STUDENT NUMBER/NAME: ....

NSW INDEPENDENT SCHOOLS

2014
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in Question 11 – 16
- Write your student number and/or name at the top of every page

Total marks – 100

Section I - Pages 3 – 5

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 6 – 11

90 marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

This paper MUST NOT be removed from the examination room

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Marks

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

1 Which of the following is an expression for the limiting sum of the geometric series $1+2\cos^2\theta+4\cos^4\theta+8\cos^6\theta+\ldots$ whenever this limiting sum exists? 1

(A) $-\cos 2\theta$

(B) $-\sec 2\theta$

(C) $\cos 2\theta$

(D) $\sec 2\theta$

2 Which of the following is the range of the function $f(x)=\sin^{-1}x+\tan^{-1}x$? 1

(A) $-\pi < y < \pi$

(B) $-\pi \le y \le \pi$

(C) $-\frac{3\pi}{4} \le y \le \frac{3\pi}{4}$

(D) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

3 If $e^x+e^y=1$, which of the following is an expression for $\frac{dy}{dx}$? 1

(A) $-e^{x-y}$

(B) e^{x-y}

(C) e^{y-x}

(D) $-e^{y-x}$

4 Which of the following graphs is the locus of the point P representing the complex number z which moves in the Argand diagram such that $|z-6|=2|z|$? 1

(A) a straight line

(B) a circle

(C) an ellipse

(D) a hyperbola

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Marks

5 $S(4,0)$ is a focus of the rectangular hyperbola $x^2-y^2=k$. Which of the following is the value of k? 1

(A) $2\sqrt{2}$

(B) $4\sqrt{2}$

(C) 8

(D) 32

6 Which of the following is an expression for $\int xe^{-x}\,dx$? 1

(A) $xe^{-x}+e^{-x}+c$

(B) $xe^{-x}-e^{-x}+c$

(C) $-xe^{-x}+e^{-x}+c$

(D) $-xe^{-x}-e^{-x}+c$

7 The region bounded by the curve $y=\sqrt{x}$ and the y axis between $y=0$ and $y=1$ is rotated through one revolution about the line $x=1$ to form a solid of volume V. Which of the following is an expression for V? 1

(A) $2\pi\int_0^1 x\sqrt{x}\,dx$

(B) $2\pi\int_0^1 (1-x)\sqrt{x}\,dx$

(C) $2\pi\int_0^1 x\left(1-\sqrt{x}\right)dx$

(D) $2\pi\int_0^1 (1-x)\left(1-\sqrt{x}\right)dx$

8 The equation $x^3-4x-2=0$ has roots α, β and γ. Which of the following equations has roots $-\frac{1}{\alpha}$, $-\frac{1}{\beta}$ and $-\frac{1}{\gamma}$? 1

(A) $2x^3-4x^2-1=0$

(B) $2x^3+4x^2-1=0$

(C) $2x^3-4x^2+1=0$

(D) $2x^3+4x^2+1=0$

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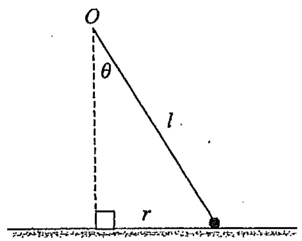
Marks

9



In the diagram a particle of mass m, attached to a string of length l, is suspended from a point O above a smooth, horizontal table with the string inclined at angle θ to the vertical. The particle moves on the table with constant angular velocity ω in a horizontal circle of radius r. The forces acting on the particle are the force due to gravity, the normal reaction N and the tension T in the string. Which of the following is an expression for T? **1**

(A) $ml\omega^2$

(B) $(mg-N)\cos\theta$

(C) $\dfrac{mg-N}{\sin\theta}$

(D) $mr\omega^2$

10 Which of the following is the value of $\lim\limits_{n\to\infty}\dfrac{{}^nC_1 \cdot {}^nC_2}{{}^nC_3}$? **1**

(A) $\frac{1}{6}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) 3

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Marks

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer the questions on your own paper, or in writing booklets if provided.
Start each question on a new page.
All necessary working should be shown in every question.

Question 11 (15 marks) **Use a SEPARATE writing booklet**

(a) If $z=1+3i$ and $w=2-i$ find in the form $a+ib$ (for real a and b) the values of

(i) $\overline{z}-w$ **1**

(ii) zw **1**

(b)(i) Express $-1+\sqrt{3}\,i$ in modulus/argument form. **2**

(ii) Hence find the value of z^8-16z^4 in the form $a+ib$ where a and b are real. **2**

(c) In the Argand diagram $OABC$ is a square, where O, A, B, C are in anti-clockwise cyclic order. The complex number z is represented by the vector $\overrightarrow{OA}$.

(i) Find in terms of z the complex numbers represented by the vectors $\overrightarrow{OC}$ and $\overrightarrow{OB}$. **2**

(ii) If the vector $\overrightarrow{OB}$ represents the complex number $4+2i$, find z in the form $a+ib$ where a and b are real. **2**

(d) The equation $x^4-2x^2-5x+3=0$ has roots α, β, γ and δ.

(i) Find the values of $\alpha^2+\beta^2+\gamma^2+\delta^2$ and $\alpha^4+\beta^4+\gamma^4+\delta^4$. **2**

(ii) State the number of real and non-real roots of the equation and give reasons for your answer. **3**

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Marks

Question 12 (15 marks) **Use a SEPARATE writing booklet**

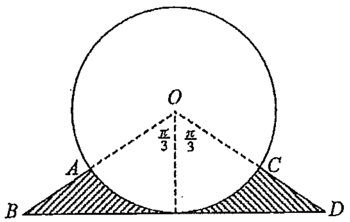
(a) By completing the square find $\int \frac{1}{\sqrt{3-(x^2-2x)}}\,dx$. 2

(b) Find $\int \frac{e^{2x}}{e^x+1}\,dx$. 2

(c) Find $\int \tan^{-1}x \; dx$. 2

(d) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x+3\cos x}\,dx$. 4

(e)



The diagram shows a two-dimensional view of a trophy comprising a metal sphere of radius R cm and with centre O, mounted on a base (shaded) so that the sphere fits snugly in the indentation in the base. (The three-dimensional trophy is the rotation of this two-dimensional view about the vertical through O). In the diagram, OAB and OCD are straight lines, each making an angle $\frac{\pi}{3}$ radians with the vertical through O.

(i) By taking annular, horizontal cross sections of thickness δy at a distance y cm below O, show that the volume V cm^3 of the solid base of the trophy (shaded) is given by $V=\pi\int_{\frac{1}{2}R}^{R}\left(4y^2-R^2\right)dy$. 3

(ii) Hence find the value of V. 2

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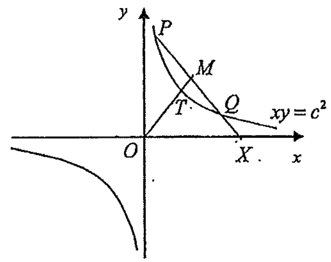
Marks

Question 13 (15 marks) **Use a SEPARATE writing booklet**

(a) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$, cuts the x axis at M and N. The ellipse has eccentricity e and $S(ae,0)$ is one focus of the ellipse. The focal chord PSQ is perpendicular to the x axis. Show that $\frac{1}{MS}+\frac{1}{NS}=\frac{4}{PQ}$. 4

(b)

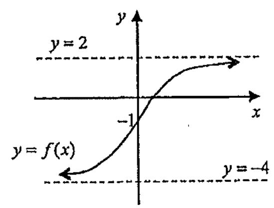


In the diagram $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$, where $q>p>0$, are two points on the rectangular hyperbola $xy=c^2$. M is the midpoint of PQ. PQ produced cuts the x axis at X. OM cuts the rectangular hyperbola at T.

(i) Show that *gradient* $MX=-$ *gradient* OM and hence show that $MX=OM$. 3

(ii) Show that the tangent to the rectangular hyperbola at T is parallel to the chord PQ. 2

(c)



The diagram shows the curve $y=f(x)$ where $f(x)=\frac{2e^x-4}{e^x+1}$. On separate diagrams sketch the following curves showing clearly the intercepts on the axes and the equations of any asymptotes.

(i) $y=|f(x)|$ 1

(ii) $y=f(|x|)$. 1

(iii) $y=\frac{1}{f(x)}$. 2

(iv) $y=f^{-1}(x)$. 2

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Marks

Question 14 (15 marks) **Use a SEPARATE writing booklet**

(a) Express $\dfrac{x^2+3x+4}{(x-2)(x^2+3)}$ in the form $\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+3}$ for some constants A, B and C. **3**

(b) A sequence of numbers is given by $T_1=6,\ T_2=27$ and $T_n=6T_{n-1}-9T_{n-2}$ for $n\geq 3$. **3**
Use Mathematical Induction to show that $T_n=(n+1)3^n$ for $n\geq 1$.

(c) A particle of mass m kg is fired vertically upwards with speed 200 ms^{-1} in a medium where the resistance is $\frac{1}{10}mv$ Newtons when the speed is $v\ \text{ms}^{-1}$. Take $g=10\ \text{ms}^{-2}$.

(i) For the upward journey, if x metres is the vertical displacement upwards from the point of projection, using the equation of motion $\ddot{x}=-\frac{1}{10}(100+v)$, show that the maximum height attained above the point of projection is H metres where **3**
$H=1000(2-\ln 3)$.

(ii) Show that the speed v of the particle on return to its point of projection satisfies **3**
$\frac{v}{100}+\ln\left(1-\frac{v}{100}\right)+(2-\ln 3)=0$.

(iii) Show that $\lambda+\ln(1-\lambda)+(2-\ln 3)=0$ has a root between $0\cdot 8$ and $0\cdot 9$, and applying Newton's method once with $0\cdot 82$ as a first approximation, find a second approximation for λ. **2**

(iv) What percentage of its terminal velocity has the particle attained on return to its point of projection? Explain your answer. **1**

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Marks

Question 15 (15 marks) **Use a SEPARATE writing booklet**

(a) Consider $I_n=\displaystyle\int_0^1 \frac{x^n}{1+x^2}\,dx$ for $n=0,1,2,\ldots$.

(i) Show that $I_n=\dfrac{1}{n-1}-I_{n-2}$ *for* $n=2,3,4,\ldots$. **2**

(ii) Hence find the value of I_4. **2**

(b)(i) If $a>0$ is a real number, show that $a+\dfrac{1}{a}\geq 2$. **1**

(ii) Hence show that if $a>0,\ b>0,\ c>0$ are real numbers, then

(α) $\dfrac{b+c}{a}+\dfrac{c+a}{b}+\dfrac{a+b}{c}\geq 6$. **2**

(β) $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\geq\dfrac{3}{2}$. **2**

(c)(i) Show that $\sin(2k+1)\theta-\sin(2k-1)\theta=2\sin\theta\cos 2k\theta$. **1**

(ii) Hence show that $\sin\theta\displaystyle\sum_{k=1}^{n}\cos 2k\theta=\sin n\theta\cos(n+1)\theta$. **3**

(iii) Hence find the value of $\displaystyle\sum_{k=1}^{10}\sin^2\left(\frac{k\pi}{10}\right)$. **2**

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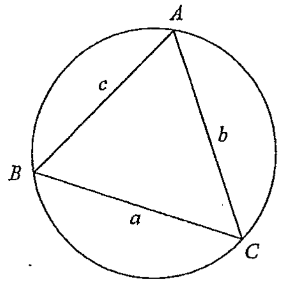
Marks

Question 16 (15 marks) **Use a SEPARATE writing booklet**

(a)



In the diagram, triangle ABC is inscribed in a circle of radius R.

(i) By constructing the diameter through B, or otherwise, show that $R = \dfrac{a}{2\sin A}$. **2**

(ii) Show that $\dfrac{Area\ \Delta ABC}{Area\ circle\ ABC} = \dfrac{2}{\pi}\sin A \sin B \sin C$. **1**

(iii) If the sizes of angles A, B, C are in radians and satisfy $A \geq B \geq C$, $\alpha = A - C$ and the function $f(\alpha)$ is defined by $f(\alpha) = \dfrac{2}{\pi}\sin A \sin B \sin C$, show that $f'(\alpha) \leq 0$ throughout the domain of f. **2**

(iv) Hence show that $\dfrac{Area\ \Delta ABC}{Area\ circle\ ABC}$ has a maximum value when ΔABC is equilateral and state this maximum value. **2**

(b)(i) For real numbers $x_1, x_2, x_3, \ldots$, if $S_n = \sum_{k=1}^{n} x_k$ show that **2**

$$\sum_{k=1}^{n} x_k^2 = \frac{1}{n}S_n^2 + \sum_{k=1}^{n}\left(x_k - \frac{1}{n}S_n\right)^2 .$$

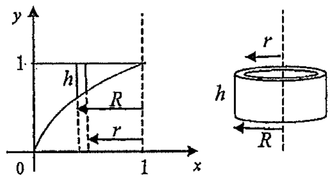
(ii) Hence show that $\sum_{k=0}^{n}\left({}^nC_k\right)^2 > \dfrac{2^{2n}}{n+1}$ for $n = 2, 3, 4, \ldots$. **3**

(iii) Use the identity $\left\{(1+x)^n\right\}^2 \equiv (1+x)^{2n}$ to show $\sum_{k=0}^{n}\left({}^nC_k\right)^2 = \dfrac{(2n)!}{(n!)^2}$. **1**

(iv) Hence or otherwise show that $0 < \ln 2 - \ln 3 + \ln 4 - \ldots + \ln(2n) < \ln(n+1)$ for $n = 2, 3, 4, \ldots$. **2**

Independent Trial HSC 2014 Mathematics Extension 2 Marking Guidelines

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1.	B	$a=1$, $r=2\cos^2\theta$ $\quad \frac{a}{1-r}=\frac{1}{1-(1+\cos 2\theta)}=\frac{1}{-\cos 2\theta}=-\sec 2\theta$	H5
2.	C	$f(x)=\sin^{-1}x+\tan^{-1}x$ is an increasing function with domain $-1\le x\le 1$. $\therefore\ \sin^{-1}(-1)+\tan^{-1}(-1)\le y\le \sin^{-1}1+\tan^{-1}1$ $(-\frac{\pi}{2})+(-\frac{\pi}{4})\le y\le \frac{\pi}{2}+\frac{\pi}{4}$ $\quad\therefore -\frac{3\pi}{4}\le y\le \frac{3\pi}{4}$	HE4
3.	A	$e^x+e^y=1$ $e^x+e^y\frac{dy}{dx}=0$ $\quad\therefore \frac{dy}{dx}=\frac{-e^x}{e^y}=-e^{x-y}$	E6
4.	B	$\lvert z-6\rvert=2\lvert z\rvert$ $\quad (x-6)^2+y^2=4(x^2+y^2)$ $\quad 36=3x^2+12x+3y^2$ $\quad (x+2)^2+y^2=16$	E3
5.	C	$e=\sqrt{2}$, $a=\sqrt{k}$ $\quad S(\sqrt{2k},0)$ $\quad\therefore \sqrt{2k}=4$ $\quad\therefore k=8$	E4
6.	D	$\int xe^{-x}\,dx=x(-e^{-x})-\int(-e^{-x})\,dx=-xe^{-x}-e^{-x}+c$	E8
7.	D	 $h=1-\sqrt{x}$ $R=1-x$ $r=1-x-\delta x$ $\delta V=\pi(R^2-r^2)h$ $=\pi(R+r)(R-r)h$ $\therefore \delta V=\pi\{2(1-x)-\delta x\}\delta x(1-\sqrt{x})$ $\delta V=2\pi(1-x)(1-\sqrt{x})\delta x$ (ignoring terms in $(\delta x)^2$) $\therefore V=\lim_{\delta x\to 0}\sum_{x=0}^{x=1}2\pi(1-x)(1-\sqrt{x})\delta x=2\pi\int_0^1(1-x)(1-\sqrt{x})\,dx$	E7
8.	C	$-\frac{1}{\alpha}$, $-\frac{1}{\beta}$ and $-\frac{1}{\gamma}$ satisfy $(-\frac{1}{x})^3-4(-\frac{1}{x})-2=0$ Rearranging gives $2x^3-4x^2+1=0$	E4
9.	A	Resolving forces horizontally and applying Newton's 2nd law gives $T\sin\theta=mr\omega^2$. Then $r=l\sin\theta$ gives $T=ml\omega^2$	E5
10.	D	$\lim_{n\to\infty}\frac{{}^nC_1\,.\,{}^nC_2}{{}^nC_3}=\lim_{n\to\infty}\left\{n.\frac{n(n-1)}{2!}.\frac{3!}{n(n-1)(n-2)}\right\}=3\lim_{n\to\infty}\frac{1}{1-\frac{2}{n}}=3$	HE3

Section II

Question 11

a. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • find the difference	1
ii • find the product	1

Answer

i. $\overline{z}-w=(1-3i)-(2-i)=-1-2i$ $\qquad$ ii. $zw=(1+3i)(2-i)=5+5i$

b. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • find the modulus	1
• find the argument	1
ii • use deMoivre's theorem	1
• simplify into required form	1

Answer

i. $-1+\sqrt{3}\,i=2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

ii. $z^8=2^8\left(\cos\frac{16\pi}{3}+i\sin\frac{16\pi}{3}\right)=2^8\left(\cos\frac{-2\pi}{3}+i\sin\frac{-2\pi}{3}\right)=2^7\left(-1-\sqrt{3}\,i\right)$

$16z^4=16.2^4\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right)=2^8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=2^7\left(-1+\sqrt{3}\,i\right)$

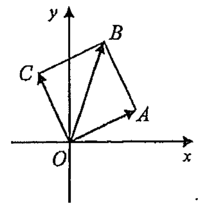
$z^8-16z^4=-2^8\sqrt{3}\,i=-256\sqrt{3}\,i$

c. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • find z represented by vector OC	1
• find z represented by vector OB	1
ii • write an expression for z using complex numbers in form $a+ib$	1
• evaluate z	1

Answer



i. $\overrightarrow{OC}$ represents iz (anticlockwise rotation of $\overrightarrow{OA}$ by $\frac{\pi}{2}$)

$\overrightarrow{OB}$ is the vector sum of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

Hence $\overrightarrow{OB}$ represents $z+iz=(1+i)z$

ii. $4+2i=(1+i)z$

$(4+2i)(1-i)=2z$

$6-2i=2z$

$z=3-i$

Q11 (cont)

d. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
i • use the relationships between roots and coefficients to evaluate the sum of squares	1
• evaluate the sum of fourth powers by writing it in terms of sums of lower powers	1
ii • use the negative value of the sum of the fourth powers to deduce at least one root is non-real	1
• deduce that there are either 4 non-real roots, or 2 real and 2 non-real roots	1
• show that one root is real by establishing the change of sign of the polynomial function	1

Answer

i. $\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha)=0-2(-2)=4$

Each of $\alpha,\beta,\gamma,\delta$ satisfies $x^4-2x^2-5x+3=0$. Hence

$(\alpha^4+\beta^4+\gamma^4+\delta^4)-2(\alpha^2+\beta^2+\gamma^2+\delta^2)-5(\alpha+\beta+\gamma+\delta)+(3+3+3+3)=0$

$(\alpha^4+\beta^4+\gamma^4+\delta^4)-2\times4-5\times0+12=0$

$\therefore \alpha^4+\beta^4+\gamma^4+\delta^4=-4$

ii. At least one of $\alpha,\beta,\gamma,\delta$ must be non-real (since the fourth powers are not all non-negative). However the non-real roots come in complex conjugate pairs (since the coefficients are real). Hence either there are 4 non-real roots, or there are 2 non-real and 2 real roots. Considering the continuous polynomial function $P(x)=x^4-2x^2-5x+3$, $P(0)=3>0$ and $P(1)=-3<0$ and hence there is a real root of $P(x)=0$ lying between 0 and 1. Hence the equation must have 2 real and 2 non-real roots.

Question 12

a. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• complete the square	1
• write the primitive function	1

Answer

$$\int\frac{1}{\sqrt{3-(x^2-2x)}}\,dx=\int\frac{1}{\sqrt{4-(x-1)^2}}\,dx=\sin^{-1}\left(\frac{x-1}{2}\right)+c$$

b. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• rearrange integrand into appropriate form	1
• write the primitive	1

Answer

$$\int\frac{e^{2x}}{e^x+1}\,dx=\int\frac{e^x(e^x+1)-e^x}{e^x+1}\,dx=\int\left\{e^x-\frac{e^x}{e^x+1}\right\}dx=e^x-\ln(e^x+1)+c$$

Q12 (cont)

c. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• apply integration by parts	1
• complete the primitive function	1

Answer

$$\int 1.\tan^{-1}x\;dx=x\tan^{-1}x-\int\frac{x}{1+x^2}\,dx=x\tan^{-1}x-\tfrac12\ln(1+x^2)+c$$

d. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• express the integrand in terms of t	1
• write the definite integral in terms of t	1
• find the primitive function	1
• evaluate using t limits	1

Answer

$t=\tan\frac{x}{2}$

$dt=\frac12\sec^2\frac{x}{2}\,dx$

$dx=\frac{2}{1+t^2}\,dt$

$x=0\ \Rightarrow\ t=0$

$x=\frac{\pi}{2}\ \Rightarrow\ t=1$

$5+4\sin x+3\cos x$

$=\frac{5(1+t^2)+8t+3(1-t^2)}{1+t^2}$

$=\frac{2(t^2+4t+4)}{1+t^2}$

$=\frac{2(t+2)^2}{1+t^2}$

$\int_0^{\frac{\pi}{2}}\frac{1}{5+4\sin x+3\cos x}\,dx$

$=\int_0^1\frac{1+t^2}{2(t+2)^2}\cdot\frac{2}{1+t^2}\,dt$

$=-\left[\frac{1}{t+2}\right]_0^1$

$=\frac16$

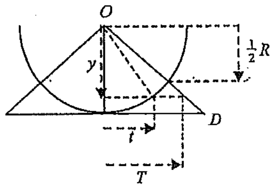
e. Outcomes assessed: E7, E8

Marking Guidelines

Criteria	Marks
i • find either inner or outer radius of the annular cross section	1
• find other radius of the annulus and then its area	1
• express V as a limiting sum of slice volumes and hence as a definite integral	1
ii • find the primitive function	1
• evaluate in terms of R.	1

Answer

i. Cross section y below O is an annulus with inner radius t and outer radius T, where $t^2=R^2-y^2$ and $T=y\tan\frac{\pi}{3}=y\sqrt3$ (since vertical through O makes angle $\frac{\pi}{3}$ with OD)



Hence area of cross section is $\pi\{3y^2-(R^2-y^2)\}$

$\therefore V=\lim_{\delta y\to0}\sum_{y=\frac12R}^{R}\pi(4y^2-R^2)\delta y=\pi\int_{\frac12R}^{R}(4y^2-R^2)\,dy$

ii. $V=\pi\left[\frac43y^3-R^2y\right]_{\frac12R}^{R}=\pi\left\{\frac43\left(R^3-(\frac12R)^3\right)-R^2\left(R-\frac12R\right)\right\}$

$\therefore V=\pi R^3\left(\frac43\times\frac78-\frac12\right)=\frac23\pi R^3$

Question 13

a. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• write expressions for MS, NS in terms of a and e	1
• find PS in terms of a and e	1
• find PQ in terms of a and e	1
• find the sum of the reciprocals of MS and NS and rearrange to obtain result	1

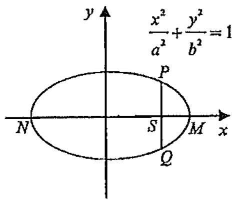
Answer

$$\frac{1}{MS}+\frac{1}{NS}=\frac{1}{a(1-e)}+\frac{1}{a(1+e)}=\frac{(1+e)+(1-e)}{a(1-e^2)}=\frac{2}{a(1-e^2)}$$

Using the locus definition of the ellipse and the directrix $x=\frac{a}{e}$,

$$PS=e\left(\frac{a}{e}-ae\right)=a\left(1-e^2\right) \text{ and hence } PQ=2PS=2a(1-e^2)$$

$$\therefore \frac{1}{MS}+\frac{1}{NS}=\frac{4}{PQ}$$



b. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
i • find gradient of PQ and hence gradient of MX	1
• find coordinates of M and gradient of OM	1
• deduce MX and OM make equal acute angles with the x-axis so that ΔMOX is isosceles	1
ii • find the parameter at T in terms of p and q	1
• compare the gradient of the tangent at T with the gradient of PQ	1

Answer

i. $gradient\ MX = Gradient\ PQ = \dfrac{c(\frac{1}{p}-\frac{1}{q})}{c(p-q)}=-\dfrac{1}{pq}$. M has coordinates $\left(\dfrac{c(p+q)}{2},\dfrac{c(p+q)}{2pq}\right)$

$gradient\ OM=\dfrac{c(p+q)}{2pq}\div\dfrac{c(p+q)}{2}=\dfrac{1}{pq}$ $\qquad \therefore gradient\ MX=-\,gradient\ OM$

If MX and OM make angles α and β respectively with the positive x axis, then $\tan\alpha=-\tan\beta$.
Hence $\beta=180°-\alpha$ and in ΔMOX, $\angle MOX=\angle MXO=\beta$. Then ΔMOX is isosceles with $MX=OM$.

ii. Let T have coordinates $\left(ct,\frac{c}{t}\right)$. Then $m_{OT}=\dfrac{c}{t}\div ct=\dfrac{1}{t^2}$. But $m_{OT}=m_{OM}$. $\therefore t^2=pq$.

At T, $\dfrac{dy}{dx}=\dfrac{dy}{dt}\div\dfrac{dx}{dt}=\dfrac{-c}{t^2}\div c=-\dfrac{1}{t^2}=-\dfrac{1}{pq}=m_{PQ}$. Hence the tangent at T is parallel to PQ.

Q13 (cont)

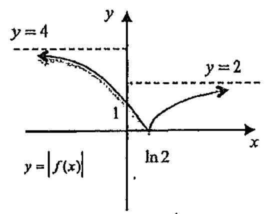
c. Outcomes assessed: E6

Marking Guidelines

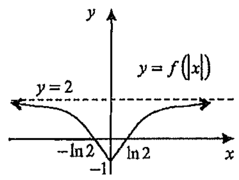
Criteria	Marks
i • reflect section of graph below x axis in x axis	1
ii • reflect section of graph to right of y axis in y axis to obtain graph for $x<0$	1
iii • sketch upper branch with asymptotes	1
• sketch lower branch with asymptotes and y intercept	1
iv • correct domain, shape and vertical asymptotes	1
• correct intercepts on axes	1

Answer

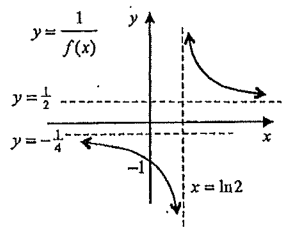
i.



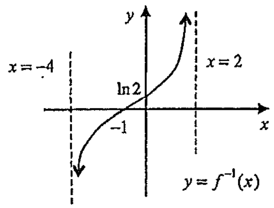
ii.



iii.



iv.



Question 14

a. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• find A	1
• find B	1
• find C	1

Answer

$$x^2+3x+4\equiv A\left(x^2+3\right)+\left(Bx+C\right)\left(x-2\right)$$

$x=2 \Rightarrow 14=7A \qquad \therefore A=2$

$x=0 \Rightarrow 4=3A-2C \qquad \therefore C=1$

$Equate\ coeff.\ of\ x^2:\ 1=A+B \qquad \therefore B=-1$

$$\frac{x^2+3x+4}{(x-2)(x^2+3)}=\frac{2}{x-2}+\frac{-x+1}{x^2+3}$$

Q14 (cont)

b. Outcomes assessed: HE2

Marking Guidelines

Criteria	Marks
• define a sequence of statements and show the first two are true	1
• use the recurrence relation to write T_{k+1} in terms of values of T_k, T_{k-1} given $S(n)$ true, $n \le k$	1
• rearrange to establish conditional truth of $S(k+1)$ and complete the induction process	1

Answer

Let $S(n)$, $n = 1, 2, 3, \dots$ be the sequence of statements defined by $S(n): T_n = (n+1)\,3^n$.

Consider $S(1)$ and $S(2)$: $\quad T_1 = 6 = (1+1)\times 3^1 \quad \therefore S(1)$ is true

$T_2 = 27 = (2+1)\times 3^2 \quad \therefore S(2)$ is true

If $S(n)$ is true for $n \le k$ (where $k \ge 2$): $\quad T_n = (n+1)\,3^n, \quad n = 1, 2, 3, \dots, k \quad$ *

Consider $S(k+1)$, $k \ge 2$:

$$\begin{aligned} T_{k+1} &= 6T_k - 9T_{k-1} \\ &= 6(k+1)\,3^k - 9k\,.3^{k-1} \quad \text{if } S(n) \text{ is true for } n \le k \text{, using *} \\ &= \{2(k+1) - k\}\,3^{k+1} \\ &= \{(k+1)+1\}\,3^{k+1} \end{aligned}$$

Hence if $S(n)$ is true for $n \le k$ (where $k \ge 2$) then $S(k+1)$ is true. But $S(n)$ is true for $n \le 2$. Hence $S(3)$ is true, then $S(n)$ true for $n \le 3 \Rightarrow S(4)$ is true and so on. Hence by Mathematical Induction, $T_n = (n+1)\,3^n$ for all integers $n \ge 1$.

c. Outcomes assessed: E5

Marking Guidelines

Criteria	Marks
i • find $\frac{dx}{dv}$ in terms of v	1
• use initial conditions to find x in terms of v	1
• find expression for H by finding x when v is zero	1
ii • find equation of motion for downward journey	1
• find distance fallen in terms of v	1
• use expression for maximum height to establish required equation	1
iii • note continuity and establish change of sign	1
• apply Newton's method	1
iv • find terminal velocity and use value of λ to obtain required percentage	1

Answer

i.

$$\begin{aligned} \ddot{x} &= -\tfrac{1}{10}(100+v) \\ v\frac{dv}{dx} &= -\tfrac{1}{10}(100+v) \\ -\tfrac{1}{10}\frac{dx}{dv} &= \frac{v}{100+v} \\ -\tfrac{1}{10}\frac{dx}{dv} &= 1 - \frac{100}{100+v} \end{aligned}$$

$$-\tfrac{1}{10}x = v - 100\ln(100+v) + c$$

$t = 0$, $x = 0$, $v = 200$ $\Big\}$ $\quad 0 = 200 - 100\ln 300 + c$

$$\tfrac{1}{10}x = (200 - v) - 100\ln\left(\tfrac{300}{100+v}\right)$$

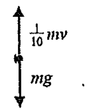
$x = H,\ v = 0 \Rightarrow \quad \tfrac{1}{10}H = 200 - 100\ln 3$

$\therefore \quad H = 1000\,(2 - \ln 3)$

Q14 c (cont)

ii. For the downward journey, let x be the distance fallen below the position of maximum height, with initial conditions $x = 0,\ v = 0$.

Forces on particle



By Newton's 2nd Law

$$\begin{aligned} m\ddot{x} &= mg - \tfrac{1}{10}mv \\ \ddot{x} &= \tfrac{1}{10}(100 - v) \\ v\frac{dv}{dx} &= \tfrac{1}{10}(100 - v) \\ \tfrac{1}{10}\frac{dx}{dv} &= \frac{v}{100-v} \\ -\tfrac{1}{10}\frac{dx}{dv} &= 1 - \frac{100}{100-v} \end{aligned}$$

$$-\tfrac{1}{10}x = v + 100\ln(100-v) + c$$

$t = 0, x = 0, v = 0 \Rightarrow \quad 0 = \quad 100\ln 100 \quad + c$

$$-\tfrac{1}{10}x = v + 100\ln\left(\tfrac{100-v}{100}\right)$$

$x = H \Rightarrow \quad -100(2 - \ln 3) = v + 100\ln\left(1 - \tfrac{v}{100}\right)$

$$\tfrac{v}{100} + \ln\left(1 - \tfrac{v}{100}\right) + \left(2 - \ln 3\right) = 0$$

iii. Let $f(\lambda) = \lambda + \ln(1-\lambda) + (2 - \ln 3)$.

$f'(\lambda) = 1 - \frac{1}{1-\lambda} = \frac{-\lambda}{1-\lambda}$

Then $f(\lambda)$ is continuous for $0 < \lambda < 1$ and $f(0\cdot 8) \approx 0\cdot 09 > 0$, $f(0\cdot 9) \approx -0\cdot 50 < 0$

Hence $f(\lambda) = 0$ for some $0\cdot 8 < \lambda < 0\cdot 9$.

Using $\lambda_0 = 0\cdot 82$, $\lambda_1 = 0\cdot 82 - \dfrac{0\cdot 82 + \ln(1 - 0\cdot 82) + 2 - \ln 3}{\left(\frac{-0\cdot 82}{1-0\cdot 82}\right)} \approx 0\cdot 82$

iv. Since Newton's method returned the same approximate root to 2 decimal places, $\frac{v}{100} \approx 0\cdot 82$ gives the speed v on return to projection point as $82\ \text{ms}^{-1}$ (to nearest 1).

For the downward journey, $\ddot{x} \to 0$ as $v \to 100$. Hence the terminal velocity is $100\ \text{ms}^{-1}$.

Hence particle has attained 82% of its terminal velocity on return to its point of projection.

Question 15

a. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
i • rearrange integrand	1
• evaluate definite integral to obtain reduction formula	1
ii • evaluate I_0	1
• reduce and evaluate I_4	1

Answer

i. $\displaystyle\int_0^1 \frac{x^n}{1+x^2}\,dx = \int_0^1 \frac{\{(1+x^2) - 1\}x^{n-2}}{1+x^2}\,dx$, $n = 2, 3, 4, \dots$

$$\begin{aligned} I_n &= \int_0^1 x^{n-2}\,dx - I_{n-2} \\ &= \tfrac{1}{n-1}\left[x^{n-1}\right]_0^1 - I_{n-2} \\ &= \tfrac{1}{n-1} - I_{n-2} \end{aligned}$$

ii. $\displaystyle I_0 = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1 = \tfrac{\pi}{4}$

$$\begin{aligned} I_4 &= \tfrac{1}{3} - I_2 \\ &= \tfrac{1}{3} - \left(1 - I_0\right) \\ &= \tfrac{\pi}{4} - \tfrac{2}{3} \end{aligned}$$

Q15(cont)

b. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
i • show the sum of a positive real number and its reciprocal is at least 2	1
ii α • rearrange given expression into such sums	1
• apply result from (i) to establish required inequality	1
β • make appropriate replacements for a, b, c	1
• rearrange to establish required inequality	1

Answer

i. $\left(a+\frac{1}{a}\right)^2=\left(a-\frac{1}{a}\right)^2+4\ge 4$, since $\left(a-\frac{1}{a}\right)^2\ge 0$ for real $a\ne 0$

$\therefore a+\frac{1}{a}\ge 2$ for real $a>0$

ii(α). $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}=\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$, where each of $\frac{a}{b},\frac{b}{c},\frac{c}{a}$ is real and positive.

$\ge 2\ +2\ +2$ using (i)

$\therefore \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\ge 6$

ii (β). Replacing $a\to b+c,\ b\to c+a,\ c\to a+b$:

$$\frac{(c+a)+(a+b)}{b+c}+\frac{(a+b)+(b+c)}{c+a}+\frac{(b+c)+(c+a)}{a+b}\ge 6$$

$$1+\frac{2a}{b+c}\quad +\quad 1+\frac{2b}{c+a}\quad +\quad 1+\frac{2c}{a+b}\quad \ge 6$$

$$2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\ge 3$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}$$

c. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • expand using compound angle trigonometric identities then simplify	1
ii • use identity from (i) to simplify sum	1
• use trigonometric identity converting difference to product	1
• simplify to obtain required result	1
iii • use appropriate trigonometric identity	1
• use result from (ii) to evaluate sum	1

Answer

i. $\sin(2k+1)\theta=\sin 2k\theta\cos\theta+\cos 2k\theta\sin\theta$ and $\sin(2k-1)\theta=\sin 2k\theta\cos\theta-\cos 2k\theta\sin\theta$

$\therefore \sin(2k+1)\theta-\sin(2k-1)\theta=2\sin\theta\cos 2k\theta$

ii. $2\sin\theta\sum_{k=1}^{n}\cos 2k\theta=\sum_{k=1}^{n}\left\{\sin(2k+1)\theta-\sin(2k-1)\theta\right\}=\sin(2n+1)\theta-\sin\theta$

Using $\sin A-\sin B=2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$, $A=2(n+1)\theta,\ B=\theta$: $\sin\theta\sum_{k=1}^{n}\cos 2k\theta=\sin n\theta\cos(n+1)\theta$

iii. $\sum_{k=1}^{10}\sin^2\frac{k\pi}{10}=\frac{1}{2}\sum_{k=1}^{10}\left(1-\cos\frac{2k\pi}{10}\right)=\frac{1}{2}\left\{10-\frac{\sin\frac{10\pi}{10}\cos\frac{11\pi}{10}}{\sin\frac{\pi}{10}}\right\}=5$

Question 16

a. Outcomes assessed: P5, H5, PE3

Marking Guidelines

Criteria	Marks
i • provide a sequence of deductions leading to required result	1
• justify deductions using geometric properties and trigonometry	1
ii • use the sine rule and (i) to obtain result	1
iii • write f explicitly as function of α for fixed A and find derivative	1
• show derivative is negative throughout domain of f	1
iv • deduce $f(\alpha)$ takes its maximum value for $\alpha=0$ and hence when ΔABC is equilateral	1
• find this maximum value	1

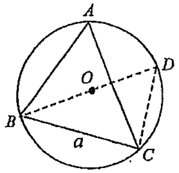
Answer

i. Construct diameter BD through centre O and construct DC.

$\angle BDC=\angle A$ (*$\angle$'s in same segment subtended by chord BC are equal*)

$\angle BCD=\frac{\pi}{2}$ (*$\angle$ in semi-circle is a right angle*)

In ΔBCD , $\sin\angle BDC=\frac{a}{BD}=\frac{a}{2R}$. $\therefore R=\frac{a}{2\sin A}$



ii. $\frac{Area\ \Delta ABC}{Area\ circle\ ABC}=\frac{\frac{1}{2}bc\sin A}{\pi R^2}=\frac{bc\sin A(2\sin A)^2}{2\pi a^2}=\frac{2}{\pi}abc\left(\frac{\sin A}{a}\right)^3$

Using the sine rule in ΔABC , $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$.

$$\frac{Area\ \Delta ABC}{Area\ circle\ ABC}=\frac{2}{\pi}abc.\frac{\sin A}{a}.\frac{\sin B}{b}.\frac{\sin C}{c}=\frac{2}{\pi}\sin A\sin B\sin C$$

iii. $C=A-\alpha,\ B=\pi-(2A-\alpha)$ (*since $\angle$ sum of ΔABC is π*)

$f(\alpha)=\frac{2}{\pi}\sin A\sin B\sin C=\frac{2}{\pi}\sin A\sin(2A-\alpha)\sin(A-\alpha)$, where $0\le\alpha<A$

Using $\cos(p-q)-\cos(p+q)=2\sin p\sin q$ with $p=2A-\alpha,\ q=A-\alpha$:

$f(\alpha)=\frac{1}{\pi}\sin A\left\{\cos A-\cos(3A-2\alpha)\right\}$

$f'(\alpha)=-\frac{2}{\pi}\sin A\sin(3A-2\alpha)$

But $3A-2\alpha=A+(A-\alpha)+(A-\alpha)=A+C+C\le A+B+C=\pi$ (*since $A\ge B\ge C$*)

Hence $\sin(3A-2\alpha)\ge 0$ and $f'(\alpha)\le 0$ for $0\le\alpha<A$.

iv. $f(\alpha)$ is a decreasing function throughout its domain $0\le\alpha<A$, hence it takes its maximum value when $\alpha=0$ and $C=A$. But then $A\ge B\ge C\Rightarrow A=B=C=\frac{\pi}{3}$ and ΔABC is equilateral.

The maximum value of $\frac{Area\ \Delta ABC}{Area\ circle\ ABC}$ is $f(0)=\frac{2}{\pi}\left(\sin\frac{\pi}{3}\right)^3=\frac{3\sqrt{3}}{4\pi}$.

Q16 (cont)

b. Outcomes assessed: PE3, HE3

Marking Guidelines

Criteria	Marks
i • expand the square and break up into separate sums	1
• manipulate sigma notation to obtain required result	1
ii • apply result to the sequence of $n+1$ binomial coefficients	1
• evaluate the sum of these binomial coefficients	1
• explain why strict inequality holds	1
iii • equate coefficients of x^n on both sides of identity using properties of binomial coefficients	1
iv • simplify factorial quotient then apply (ii) and (iii) to obtain appropriate inequality	1
• take logarithms to complete proof	1

Answer

i.

$$\sum_{k=1}^{n}\left(x_k-\tfrac{1}{n}S_n\right)^2=\sum_{k=1}^{n}\left\{x_k^{\,2}-2\left(\tfrac{1}{n}S_n\right)x_k+\left(\tfrac{1}{n}S_n\right)^2\right\}$$

$$=\sum_{k=1}^{n}x_k^{\,2}-2\left(\tfrac{1}{n}S_n\right)\sum_{k=1}^{n}x_k+\left(\tfrac{1}{n}S_n\right)^2\sum_{k=1}^{n}1$$

$$=\sum_{k=1}^{n}x_k^{\,2}-2\left(\tfrac{1}{n}S_n\right)S_n+n\left(\tfrac{1}{n}S_n\right)^2$$

$$=\sum_{k=1}^{n}x_k^{\,2}-\tfrac{1}{n}S_n^{\,2}$$

$$\sum_{k=1}^{n}x_k^{\,2}=\tfrac{1}{n}S_n^{\,2}+\sum_{k=1}^{n}\left(x_k-\tfrac{1}{n}S_n\right)^2$$

ii. $x_{k+1}={}^nC_k$, $k=0,1,2,\ldots,n$. Then $(1+x)^n\equiv\sum_{k=0}^{n}{}^nC_kx^k$ gives $2^n=\sum_{k=0}^{n}{}^nC_k$. $\therefore S_{n+1}=2^n$

From (i), for real x_k , $\sum_{k=1}^{n+1}x_k^{\,2}\ge\frac{1}{n+1}S_{n+1}^{\,2}$, since each of $(x_k-\frac{1}{n+1}S_{n+1})^2\ge0$.

Since equality only holds for $x_1=x_2=\ldots=x_{n+1}$, $\sum_{k=0}^{n}\left({}^nC_k\right)^2>\dfrac{2^{2n}}{n+1}$ for $n=2,3,4,\ldots$

iii. Considering the coefficient of x^n on both sides of the identity $\left\{(1+x)^n\right\}^2\equiv(1+x)^{2n}$,

$\sum_{k=0}^{n}{}^nC_k\,{}^nC_{n-k}={}^{2n}C_n$. But ${}^nC_{n-k}={}^nC_k$ and ${}^{2n}C_n=\dfrac{(2n)!}{n!\,n!}$. Hence $\sum_{k=0}^{n}\left({}^nC_k\right)^2=\dfrac{(2n)!}{(n!)^2}$.

iv. $$\frac{(2n)!}{(n!)^2}=\frac{2n(2n-1)\{2(n-1)\}(2n-3)\{2(n-2)\}\ldots2.1}{\{n(n-1)(n-2)\ldots1\}^2}=\frac{2^n(2n-1)(2n-3)\ldots3.1}{n(n-1)(n-2)\ldots1}$$

Using (ii) and (iii), $\dfrac{2^n(2n-1)(2n-3)\ldots3.1}{n(n-1)(n-2)\ldots1}>\dfrac{2^{2n}}{n+1}$, giving $\dfrac{1.3.5\ldots(2n-3)(2n-1)}{2.4.6\ldots\{2(n-1)\}\{2n\}}>\dfrac{1}{n+1}$

Taking logs of both sides gives $\ln1-\ln2+\ln3-\ln4+\ldots+\ln(2n-1)-\ln(2n)>-\ln(n+1)$

$$\ln2-\ln3+\ln4-\ldots+\ln(2n)<\ln(n+1)$$

Also $\ln2-\ln3+\ln4-\ldots+\ln(2n)=\ln2+\ln\frac{4}{3}+\ln\frac{6}{5}+\ldots+\ln\frac{2n}{2n-1}>0$

$\therefore\ 0<\ln2-\ln3+\ln4-\ldots+\ln(2n)<\ln(n+1)$ for $n=2,3,4,\ldots$