2 UNIT TEST NUMBER 7

1996

Logarithmic and Exponential Functions.

QUESTION 1. (12 marks) — Marks

(a) State the range of the function $y = e^x$. — 1

(b) State the domain of the function $y = \log_2 x$. — 1

(c) Write an equivalent equation to $\log_x y = \frac{1}{2}$, without logarithmic notation. — 1

(d) Factorise $10^n - 2^n$. — 1

(e) Evaluate $\log_3 5$ correct to 2 decimal places. — 2

(f) Simplify $\dfrac{p^{-\frac{1}{2}} \times p^{\frac{3}{4}}}{p^{-\frac{1}{4}}}$. — 2

(g) Evaluate $\log_{\frac{1}{2}} 8$. — 2

(h) Sketch the graph of $y = \log_e(x-2)$ showing all important features. — 2

QUESTION 2. (13 marks)

(a) Differentiate: — 4

(i) e^{x^2},

(ii) $\ln(x^2+5)$,

(iii) $e^{-x}\ln(2x)$.

(b) Show that $\int_{-1}^{1} (e^{2x} + e^{-2x})dx = \left(e - \frac{1}{e}\right)\left(e + \frac{1}{e}\right)$. — 3

(c) The tangent at any point A, whose x-coordinate is a, on the curve $y = e^x$, cuts the x-axis at K. A straight line through A meets the x-axis at right-angles at L. — 6

(i) Represent this information on a diagram and find the equation of the tangent AK.

(ii) Find the length of KL.

QUESTION 3. (15 marks) **Marks**

(a) Find : **4**

(i) $\int \frac{3}{3x-2}dx,$

(ii) $\int \frac{e^x}{e^x-1}dx\,,$

(iii) $\int \frac{x^2}{5-x^3}\,dx\,.$

(b) If $\int_1^5 \frac{dx}{2x-1} = \ln(z)$, find the value of z. **3**

(c) Consider the function $y = \frac{1}{x} + \ln x$. **8**

(i) Find the first derivative.

(ii) Show that it has a minimum turning point at $(1,1)$.

(iii) For what values of x is the function increasing?

(iv) For what values of x is the function decreasing?

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SUGGESTED SOLUTIONS

QUESTION 1

(a) $y = e^x$. Range: $y > 0$. **1** *Hint :* Sketch the graph.

(b) $y = \log_2 x$. Domain: $x > 0$ **1** *Hint :* Sketch the graph.

(c) $\log_x y = \frac{1}{2}$ *Note :* $y = a^x$, $x = \log_a y$ are equivalent equations.

$y = x^{\frac{1}{2}}$ or $\sqrt{x}$ **1**

(d) $10^n - 2^n = 2^n \times 5^n - 2^n$ *Note :* Using the law: $(ab)^n = a^n b^n$.

$= 2^n(5^n - 1)$ **1**

(e) $\log_3 5 = \dfrac{\log_{10} 5}{\log_{10} 3}$ **1** *Note :* Using the 'change of base' formula. Could also use base e.

$= \dfrac{0.6990}{0.4771}$

$= 1.46$ (2 dec. places) **1** **Total = 2**

(f) $\dfrac{p^{-\frac{1}{2}} \times p^{\frac{3}{4}}}{p^{-\frac{1}{4}}} = \dfrac{p^{\frac{1}{4}}}{p^{-\frac{1}{4}}}$ **1** *Note :* Using $a^m \times a^n = a^{m+n}$.

$= p^{\frac{1}{4}-(-\frac{1}{4})}$ *Note :* Using $a^m \div a^n = a^{m-n}$.

$= p^{\frac{1}{2}}$ **1** **Total = 2**

(g) Let $\log_{\frac{1}{2}} 8 = x$

$$8 = \left(\frac{1}{2}\right)^x$$ 1 — See note to (c) above.

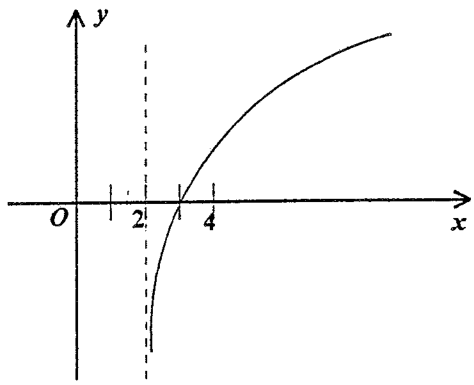
$$2^3 = 2^{-x}$$

$$3 = -x$$

$$x = -3$$

$$\therefore \quad \log_{\frac{1}{2}} 8 = -3$$ 1 — **Total = 2**

(h)



Note : Domain of $y = \log_e x$ is $x > 0$.
$\therefore$ for $y = \log_e(x-2)$,
$x - 2 > 0$ i.e. $x > 2$.

Also, $\log 1 = 0$
$\therefore$ curve cuts the x-axis when $x - 2 = 1$,
i.e. $x = 3$.

2

QUESTION 2

(a) (i) $\frac{d}{dx} e^{x^2} = 2x\, e^{x^2}$ 1 — *Note :* Using $\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$.

(ii) $\frac{d}{dx} \ln\left(x^2 + 5\right) = \frac{2x}{x^2 + 5}$ 1 — *Note :* Using $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$.

(iii) Using the product rule:

$$\frac{d}{dx} e^{-x} \ln 2x = (\ln 2x) \times (-e^{-x}) + (e^{-x}) \times \frac{2}{2x}$$ 1

$$= -e^{-x} \ln 2x + \frac{e^{-x}}{x}$$ 1 — **Total = 2**

(b) $$\int_{-1}^{1} (e^{2x} + e^{-2x})\, dx = \left[\frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}\right]_{-1}^{1}$$ 1 — *Note :* Using $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$.

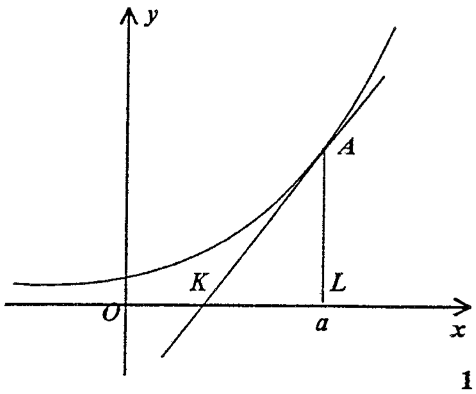
$$= \left(\frac{1}{2}e^2 - \frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^{-2} - \frac{1}{2}e^2\right)$$ 1

$$= e^2 - e^{-2}$$

$$= e^2 - \frac{1}{e^2}$$ *Note :* Difference of two squares factorisation.

$$= \left(e - \frac{1}{e}\right)\left(e + \frac{1}{e}\right)$$ 1 — **Total = 3**

(c) (i) $y = e^x$



1

A is the point (a, e^a).

$\frac{dy}{dx} = e^x$. When $x = a$, $\frac{dy}{dx} = e^a$ 1

Hence gradient of $AK = e^a$

Equation of AK: $y - e^a = e^a(x - a)$ 1 *Note :* Using $y - y_1 = m(x - x_1)$.

Total = 3

(ii) To find K, substitute $y = 0$:

$0 - e^a = e^a(x - a)$ 1

$-e^a = e^a(x - a)$

$-1 = x - a$ *Note :* Dividing both sides by e^a.

$x = a - 1$ 1

K is $(a - 1, 0)$

Since L is $(a, 0)$, $KL = 1$ unit 1 **Total = 3**

QUESTION 3

(a) *Note :* For all parts in (a), we use $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$

(i) $\int \frac{3}{3x-2} dx = \ln(3x - 2) + C$ 1

(ii) $\int \frac{e^x}{e^x - 1} dx = \ln(e^x - 1) + C$ 1

(iii) $\int \frac{x^2}{5 - x^3} dx = -\frac{1}{3} \int \frac{-3x^2}{5 - x^3} dx$ 1

$= -\frac{1}{3} \ln\left(5 - x^3\right) + C$ 1 **Total = 2**

(b) $\int_1^5 \frac{dx}{2x-1} = \left[\frac{1}{2}\ln(2x-1)\right]_1^5$ 1

$= \frac{1}{2}\ln 9 - \frac{1}{2}\ln 1$

$= \ln 9^{\frac{1}{2}} - \frac{1}{2}\times 0$

$= \ln\sqrt{9}$ 1

$= \ln 3$

$\therefore z = 3$ 1 Total = 3

(c) (i) $y = \frac{1}{x} + \ln x$

$= x^{-1} + \ln x$ 1

$\frac{dy}{dx} = -x^{-2} + \frac{1}{x}$

$= \frac{-1}{x^2} + \frac{1}{x}$ 1 Total = 2

(ii) Let $\frac{dy}{dx} = 0$:

$-\frac{1}{x^2} + \frac{1}{x} = 0$

$-1 + x = 0$ (Mult. by x^2) 1

$x = 1$

When $x = 1$, $y = \frac{1}{1} + \ln 1 = 1 + 0 = 1$ 1

Stationary point at $(1,1)$

$\frac{d^2y}{dx^2} = -(-2)x^{-3} + (-1)x^{-2}$ 1

$= \frac{2}{x^3} - \frac{1}{x^2}$

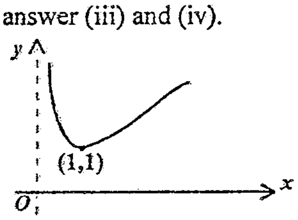
When $x = 1$, $\frac{d^2y}{dx^2} = 2 - 1 > 0$ $\therefore$ concave up. 1

Note: The stationary point is a minimum because $\frac{d^2y}{dx^2} > 0$, i.e. the curve is concave up.

$\therefore$ minimum turning point at $(1,1)$ Total = 4

(iii) Function is increasing for $x > 1$. 1

Note: Sketching the graph is necessary to answer (iii) and (iv).



(iv) Function is decreasing for $0 < x < 1$, since domain is $x > 0$. 1