2 UNIT TEST NUMBER 9

1996

Applications of Calculus to the Physical World.

QUESTION 1. (11 marks) **Marks**

The position x cm of an object moving in a straight line at time t seconds, is given by: $x = 6t^2 - t^3 + 4$.

(a) Find the times at which the object is at rest. 3

(b) Find the distance travelled between these stationary times. 2

(c) Find the total distance travelled in the first six seconds. 3

(d) Find the velocity when the acceleration is zero. 3

QUESTION 2. (6 marks)

Water in an electric jug is being heated. The rate of increase of temperature R (°C/minute) at time t minutes is given by: $R(t) = 24 - 2t$. Initially, the water is at a temperature of 5°C.

(a) Find the initial rate of increase of temperature. 1

(b) Find an expression for the temperature (T °C) at time t minutes. 2

(c) How long does it take for the water to boil (i.e. reach 100°C)? 2

(d) Is the rate of increase of temperature increasing or decreasing during the heating process. Explain. 1

QUESTION 3. (9 marks)

The population P of a colony at time t years is given by: $P = P_0 e^{kt}$.

(a) Explain why the initial population is P_0. 1

(b) Show that the rate of increase of the population at any time is proportional to the population at that time. 1

(c) The population increases from an initial number of 5000 to 9000 in 4 years. Find the value of k (to 3 significant figures). 2

(Question 3 Continued)

Marks

(d) What will the population be after 10 years? 1

(e) In how many years will the initial population be trebled? 2

(f) What is the rate of population increase after 4 years? 2

QUESTION 4. (4 marks)

The concentration (C) of a drug in the bloodstream, at time t hours after being injected, is given by the formula $C = Ae^{-kt}$.

(a) For one particular drug, it takes 5.5 hours for its concentration to decrease by half. Find the value of k correct to 3 decimal places. 2

(b) What percentage of the drug has been used up in the first hour? 2

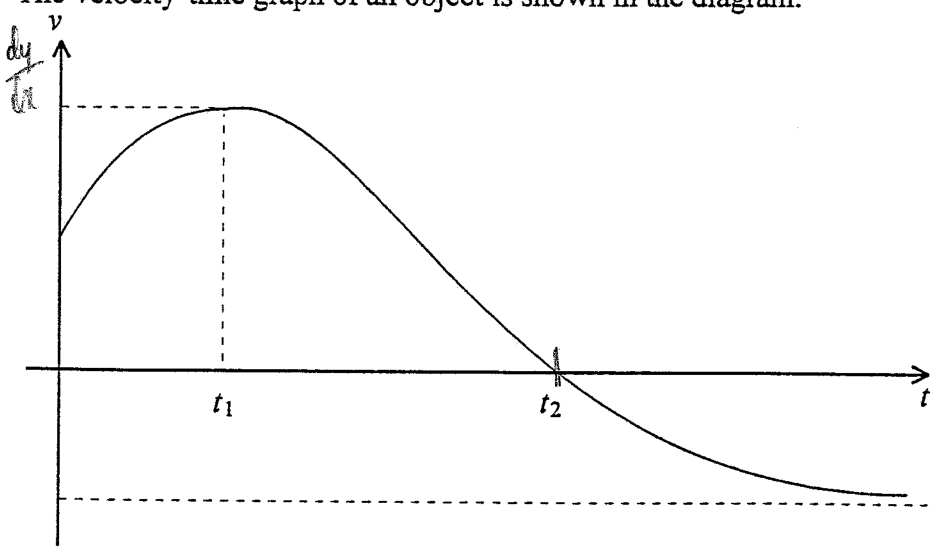
QUESTION 5. (10 marks)

(a) The speed of an object is measured at 2 second intervals and the results recorded. 4

Time (sec)	0	2	4	6	8
Speed (cm/s)	0	3.2	4.8	7.2	6.8

Determine, using the Trapezoidal Rule, the approximate distance travelled by the object in the first 8 seconds.

(b) The velocity-time graph of an object is shown in the diagram. 6



(i) Draw a displacement-time graph for this object, given that it starts at the origin. Mark t_1 and t_2 on your t-axis.

(ii) Describe the motion of the object.

2 UNIT TEST NUMBER 9

1996

SUGGESTED SOLUTIONS

QUESTION 1

(a) $x = 6t^2 - t^3 + 4$

$v = \frac{dx}{dt} = 12t - 3t^2$ **1**

At rest, $v = 0$

$12t - 3t^2 = 0$ *Note* : Quadratic equation – factorise.

$3t(4 - t) = 0$ **1**

$t = 0$ or $t = 4$

Object is at rest at 0 seconds and 4 seconds. **1** **Total = 3**

(b) When $t = 0$, $x = 4$

When $t = 4$, $x = 6 \times 4^2 - 4^3 + 4$ **1**

$= 36$

Distance $= 36 - 4$

Distance travelled is 32 cm. **1** **Total = 2**

(c) When $t = 6$, $x = 6 \times 6^2 - 6^3 + 4$

$= 4$ **1**

Between $t = 4$ and $t = 6$, object travels 32 cm. **1**

Total distance travelled is 32 + 32 = 64 cm. **1** **Total = 3**

(d) $a = \frac{dv}{dt} = 12 - 6t$ **1**

When $a = 0$, $t = 2$. **1**

When $t = 2$, $v = 12 \times 2 - 3 \times 2^2$

$= 12$ **1**

Velocity is 12 cm/s when acceleration is zero. **Total = 3**

QUESTION 2

(a) $R(t) = 24 - 2t$

Note : $R(t)$ is the <u>rate</u> of increase of temperature (T), i.e. $R(t) = \frac{dT}{dt}$.

$\frac{dT}{dt} = 24 - 2t$

where T is temperature.

When $t = 0$, $R = 24$

Temperature is initially increasing at 24°C/minute. **1**

(b) $T = 24t - t^2 + C$ **1**

Note : By integration from (a).

When $t = 0$, $T = 5$ $\therefore C = 5$

$\therefore T = 24t - t^2 + 5$ **1** **Total = 2**

(c) When $T = 100$,

$24t - t^2 + 5 = 100$

$t^2 - 24t + 95 = 0$ **1**

$(t - 5)(t - 19) = 0$

$t = 5, \quad t = 19$

Water boils after 5 minutes. **1** **Total = 2**

(d) Rate of increase of temperature is decreasing, because R gets smaller as t increases. **1**

ESTION 3

$P = P_0 e^{kt}$

When $t = 0$, $P = P_0 e^0 = P_0 \times 1 = P_0$

Hence initial population is P_0. 1

$\frac{dP}{dt} = P_0 k e^{kt}$

$= k P_0 e^{kt}$

$= kP$

$\therefore$ rate of increase of population at any time is proportional to the population at that time. 1

Substitute $t = 4$, $P = 9000$, $P_0 = 5000$

$9000 = 5000 e^{k \times 4}$

$e^{4k} = 1.8$ 1

$4k = \log_e 1.8$

$k = \frac{\log_e 1.8}{4}$

$k = 0.147$ (correct to 3 sig. fig.) 1 **Total = 2**

) When $t = 10$, $P = 5000 \times e^{0.147 \times 10}$

$= 21746$ 1 *Note*: Nature of question requires a whole number.

) Population is trebled when $P = 15\,000$

$15\,000 = 5000 e^{0.147 t}$

$e^{0.147 t} = 3$ 1

$0.147 t = \log_e 3$

$t = \frac{\log_e 3}{0.147}$

$= 7.5$ (correct to 1 dec. place)

Population trebles in 7.5 years. 1 **Total = 2**

$$\frac{dP}{dt} = kP$$

When $t = 4$, $P = 9000$

$$\frac{dP}{dt} = 0.147 \times 9000 \qquad 1$$

$$= 1323$$

Alternative solution :

$$\frac{dP}{dt} = k\,P_0\,e^{kt}$$

$$= 0.147 \times 5000 \times e^{0.147 \times 4}$$

$$= 1323$$

Population is increasing at 1323 per year after 4 years. 1 **Total = 2**

STION 4

$C = Ae^{-kt}$

Note : A is the initial concentration of drug in the bloodstream.

Substitute $t = 5.5$, $C = \frac{1}{2}A$:

$$\frac{1}{2}A = Ae^{-k \times 5.5}$$

$$e^{-5.5k} = 0.5 \qquad 1$$

$$-5.5\,k = \log_e 0.5$$

$$k = \frac{\log_e 0.5}{-5.5}$$

$k = 0.126$ (to 3 dec. places) 1 **Total = 2**

When $t = 1$,

$$C = A\,e^{-0.126 \times 1}$$

$$= A \times 0.88 \qquad 1$$

i.e. 88% of the drug remains.

$\therefore$ 12% of the drug has been used up. 1 **Total = 2**

QUESTION 5

(a) Dist = $\int$ speed dt

Using the trapezoidal rule,

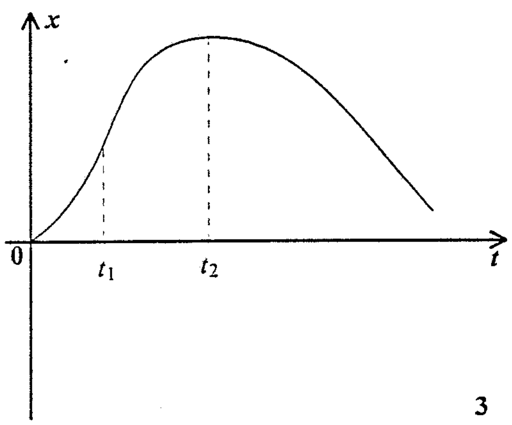
$d \approx \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4]$ **1**

$\approx \frac{2}{2}[0 + 2(3.2 + 4.8 + 7.2) + 6.8]$ **2** *Note :* From table.

≈ 37.2

Distance travelled is 37.2 cm. (1 d.p.) **1** **Total = 4**

(b) (i)



3

(ii) The object moves in a positive direction, increasing speed to $t = t_1$, and then it slows down to zero speed at $t = t_2$.
It then moves in a negative direction with speed increasing until it approaches a constant speed in the negative direction. **3**