2 UNIT TEST NUMBER 6

1996

Integration.

QUESTION 1. (10 marks)

Marks

(a) Find: **6**

(i) $\int (1-x^2)dx$

(ii) $\int \left(x+\frac{1}{x^2}\right)dx$

(iii) $\int (x^{\frac{3}{2}}+x^{-\frac{1}{3}})\,dx$.

(b) Show that $\int_0^4 \sqrt{2x+1}\,dx \;=\; 8\frac{2}{3}$. **4**

QUESTION 2. (14 marks)

(a) The region, enclosed by the parabola $y^2 = 4ax$ and the line $x = a$, is rotated about the x-axis. Find the volume of the solid formed. **3**

(b) Consider the functions $y = 16 - x^2$ and $y = 6x$. **11**

(i) Find the coordinates of the points of intersection of the graphs of these functions.

(ii) Sketch, on the same axes, those parts of the curve $y = 16 - x^2$ and the straight line $y = 6x$ which lie in the first quadrant.
Hence shade the region which satisfies $y \leq 16 - x^2$, $y \geq 6x$, and $x \geq 0$.

(iii) By considering the regions above and below the line $y = 12$ separately, find the volume generated when the region described in part (ii) is rotated completely about the y-axis. Give your answer to the nearest cubic unit.

QUESTION 3. (16 marks) **Marks**

(a) The function $f(x)$ describes a continuous curve, some of whose coordinates are given in the following table: **3**

x	0	0.5	1	1.5	2
$f(x)$	1	1.2	1.6	2.2	3

Use this information to evaluate $\int_0^2 f(x)\,dx$ by applying Simpson's rule with five function values.

(b) Consider the function $y = x^2 - 6x + 5$. **10**

(i) Sketch the curve for the domain $0 \le x \le 7$, showing the intercepts on the axes, and the endpoints.

(ii) Find the area between the x-axis and the section of the curve below the x-axis.

(iii) Show that $\int_1^7 (x^2 - 6x + 5)dx = 0$.

(iv) Explain the result for part (iii) in terms of areas.

(c) If $y^2 = x$, for $y \ge 0$, show by means of sketch graphs, and <u>not</u> by means of evaluating definite integrals, that $\int_0^1 x\,dy = 1 - \int_0^1 y\,dx$. **3**

2 UNIT TEST NUMBER 6
1996
SUGGESTED SOLUTIONS

QUESTION 1

(a) (i) $\int\left(1-x^2\right)\,dx = x-\frac{1}{3}x^3+C$ **2**

Note : $\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C.$

(ii) $\int\left(x+\frac{1}{x^2}\right)dx = \int\left(x+x^{-2}\right)dx$ **1**

Note : "$+C$" needs to be included in all indefinite integrals.

$= \frac{1}{2}x^2 - x^{-1}+C$ **1**

or $\frac{1}{2}x^2-\frac{1}{x}+C$

Total = 2

(iii) $\int\left(x^{\frac{3}{2}}+x^{-\frac{1}{3}}\right)dx = \frac{2}{5}x^{\frac{5}{2}}+\frac{3}{2}x^{\frac{2}{3}}+C$ **2**

(b) $\int_0^4 \sqrt{2x+1}\,dx = \int_0^4 (2x+1)^{\frac{1}{2}}dx$ **1**

Note : $\int (ax+b)^n dx = \frac{1}{n+1}\times\frac{1}{a}(ax+b)^{n+1}+C.$

$= \left[\frac{2}{3}(2x+1)^{\frac{3}{2}}\times\frac{1}{2}\right]_0^4$

$= \left[\frac{1}{3}(2x+1)^{\frac{3}{2}}\right]_0^4$ **1**

$= \left[\frac{1}{3}(2\times4+1)^{\frac{3}{2}}\right]-\left[\frac{1}{3}(2\times0+1)^{\frac{3}{2}}\right]$ **1**

$= \left[\frac{1}{3}\times9^{\frac{3}{2}}\right]-\left[\frac{1}{3}\times1^{\frac{3}{2}}\right]$

Note : $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$

$\therefore 9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 3^3 = 27.$

$= 9-\frac{1}{3}$

$= 8\frac{2}{3}$ **1**

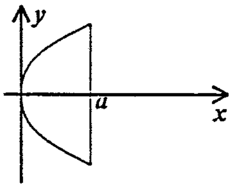
Total = 4

QUESTION 2

(a) $V = \pi \int_0^a y^2 \, dx$ **1**

$= \pi \int_0^a 4ax \, dx$

$= \pi \left[2ax^2 \right]_0^a$ **1**



$= \pi \left[2a \times a^2 - 0 \right]$

Volume $= 2\pi a^3$ units^3. **1** **Total = 3**

(b) (i) $y = 16 - x^2, \quad y = 6x$

$16 - x^2 = 6x$ **1**

$x^2 + 6x - 16 = 0$

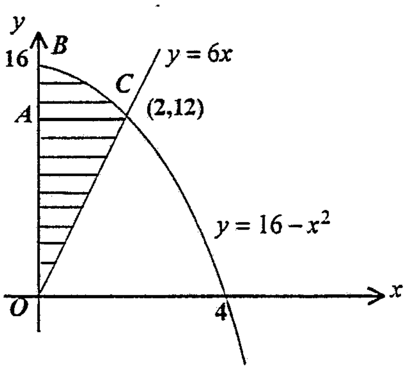
$(x + 8)(x - 2) = 0$

$x = -8, \quad x = 2$ **1**

Note : Substitute each value of x into one of the equations to find the y-coordinates of each point.

Points of intersection $(-8, -48), \quad (2, 12)$ **1** **Total = 3**

(ii)



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(iii) When region OAC is rotated about y-axis,

$V = \pi \int_0^{12} x^2 \, dy$

Note : The 'dy' means we must integrate a function of 'y'. Hence from $y = 6x$, we get $x = \frac{y}{6}$ and $x^2 = \frac{y^2}{36}$.

$= \pi \int_0^{12} \frac{y^2}{36} \, dy$ **1**

$= \pi \left[\frac{y^3}{108} \right]_0^{12}$

$= \pi \left[\frac{12^3}{108} - 0 \right]$

$= \pi [16 - 0]$

$= 16\pi$ **1**

When region ABC is rotated about y-axis,

$$V = \pi \int_{12}^{16} x^2\,dy$$

Note : $y = 16 - x^2$
$x^2 = 16 - y$

$$= \pi \int_{12}^{16} (16 - y)\,dy \quad 1$$

$$= \pi \left[16y - \frac{1}{2}y^2 \right]_{12}^{16}$$

$$= \pi \left[\left(16 \times 16 - \frac{1}{2} \times 16^2\right) - \left(16 \times 12 - \frac{1}{2} \times 12^2\right) \right]$$

$$= \pi[(256 - 128) - (192 - 72)]$$

$$= 8\pi \quad 1$$

Total Volume $= (16\pi + 8\pi)$ unit^3

Volume $= 24\pi$ unit^3

Volume $= 75$ unit^3 (nearest cubic unit). **1** **Total = 5**

QUESTION 3

(a) Simpson's Rule:

$$\int_a^b f(x)dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4] \quad 1$$

$$\int_0^2 f(x)dx \approx \frac{0.5}{3}[1 + 4 \times 1.2 + 2 \times 1.6 + 4 \times 2.2 + 3] \quad 1$$

$$\approx 3.47 \text{ (to 2 decimal places)} \quad 1$$

Note : We do not put any units after the answer. A definite integral represents a number. It is only when we find area, volume, etc. using definite integrals, that we place the appropriate units in the answer.

Total = 3

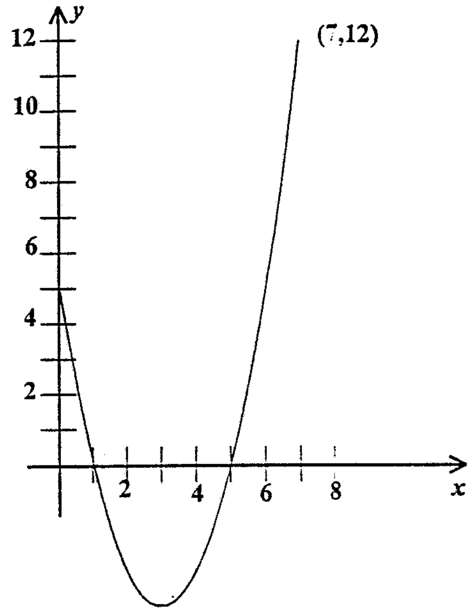
(b) (i) $y = x^2 - 6x + 5$

$y = (x-1)(x-5)$

Cuts x-axis at $x = 1,\ x = 5$ **1**

When $x = 0,\ y = 5$

When $x = 7,\ y = 12.$ **1**



1 **Total = 3**

(ii) $A = \left|\int_1^5 (x^2 - 6x + 5)\,dx\right|$ **1**

$= \left|\left[\frac{1}{3}x^3 - 3x^2 + 5x\right]_1^5\right|$

$= \left|\left(41\frac{2}{3} - 75 + 25\right) - \left(\frac{1}{3} - 3 + 5\right)\right|$ **1**

$= \left|-8\frac{1}{3} - 2\frac{1}{3}\right|$

Note : The definite integral is a negative number because $y < 0$ in this domain.

$= \left|-10\frac{2}{3}\right|$

Area is $10\frac{2}{3}$ unit^2. **1** **Total = 3**

(iii) $\int_1^7 \left(x^2 - 6x + 5\right)\, dx$

$= \left[\frac{1}{3}x^3 - 3x^2 + 5x\right]_1^7$ 1

$= \left(114\frac{1}{3} - 147 + 35\right) - \left(\frac{1}{3} - 3 + 5\right)$

$= 2\frac{1}{3} - 2\frac{1}{3}$ 1

$= 0$ **Total = 2**

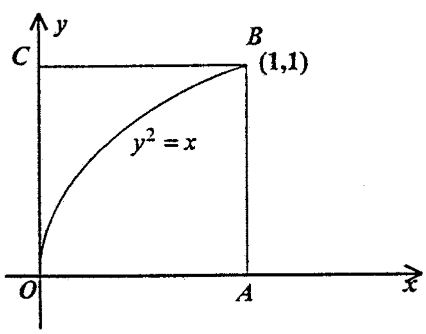
(iv) $\int_1^5 \left(x^2 - 6x + 5\right)\, dx$ is a negative number because the values of $\left(x^2 - 6x + 5\right)$ are negative in the domain $1 \le x \le 5$.

$\int_5^7 \left(x^2 - 6x + 5\right)\, dx$ is a positive number because the values of $\left(x^2 - 6x + 5\right)$ are positive in the domain $5 \le x \le 7$. 1

Since $\int_1^7 \left(x^2 - 6x + 5\right)\, dx = 0$, the area between the curve and the x-axis for $1 \le x \le 5$ is the same as the area between the curve and the x-axis for $5 \le x \le 7$. 1

Total = 2

(c)



1

$\int_0^1 x\, dy$ represents the area of OBC.

Area OBC = area $OABC$ - area OAB 1

$= 1 \times 1 - \int_0^1 y\, dx$ 1

$= 1 - \int_0^1 y\, dx$ **Total = 3**