.	B	$y=x^3$ $\therefore \frac{dy}{dt}=3x^2\frac{dx}{dt}=12\times 0\cdot 05=0\cdot 6$ when $x=2$	HE5

Section II

Question 11

a. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• count the arrangements of the consonants with the repetition of C	1
• multiply by 2 for the arrangement of the vowels	1

Answer

$2\times\frac{4!}{2!}=24$ arrangements

b. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
i • find gradient	1
ii • find the minimum value of the gradient and the corresponding value of p	1
• find coordinates of P for this value of p	1

Answer

i. $m_{PQ}=\frac{ap^2(p^2-1)}{2ap(p-1)}=\frac{p(p-1)(p+1)}{2(p-1)}=\frac{1}{2}(p^2+p)$

ii. $m_{PQ}=\frac{1}{2}\left(p+\frac{1}{2}\right)^2-\frac{1}{8}$. Hence minimum value of m_{PQ} is $-\frac{1}{8}$ when $p=-\frac{1}{2}$ and P has coordinates $\left(-a,\frac{1}{4}a\right)$.

c. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
• apply an appropriate method for solving a variable-denominator inequality	1
• show solution includes x such that $x\ge 2$	1
• find inequality determining the remaining solutions for x	1

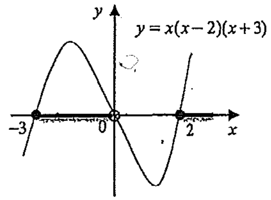
Answer

$\frac{x^2+x-6}{x}\ge 0$

$x(x^2+x-6)\ge 0$ and $x\ne 0$

$x(x-2)(x+3)\ge 0$ and $x\ne 0$

$-3\le x<0$ or $x\ge 2$

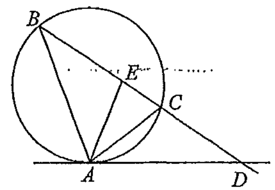


d. Outcomes assessed: PE2, PE3

Marking Guidelines

Criteria	Marks
• use the alternate segment theorem to find one pair of equal angles	1
• complete a sequence of deductions to arrive at the required result	1
• justify these deductions by quoting appropriate geometric properties	1

Answer



$\angle DAC=\angle CBA$ (*Alternate segment theorem*)

$\angle DEA=\angle DAE$ (*$\angle$'s opp. equal sides are equal in ΔADE*)

$\therefore \angle\_\_\_=\angle\_\_\_+\angle\_\_\_$ (*by addition of adjacent angles*)

$\therefore \angle DEA=\angle CBA+\angle CAE$

$\angle\_\_\_=\angle\_\_\_+\angle\_\_\_$ (*Ext. $\angle$ equals sum of int. opp. $\angle$'s in ΔABE*)

$\therefore \angle\_\_\_=\angle\_\_\_$

Hence EA bisects $\angle BAC$

Q 11 (cont)
e. Outcomes assessed: H5, HE4

Marking Guidelines

Criteria	Marks
i • explain why the curves intersect at the stated value of x	1
ii • evaluate both derivatives at this value of x	1
• write an expression for the tangent of the required angle	1
• evaluate this angle to the nearest degree	1

Answer

i. $\sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$. Hence the curves $y=\sin^{-1}x$ and $y=\cos^{-1}x$ intersect at $\left(\frac{1}{\sqrt{2}},\frac{\pi}{4}\right)$.

ii. $\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}=\frac{1}{\sqrt{\frac{1}{2}}}=\sqrt{2}$ and $\frac{d}{dx}\cos^{-1}x=\frac{-1}{\sqrt{1-x^2}}=\frac{-1}{\sqrt{\frac{1}{2}}}=-\sqrt{2}$ when $x=\frac{1}{\sqrt{2}}$.

Hence the acute angle θ between the tangents to the curves at the point of intersection is given by

$$\tan\theta=\left|\frac{\sqrt{2}-\left(-\sqrt{2}\right)}{1+\sqrt{2}\left(-\sqrt{2}\right)}\right|=2\sqrt{2}\ . \qquad \therefore \theta\approx 71^\circ \text{ (to the nearest degree)}$$

Question 12
a. Outcomes assessed: HE2

Marking Guidelines

Criteria	Marks
• define an appropriate sequence of statements and establish the truth of the first statement	1
• rearrange the expression of the $(k+1)^{st}$ statement to incorporate the k^{th} statement explicitly	1
• explain why the truth of the k^{th} statement implies the truth of $(k+1)^{st}$ and complete the induction	1

Answer

Let the sequence of statements S_n, $n=2,3,4,\ldots$ be defined by $S_n\ :\ 5^n-4n-1=4I$ *for some integer I.*

Consider S_2 : $5^2-4\times2-1=16=4\times4$ Hence S_2 is true.

If S_k is true : $5^k-4k-1=4I$ *for some integer I.* *

Consider S_{k+1} :

$$\begin{aligned}5^{k+1}-4(k+1)-1&=(4+1)5^k-4k-4-1\\&=\left(5^k-4k-1\right)+4\left(5^k-1\right)\\&=4I+4\left(5^k-1\right) && \text{if } S_k \text{ is true using } *\\&=4\times\left(I+5^k-1\right) && \text{where } \left(I+5^k-1\right) \text{ is integral}\end{aligned}$$

Hence if S_k is true then S_{k+1} is true. But S_2 is true, hence S_3 is true, and so on. Therefore by Mathematical Induction, 5^n-4n-1 is divisible by 4 for all integers $n\geq2$.

Q12 (cont)
b. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• use an appropriate trigonometric identity to enable integration	1
• identify the family of primitive functions	1
• select the particular member of this family that passes through the stated point	1

Answer

$$\begin{aligned}\frac{dy}{dx}&=2\cos^2 2x\\&=1+\cos 4x\end{aligned}$$

$\therefore y=x+\frac{1}{4}\sin 4x+c$, c constant

$$\left.\begin{aligned}x&=\tfrac{\pi}{4}\\y&=0\end{aligned}\right\}\Rightarrow \begin{aligned}&0=\tfrac{\pi}{4}+\tfrac{1}{4}\sin\pi+c\\&\therefore c=-\tfrac{\pi}{4}\\&\therefore y=x+\tfrac{1}{4}\sin 4x-\tfrac{\pi}{4}\end{aligned}$$

c. Outcomes assessed: HE6

Marking Guidelines

Criteria	Marks
• transform the definite integral by applying the given substitution	1
• find the corresponding primitive function	1
• evaluate the definite integral using the transformed limits	1

Answer

$u=x-1$
$du=dx$

$x=2\Rightarrow u=1$
$x=5\Rightarrow u=4$

$$\begin{aligned}\int_2^5\frac{x}{\sqrt{x-1}}\,dx&=\int_1^4\frac{u+1}{\sqrt{u}}\,du\\&=\int_1^4\left(u^{\frac{1}{2}}+u^{-\frac{1}{2}}\right)du\\&=\left[\tfrac{2}{3}u^{\frac{3}{2}}+2u^{\frac{1}{2}}\right]_1^4\\&=\tfrac{2}{3}(8-1)+2(2-1)\\&=6\tfrac{2}{3}\end{aligned}$$

d. Outcomes assessed: H5, PE3

Marking Guidelines

Criteria	Marks
i • differentiate and set derivative equal to zero	1
• justify rearrangement to give the required equation	1
ii • note the continuity of the function over the stated interval	1
• establish the change of sign over the stated interval	1
iii • substitute correctly into formula associated with Newton's method	1
• evaluate to obtain the next approximation	1

Answer

i. $y=x\cos x$

$\frac{dy}{dx}=\cos x-x\sin x$

$\frac{dy}{dx}=0 \Rightarrow \begin{aligned}x\sin x&=\cos x\\ \tan x&=\tfrac{1}{x}\end{aligned}$ (since $\frac{dy}{dx}\neq0$ for $x=0$ nor $\cos x=0$).

Hence x coordinates of stationary points satisfy $\tan x-\frac{1}{x}=0$.

ii. $f(x)=\tan x-\frac{1}{x}$ is continuous for $\frac{\pi}{8}\leq x\leq\frac{3\pi}{8}$ and $f(\frac{\pi}{8})\approx-2\cdot1<0$, $f(\frac{3\pi}{8})\approx1\cdot6>0$.

iii. $f'(x)=\sec^2 x+\frac{1}{x^2}$. Hence next approximation is $\alpha_1=\frac{\pi}{4}-\dfrac{1-\frac{4}{\pi}}{2+\left(\frac{4}{\pi}\right)^2}\approx0\cdot9$

Question 13

a. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • differentiate and rearrange	1
ii • correct shape with correct initial population size	1
• asymptote to show limiting population size	1
iii • write equation for t using initial rate of decrease	1
• solve for t and interpret solution	1

Answer

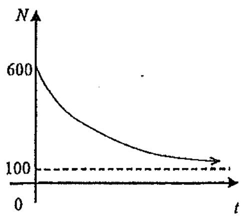
i.

$$N = 100 + 500\, e^{-0{\cdot}1t}$$

$$\frac{dN}{dt} = -0{\cdot}1\left(500\, e^{-0{\cdot}1t}\right)$$

$$= -0{\cdot}1\left(N - 100\right)$$

ii.



iii. $t = 0 \Rightarrow N = 600$ and $\frac{dN}{dt} = -0{\cdot}1 \times 500 = -50$.

$$\frac{dN}{dt} = -25 \Rightarrow N - 100 = 250$$

$$\therefore e^{-0{\cdot}1t} = \tfrac{1}{2}$$

$$-\tfrac{1}{10}t = -\ln 2$$

$$t = 10 \ln 2$$

Time taken is $6{\cdot}93$ years

i.e. 6 years $11{\cdot}17$ months.

$\therefore$ December, 2016.

b. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • use appropriate expression for $\ddot{x}$ and obtain primitive	1
• use initial conditions to evaluate the constant of integration and rearrange	1
ii • state value of b	1
• find value of a	1
iii • state value of n	1
• use initial conditions to evaluate α	1

Answer

i. $\ddot{x} = -2(x-1)$

$$\frac{dv^2}{dx} = -4(x-1)$$

$$v^2 = -2(x-1)^2 + c$$

$$\left.\begin{aligned} t &= 0 \\ x &= 2 \\ v &= \sqrt{6} \end{aligned}\right\} \quad \begin{aligned} 6 &= -2 + c \\ v^2 &= -2(x-1)^2 + 8 \\ v^2 &= 2\left(3 + 2x - x^2\right) \end{aligned}$$

ii. $x = b + a\cos(nt + \alpha)$

Centre of motion is at $x = 1$ $\therefore b = 1$

$$v = 0 \Rightarrow \begin{aligned} 2(x-1)^2 &= 8 \\ x = -1,\ x &= 3 \end{aligned}$$

$\therefore$ oscillation between these x values with amplitude 2 $\quad \therefore a = 2$

iii. ${}^2 = 2 \quad \therefore\ = \sqrt{2}$.

$= +\ \sqrt{\ } + \alpha$

$= -\ \sqrt{\ }\ \sqrt{\ } + \alpha$

$= 0$, $= 2$, > 0 $\Big\}$ $\cos\alpha = \frac{1}{2}$, $\sin\alpha < 0$, $\alpha = \frac{5\pi}{3}$

Q13 (cont)

c. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • count the number of arrangements comprising the event	1
• calculate the probability	1
ii • count the possible outcomes	1
• calculate the probability	1

Answer

i. $\dfrac{{}^6P_4}{6^4} = \frac{5}{18}$ $\qquad$ ii. $\dfrac{6}{6^4} = \frac{1}{216}$

Question 14

a. Outcomes assessed: HE5

Marking Guidelines

Criteria	Marks
i • use an appropriate expression for a	1
ii • rearrange to show result	1
iii • find primitive function for t in terms of x	1
• use initial conditions to evaluate constant of integration	1
• rearrange to obtain x in terms of t	1
iv • find limiting value of x as $t \to \infty$	1

Answer

i. $v = \frac{1}{2}\left(1 - x^2\right)$

$$\frac{dv}{dx} = -x$$

$$v\frac{dv}{dx} = \tfrac{1}{2}\left(1-x^2\right)\left(-x\right)$$

$$\therefore a = -\tfrac{1}{2}x\left(1-x^2\right)$$

ii. $1 - x^2 = \left(1+x\right)\left(1-x\right)$

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{(1-x)+(1+x)}{1-x^2}$$

$$= \frac{2}{1-x^2}$$

iii. $\frac{dx}{dt} = \frac{1}{2}\left(1-x^2\right)$

$$\frac{dt}{dx} = \frac{2}{1-x^2}$$

$$= \frac{1}{1+x} + \frac{1}{1-x}$$

$$t = \ln\left(1+x\right) - \ln\left(1-x\right) + c$$

$$t = \ln\left(\frac{1+x}{1-x}\right) + c$$

$$\left.\begin{aligned} t &= 0 \\ x &= 0 \end{aligned}\right\} \quad \begin{aligned} 0 &= \ln 1 + c \quad \therefore c = 0 \\ t &= \ln\left(\frac{1+x}{1-x}\right) \end{aligned}$$

$$e^t = \frac{1+x}{1-x}$$

$$e^t - xe^t = 1 + x$$

$$e^t - 1 = x\left(e^t + 1\right)$$

$$x = \frac{e^t - 1}{e^t + 1}$$

iv $x = \dfrac{1 - e^{-t}}{1 + e^{-t}}$ where $\lim\limits_{t\to\infty} \dfrac{1-e^{-t}}{1+e^{-t}} = \dfrac{1-0}{1+0} = 1$.

Hence moves from the origin O towards a limiting position 1 metre to the right of O.

Q14 (cont)

b. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • find x as a function of t by integration	1
• find y as a function of t by integration	1
ii • write simultaneous equations for T and α	1
• eliminate α to find equation for T.	1
iii • factor as quadratic in T^2 or use quadratic formula	1
• find two values of T	1

Answer

i.

$$\ddot{x}=0 \qquad \ddot{y}=-10$$

$$\dot{x}=c_1 \qquad \dot{y}=-10t+c_3$$

$$\left.\begin{array}{l}t=0\\ \dot{x}=35\cos\alpha\end{array}\right\}\Rightarrow \begin{array}{l}c_1=35\cos\alpha\\ \dot{x}=35\cos\alpha\end{array} \qquad \left.\begin{array}{l}t=0\\ \dot{y}=35\sin\alpha\end{array}\right\}\Rightarrow \begin{array}{l}c_3=35\sin\alpha\\ \dot{y}=35\sin\alpha-10t\end{array}$$

$$x=35t\cos\alpha+c_2 \qquad y=35t\sin\alpha-5t^2+c_4$$

$$\left.\begin{array}{l}t=0\\ x=0\end{array}\right\}\Rightarrow \begin{array}{l}c_2=0\\ x=35t\cos\alpha\end{array} \qquad \left.\begin{array}{l}t=0\\ y=0\end{array}\right\}\Rightarrow \begin{array}{l}c_4=0\\ y=35t\sin\alpha-5t^2\end{array}$$

ii. When $t=T$, $y=-20$ and $x=140$.

$$35T\sin\alpha=5T^2-20 \qquad 7T\sin\alpha=T^2-4$$

$$35T\cos\alpha=140 \qquad 7T\cos\alpha=28$$

$$49T^2\left(\sin^2\alpha+\cos^2\alpha\right)=\left(T^2-4\right)^2+28^2$$

$$T^4-57T^2+800=0$$

iii. $\left(T^2-32\right)\left(T^2-25\right)=0 \quad \therefore T=4\sqrt{2}$ or $T=5$. Time of flight is $4\sqrt{2}$ s or 5 s.

c. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
• write binomial expansion and differentiate once	1
• differentiate a second time and substitute $x=1$	1
• use definition of binomial coefficient to complete proof	1

Answer

$$\frac{d^2}{dx^2}(1+x)^n\equiv\frac{d^2}{dx^2}\left({}^nC_0+{}^nC_1x+{}^nC_2x^2+\dots{}^nC_rx^r+\dots+{}^nC_nx^n\right)$$

$$\frac{d}{dx}n(1+x)^{n-1}\equiv\frac{d}{dx}\left({}^nC_1+2\,{}^nC_2x+\dots+r\,{}^nC_rx^{r-1}+\dots+n\,{}^nC_nx^{n-1}\right)$$

$$n(n-1)(1+x)^{n-2}\equiv\sum_{r=2}^{n}r(r-1)\,{}^nC_rx^{r-2}$$

Then $x=1\Rightarrow n(n-1)2^{n-2}=\sum_{r=2}^{n}r(r-1)\,{}^nC_r$, where $\frac{r(r-1)}{2}={}^rC_2$. $\quad\therefore\sum_{r=2}^{n}{}^rC_2\,{}^nC_r={}^nC_2\,2^{n-2}$