Neap.

HSC Trial Examination 2000

Mathematics

4 Unit (Additional)

Time Allowed: Three hours
(Plus 5 minutes' reading time)

This paper must be kept under strict security and may only be used on or after the morning of Tuesday 1 August, 2000, as specified in the NEAP Examination Timetable.

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 11.
- Board-approved calculators may be used.
- Each question is to be returned in a separate Writing Booklet clearly labelled, showing your Student Name or Number.
- You may ask for extra Writing Booklets if you need them.

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2000 Mathematics 4 Unit (Additional) Higher School Certificate Examination.

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QUESTION 1. Use a SEPARATE writing booklet.

Marks

(a) (i) Show that $\dfrac{1}{1+e^x} = \dfrac{e^{-x}}{1+e^{-x}}$. **2**

(ii) Hence find $\displaystyle\int \frac{dx}{1+e^x}$.

(b) Find $\displaystyle\int \sin^4 x \cos^3 x \, dx$. **3**

(c) (i) Find the constants A, B, C and D such that **4**

$$\frac{x^3+x^2+x+2}{x^4+3x^2+2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}, \text{ for all } x.$$

(ii) Hence find $\displaystyle\int \frac{x^3+x^2+x+2}{x^4+3x^2+2} \, dx$.

(d) (i) Find $\displaystyle\int x \sin x \, dx$. **3**

(ii) Hence find $\displaystyle\int \sin\sqrt{x} \, dx$.

(e) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+4\cos\theta} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$. **3**

QUESTION 2. Use a SEPARATE writing booklet. **Marks**

(a) Let $z = -1 + i\sqrt{3}$. **5**

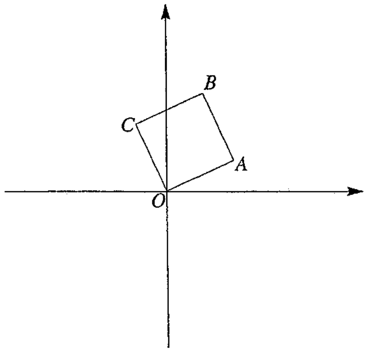
(i) Write z in modulus-argument form.

(ii) Hence or otherwise, find in real-imaginary form:

(α) z^5,

(β) $z\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

(b) **3**



The point A in the Argand diagram sketched above represents the complex number $z = a + ib$, in the first quadrant. The point B represents the complex number $4 + 7i$.

(i) If $OABC$ is a square, find in terms of a and b the complex number represented by the point C.

(ii) Hence or otherwise evaluate a and b.

(c) (i) Find the Cartesian equation and sketch the locus of z if $|z - i| = \text{Im}(z)$. **4**

(ii) What is the least value of $\arg(z)$ in (i) above?

(d) (i) Solve $z^3 - 1 = 0$, leaving your answers in modulus-argument form. **3**

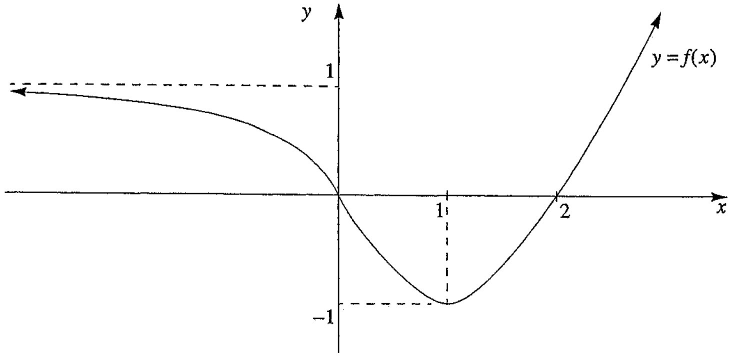
(ii) Let ω be one of the non-real roots of $z^3 - 1 = 0$.

(α) Show that $1 + \omega + \omega^2 = 0$.

(β) Hence simplify $(1 + \omega)^8$.

QUESTION 3. Use a SEPARATE writing booklet. **Marks**

(a) **10**



Given the function $y = f(x)$ in the diagram above, sketch on separate diagrams, showing all intercepts, turning points and asymptotes:

(i) $y = f(-x)$,

(ii) $y = f(|x|)$,

(iii) $|y| = f(x)$,

(iv) $y = f(2x)$,

(v) $y \times f(x) = 1$,

(vi) $y = e^{f(x)}$.

(b) (i) Graph the function $y = x^3(x-2)$. You need not use calculus, but you must show the behaviour near any x-intercepts. **5**

(ii) Differentiate $y^2 = x^3(x-2)$ implicitly, and hence show that for $y > 0$,

$$\frac{dy}{dx} = (2x-3)\sqrt{\frac{x}{x-2}}.$$

(iii) Sketch $y^2 = x^3(x-2)$, paying particular attention to the behaviour of the curve near its x-intercepts. (You do not need to find the coordinates of any inflections.)

QUESTION 4. Use a SEPARATE writing booklet. **Marks**

(a) Let α, β and γ be the roots of $x^3 - x^2 + 2x - 1 = 0$. **4**

(i) Show that $\alpha + \beta = 1 - \gamma$, and hence find an equation with roots $-(\alpha + \beta)$, $-(\beta + \gamma)$ and $-(\gamma + \alpha)$.

(ii) (α) Find an equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

(β) Hence, or otherwise, evaluate $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

(b) (i) By considering $z^9 - 1$ as a difference of two cubes, or otherwise, write **5**

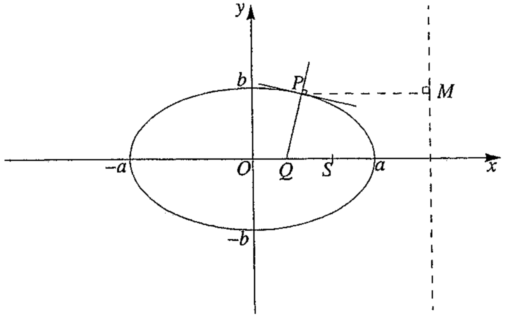
$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8$$

as a product of two polynomials with real coefficients, one of which is a quadratic.

(ii) Solve $z^9 - 1 = 0$ and hence write down the six solutions of $z^6 + z^3 + 1 = 0$.

(iii) Hence deduce that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$.

(c) **6**



Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, drawn above, with eccentricity e.

(i) Write down in terms of a and e the coordinates of the focus S, and the equation of the associated directrix.

(ii) Show that the equation of the normal to the ellipse at the point $P(x_1, y_1)$ is given by

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

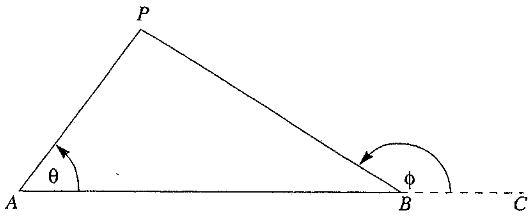
(iii) Let Q be the x-intercept of the normal and let M be the foot of the perpendicular from P to the directrix as shown in the diagram. Show that $QS = e^2PM$.

QUESTION 5. Use a SEPARATE writing booklet. **Marks**

(a) (i) By considering $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$, where $x > 0$, prove that the sum of a positive real number and its reciprocal is never less than 2, and is only equal to 2 when $x = 1$. **5**

(ii) Hence or otherwise find the smallest value of $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right)$ where a and b are positive real numbers.
For what values of a and b does this minimum value occur?

(b) The diagram shows two fixed points A and B in a plane, and a variable point P. **3**



Let $\angle PAB = \theta$ and $\angle PBC = \phi$ as shown, where AB has been produced to C.
P moves such that its angular velocity about A is equal to its angular velocity about B.
Show that P describes an arc of a circle passing through A and B.

(c) Consider the hyperbola $xy = c^2$ and the distinct points $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ on it. **7**

(i) Show that the equation of the tangent at $\left(ct, \frac{c}{t}\right)$, where $t \neq 0$, is $x + t^2y = 2ct$.

(ii) Show that the tangents at P and Q intersect at $M\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

(iii) Show that if $t_1t_2 = k$, where k is a non-zero constant, then the locus of M is a line passing through the origin.

QUESTION 6. Use a SEPARATE writing booklet.

Marks

(a) (i) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$. 5

(ii) Hence show that $2\sin 2\theta\cos 3\theta+\sin\theta=\sin 5\theta$.

(iii) Hence prove that $4\cos\theta\cos 3\theta+1=\dfrac{\sin 5\theta}{\sin\theta}$ (provided $\sin\theta\neq 0$).

(iv) Hence write down the general solution of $\cos\theta\cos 3\theta=-\dfrac{1}{2}$.

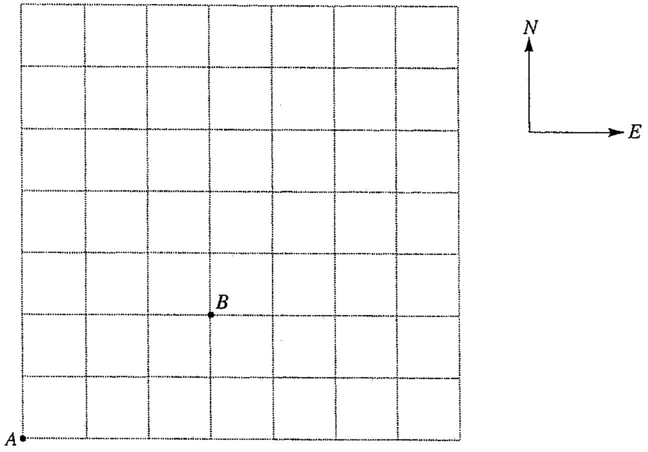
(b) The letters of the name JESSICA are arranged in a line. 4

(i) How many such arrangements are possible?

(ii) What is the probability that the *S*s are at the ends?

(iii) What is the probability that at least two other letters separate the *S*s?

(c) 3



The grid drawn above represents the street map of the town of Paen. Rahul walks from his home, *A*, to school, *B*, which is 3 blocks east and 2 blocks north of his home.

(i) In how many ways can Rahul walk to school if he always walks north or east?

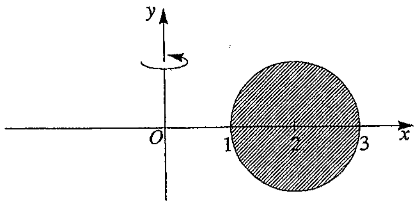
(ii) If a new school is built *p* blocks east and *q* blocks north of his home, where *p* and *q* are positive integers, how many ways can Rahul walk to the new school if he always walks north or east?

(d) Solve $\cos^{-1}(\sqrt{1-x^2})=\sin^{-1}(2x-1)$. 3

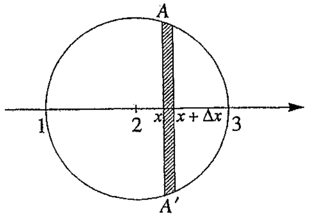
QUESTION 7. Use a SEPARATE writing booklet.

Marks

(a) 5



The circular region $(x-2)^2+y^2\leq 1$ is rotated about the *y*-axis. (The resulting doughnut-shaped solid is called a *torus*.)



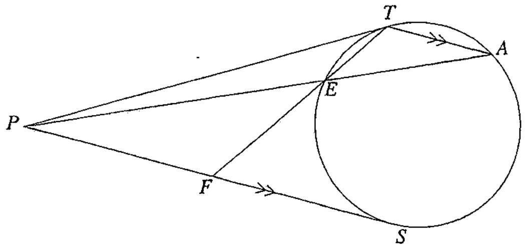
(i) Show that $AA'\doteqdot 2\sqrt{1-(x-2)^2}$.

(ii) Show that the volume ΔV obtained when a typical strip of height AA' and thickness Δx is rotated about the *y*-axis is given by

$$\Delta V\doteqdot 4\pi x\sqrt{1-(x-2)^2}\,\Delta x$$

(iii) Find the total volume of the solid generated.

(b) 5



The diagram shows two tangents *PT* and *PS* drawn to a circle from a point *P* outside the circle. Through *T*, a chord *TA* is drawn parallel to the other tangent *PS*. The secant *PA* meets the circle at *E*, and *TE* produced meets *PS* at *F*.

(i) Prove that $\Delta EFP\ |||\ \Delta PFT$.

(ii) Hence show that $PF^2=TF\times EF$.

(iii) Hence or otherwise prove that *F* is the midpoint of *PS*.

Question 7 continues on page 9.

QUESTION 7. (Cont.)

Marks

(c) A mass of 35 kilograms is dropped from a balloon descending vertically at 30 m s^{-1}. 5

Take the force of air resistance as having magnitude $70v$ newtons, where v m s^{-1} is the velocity of the mass. Take g, the acceleration due to gravity, as 10 m s^{-2}.

(i) Show that the velocity of the mass t seconds after being dropped (prior to hitting the ground) is given by

$$v = 5 + 25e^{-2t}.$$

(ii) Describe what happens to the velocity of the mass as $t \to \infty$.

(iii) If the mass was dropped at 400 metres above the ground, how close to the ground will it be after 1 minute?

QUESTION 8. Use a SEPARATE writing booklet.

Marks

(a) (i) Show that $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\frac{2}{n^2}$, where n is a positive integer. 5

(ii) Hence or otherwise show that for $n \geq 1$

$$\sum_{r=1}^{n} \tan^{-1}\frac{2}{r^2} = \frac{3\pi}{4} + \tan^{-1}\frac{2n+1}{1-n-n^2}.$$

(iii) Hence write down $\sum_{r=1}^{\infty} \tan^{-1}\frac{2}{r^2}$.

(b) (i) Use the principle of mathematical induction to prove that $3^n > n^3$ for all integers $n \geq 4$. 5

(ii) Hence or otherwise show that $\sqrt[3]{3} > \sqrt[n]{n}$ for all integers $n \geq 4$.

(c) (i) Explain why $S = (\sqrt{50}+7)^{2000} + (\sqrt{50}-7)^{2000}$ is an even integer. 5

(ii) Without using a calculator, show that $0 < \sqrt{50} - 7 < 0.1$.

(iii) Hence or otherwise prove that the first two thousand digits after the decimal point in the decimal expansion of $(\sqrt{50}+7)^{2000}$ are all 9s.

End of paper

Standard integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq 1;\ x \neq 0, \ \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}).$$

Note: $\ln x = \log_e x, \quad x > 0$

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HSC Trial Examination 2000

Mathematics

4 Unit (Additional)

Solutions and suggested marking scheme

QUESTION 1.

(a) (i) $\frac{1}{1+e^{x}} \times \frac{e^{-x}}{e^{-x}} = \frac{e^{-x}}{1+e^{-x}}$ ✓

(ii)
$$\begin{aligned} \int \frac{dx}{1+e^{x}} &= \int \frac{e^{-x}\,dx}{1+e^{-x}} \\ &= -\int \frac{-e^{-x}\,dx}{1+e^{-x}} \\ &= -\ln(1+e^{-x}) + C \quad ✓ \end{aligned}$$

(b)
$$\begin{aligned} \int \sin^4 x \cos^3 x\,dx &= \int \sin^4 x \cos^2 x \cos x\,dx \\ &= \int \sin^4 x(1-\sin^2 x)\cos x\,dx \\ &= \int (\sin^4 x - \sin^6 x)\cos x\,dx \quad ✓ \end{aligned}$$

Let $u = \sin x$. Then $du = \cos x\,dx$.

Hence
$$\begin{aligned} \int \sin^4 x \cos^3 x\,dx &= \int (u^4 - u^6)du \quad ✓ \\ &= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C \\ &= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C \quad ✓ \end{aligned}$$

(c) (i) $\frac{x^3+x^2+x+2}{x^4+3x^2+2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$

$$\begin{aligned} x^3+x^2+x+2 &= (Ax+B)(x^2+2) + (x^2+1)(Cx+D) \\ &= Ax^3 + 2Ax + Bx^2 + 2B + Cx^3 + Dx^2 + Cx + D \\ &= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D) \end{aligned}$$

Equating coefficients:

$A + C = 1$ —(1)

$B + D = 1$ —(2)

$2A + C = 1$ —(3)

$2B + D = 2$ —(4) ✓

Solving (4) and (2) simultaneously, and (1) and (3) simultaneously,

$A = 0,\ B = 1,\ C = 1,\ D = 0.$ ✓

(ii) $$\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx = \int \frac{1}{x^2+1}dx + \int \frac{x}{x^2+2}dx$$

$$= \int \frac{dx}{x^2+1} + \frac{1}{2}\int \frac{2x\,dx}{x^2+2} \quad ✓$$

$$= \tan^{-1}x + \frac{1}{2}\ln(x^2+2) + C \quad ✓$$

(d) (i) $\int x\sin x\,dx$

Let $u = x \quad v' = \sin x$

$\therefore\ u' = 1 \quad v = -\cos x$

$\left[\int u'v\,dx = uv - \int uv'\,dx\right]$

$$\therefore \int x\sin x\,dx = -x\cos x - \int 1\times(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C \quad ✓$$

(ii) $\int \sin(\sqrt{x})dx$

Let $u = \sqrt{x}$. Then $u^2 = x$, so $2u\,du = dx$

Hence $\int \sin\sqrt{x}\,dx = \int 2u\sin u\,du$ ✓

$$= 2(-u\cos u + \sin u) + C \quad \text{from (i)}$$

$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C \quad ✓$$

(e) $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+4\cos\theta}$$

Let $t = \tan\frac{\theta}{2}$.

Then $dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$

$$= \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta$$

$$= \frac{1}{2}(1+t^2)d\theta$$

so $d\theta = \frac{2dt}{1+t^2}$.

$\cos\theta = \frac{1-t^2}{1+t^2}$

When $\theta = 0$, $t = 0$ and when $\theta = \frac{\pi}{2}$, $t = 1$. ✓

Hence $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+4\cos\theta} = \int_0^1 \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$

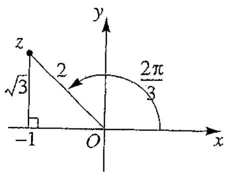
$$= \int_0^1 \frac{2dt}{5(1+t^2)+4(1-t^2)}$$

$$= \int_0^1 \frac{2dt}{9+t^2} \quad ✓$$

$$= \left[\frac{2}{3}\tan^{-1}\frac{t}{3}\right]_0^1$$

$$= \frac{2}{3}\tan^{-1}\frac{1}{3} \quad ✓$$

QUESTION 2.



(a) (i) $z=-1+i\sqrt{3}$

$z=2\left(\text{cis}\frac{2\pi}{3}\right)$ ✓✓

(ii) $z^5=\left(2\text{cis}\frac{2\pi}{3}\right)^5$

$=32\text{cis}\frac{10\pi}{3}$ (de Moivre's theorem)

$=32\text{cis}\left(-\frac{2\pi}{3}\right)$

$=32\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$

$=-16-16\sqrt{3}i$ ✓

(iii) $z\text{cis}\frac{\pi}{6}=2\text{cis}\frac{2\pi}{3}\times\text{cis}\frac{\pi}{6}$

$=2\text{cis}\frac{5\pi}{6}$ ✓

$=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$=2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

$=-\sqrt{3}+i$ ✓

(b) (i) The point C represents the complex number $i(a+ib)=-b+ia$ ✓

(Multiplication by i represents a rotation through $\frac{\pi}{2}$ in an anticlockwise direction.)

OR: $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}$

$=\overrightarrow{OB}-\overrightarrow{OA}$

$=(4+7i)-(a+bi)$

$=(4-a)+(7-b)i$

(ii) $\overrightarrow{OB}=\overrightarrow{OC}+\overrightarrow{OA}$

$\therefore 4+7i=-b+ia+a+ib$ ✓

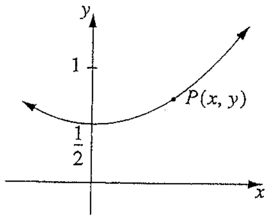
$=a-b+(a+b)i$

$\therefore a-b=4$ —(1)

$a+b=7$ —(2)

$a=5\frac{1}{2}$

$b=1\frac{1}{2}$ ✓



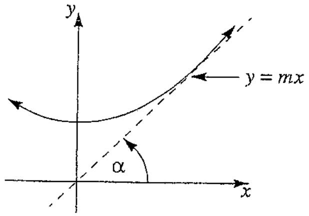
(c) (i) $|z-i|=\text{Im}(z)$

$\sqrt{x^2+(y-1)^2}=y$ $y>0$ ✓

$\therefore x^2+(y-1)^2=y^2$

$x^2+y^2-2y+1=y^2$

$y=\frac{x^2+1}{2}$ ✓



(ii) Solve simultaneously $y=\frac{x^2+1}{2}$ and $y=mx$.

$mx=\frac{x^2+1}{2}$ ✓

$x^2-2mx+1=0$

For the line to be a tangent,

$\Delta=0$

$4m^2-4=0$

$m=1$ or -1

So from the diagram, $\arg z$ lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

So the least value of $\arg z$ is $\frac{\pi}{4}$. ✓

(d) (i) $z^3-1=0$

Let $z=\text{cis}\theta$. (clearly $|z|=1$)

$z^3=1$

$\text{cis}3\theta=1$

$3\theta=0$ or 2π or 4π

Hence $z=1$, $\text{cis}\frac{2\pi}{3}$ or $\text{cis}\frac{4\pi}{3}$ $\left(\text{OR cis}-\frac{2\pi}{3}\right)$. ✓

(ii) (α) The roots of $z^3 - 1 = 0$ are 1, ω, and ω^2.

$$\text{Sum of roots} = -\frac{\text{coefficient of } z^2}{\text{coefficient of } z^3}$$

$$1 + \omega + \omega^2 = 0 \quad ✓$$

$$\text{OR } 1 + \omega + \omega^2 = 1 + \text{cis}\frac{2\pi}{3} + \text{cis}\frac{-2\pi}{3}$$

$$= 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

$$= 1 + 2\cos\frac{2\pi}{3}$$

$$= 1 + 2 \times \left(-\frac{1}{2}\right)$$

$$= 1 - 1$$

$$= 0$$

(β) $(1 + \omega)^8 = (-\omega^2)^8$

$$= \omega^{16}$$

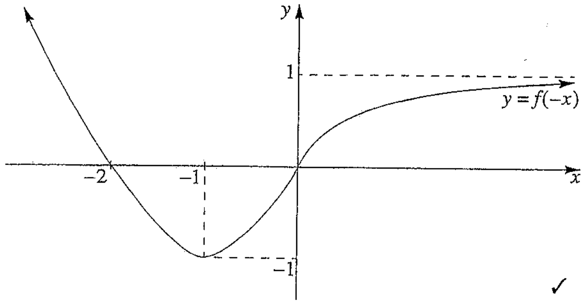
$$= (\omega^3)^5 \times \omega$$

$$= 1^5 \times \omega$$

$$= \omega \quad ✓$$

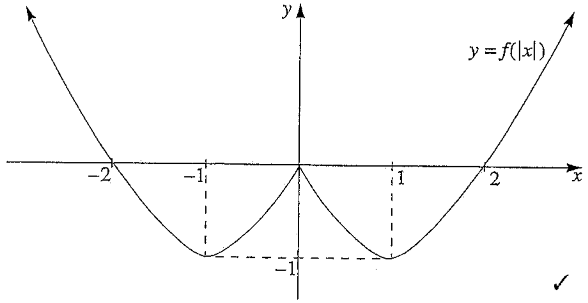
QUESTION 3.

(a) (i)



✓

(ii)



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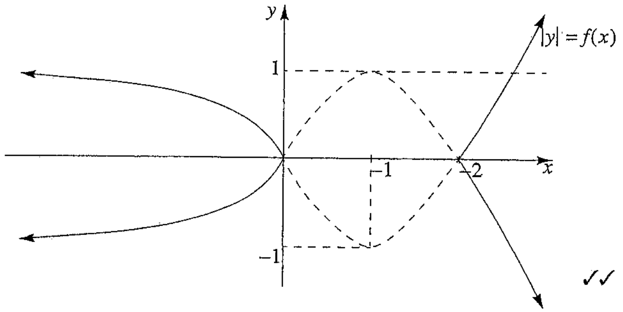
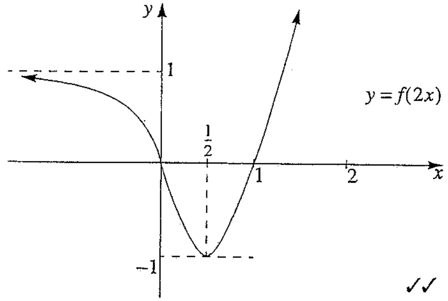
(iii)



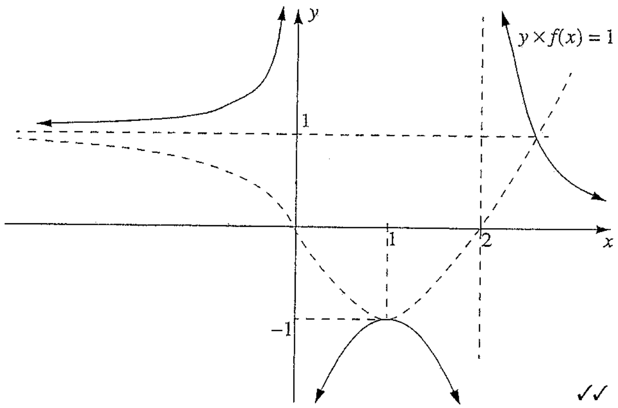
✓✓

(iv)



✓✓

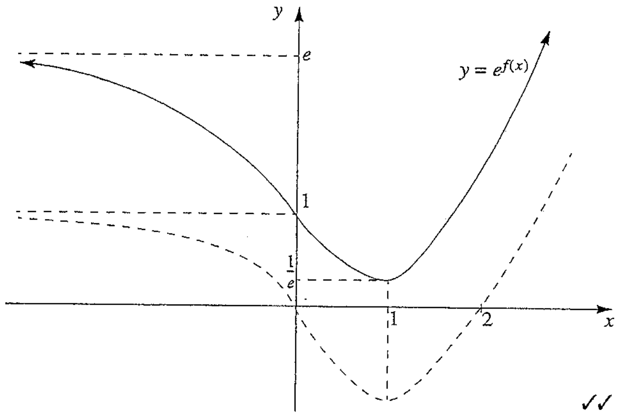
(v)



✓✓

Horizontal asymptotes at $y = 0$, $y = 1$, vertical asyptotes at $x = 0$, $x = 2$.

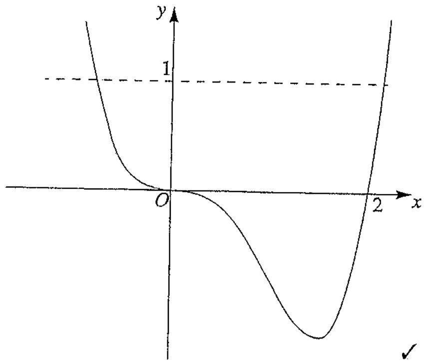
(vi)



✓✓

(b) (i) $y = x^3(x-2)$

$x = 0$ is a triple root, i.e. a horizontal point of inflexion.



✓

(ii)

$$y^2 = x^3(x-2)$$

$$y^2 = x^4 - 2x^3$$

$$2y\frac{dy}{dx} = 4x^3 - 6x^2$$

$$y\frac{dy}{dx} = 2x^3 - 3x^2$$

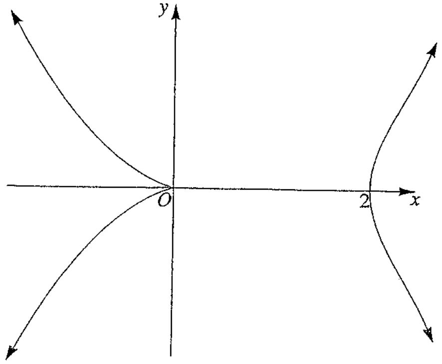
$$\frac{dy}{dx} = \frac{x^2(2x-3)}{y} \quad ✓$$

$$= \frac{x^2(2x-3)}{\sqrt{x^3(x-2)}} \quad \text{(Note: } y > 0\text{)}$$

$$= \frac{\sqrt{x}(2x-3)}{\sqrt{x-2}}$$

$$= (2x-3)\sqrt{\frac{x}{x-2}} \quad ✓$$

(iii)



✓✓

(one mark for each branch of the graph)

QUESTION 4.

(a) (i) $x^3 - x^2 + 2x - 1 = 0$ —(1)

(α) $\alpha + \beta + \gamma = 1$ $\left(\text{Sum of roots} = -\dfrac{\text{coefficient of } z^2}{\text{coefficient of } z^3}\right)$

$\alpha + \beta = 1 - \gamma$

Hence $-(\alpha + \beta) = \gamma - 1$

and $-(\alpha + \gamma) = \beta - 1$

and $-(\beta + \gamma) = \alpha - 1$ ✓

Let $y = x - 1$.

So $x = y + 1$.

Hence $(y+1)^3 - (y+1)^2 + 2(y+1) - 1 = 0$

$y^3 + 3y^2 + 3y + 1 - y^2 - 2y - 1 + 2y + 2 - 1 = 0$

$y^3 + 2y^2 + 3y + 1 = 0$ ✓

(β) To find equation with roots $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ given that equation (1) has roots α, β, γ:

Let $y = \dfrac{1}{x}$. Then $x = \dfrac{1}{y}$ in equation (1).

$\left(\dfrac{1}{y}\right)^3 - \left(\dfrac{1}{y}\right)^2 + 2 \times \dfrac{1}{y} - 1 = 0$

$1 - y + 2y^2 - y^3 = 0$ —(2) ✓

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = -\dfrac{2}{-1}$ from equation (2)

$= 2$ ✓

(b) (i) $(z^3)^3 - 1^3 = (z^3 - 1)(z^6 + z^3 + 1)$

$= (z-1)(z^2 + z + 1)(z^6 + z^3 + 1)$

Since $z^9 - 1 = (z-1)(z^8 + z^7 + \ldots + z + 1)$,

$z^8 + z^7 + \ldots + z + 1 = (z^2 + z + 1)(z^6 + z^3 + 1)$ ✓

(ii) $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$

Let $z^9 = 1$. Let $z = \text{cis}\,\theta$ (where $|z| = 1$)

$\text{cis}\,9\theta = 1$

$9\theta = 2k\pi$, $k \in J$.

$\theta = \dfrac{2k\pi}{9}$, $k \in J$.

$\therefore\ \theta = 0, \pm\dfrac{2\pi}{9}, \pm\dfrac{4\pi}{9}, \pm\dfrac{6\pi}{9}, \pm\dfrac{8\pi}{9}$ ✓

$\theta = 0, \pm\dfrac{6\pi}{9}$ are roots of $z^3 - 1 = 0$. —(1)

$\therefore$ roots of $z^6 + z^3 + 1 = 0$ are $\theta = \pm\dfrac{2\pi}{9}, \pm\dfrac{4\pi}{9}, \pm\dfrac{8\pi}{9}$ ✓

(iii) $\text{cis}\,0 + \text{cis}\dfrac{2\pi}{9} + \text{cis}\dfrac{4\pi}{9} + \text{cis}\dfrac{6\pi}{9} + \text{cis}\dfrac{8\pi}{9} + \text{cis}\left(-\dfrac{2\pi}{9}\right) + \text{cis}\left(-\dfrac{4\pi}{9}\right) + \text{cis}\left(-\dfrac{6\pi}{9}\right) + \text{cis}\left(-\dfrac{8\pi}{9}\right)$ is the sum of the roots of $z^9 - 1 = 0$ which equals zero, since the coefficient of z^8 is zero.

$\text{cis}\,0 + \text{cis}\left(\dfrac{6\pi}{9}\right) + \text{cis}\left(-\dfrac{6\pi}{9}\right) = 0$ (from (1)) ✓ (sum of roots of $z^3 - 1 = 0$)

$$\text{Hence } \text{cis}\frac{2\pi}{9} + \text{cis}\frac{4\pi}{9} + \text{cis}\frac{8\pi}{9} + \text{cis}\left(-\frac{2\pi}{9}\right) + \text{cis}\left(-\frac{4\pi}{9}\right) + \text{cis}\left(-\frac{8\pi}{9}\right) = 2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} + 2\cos\frac{8\pi}{9}$$

$$= 2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}\right)$$

$$= 0$$

so $\cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9} + \cos\dfrac{8\pi}{9} = 0$, i.e. $\cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9} = \cos\dfrac{\pi}{9}$ (since $\cos(\pi - x) = -\cos x$) ✓

(c) (i) The focus is $S(ae, 0)$. The directrix is $x = \dfrac{a}{e}$. ✓

(ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-b^2 x}{a^2 y}$

So the slope of the normal at $P(x_1, y_1)$ is $\dfrac{a^2 y_1}{b^2 x_1}$ ✓

The equation of the normal is $y - y_1 = \dfrac{a^2 y_1}{b^2 x_1}(x - x_1)$

$b^2 x_1 (y - y_1) = a^2 y_1 (x - x_1)$ ✓

$a^2 x y_1 - b^2 x_1 y = a^2 x_1 y_1 - b^2 x_1 y_1$

$\dfrac{a^2 x}{x_1} - \dfrac{b^2 y}{y_1} = a^2 - b^2$ ✓ (divide both sides by $x_1 y_1$)

(iii) Q is the point $\left(\dfrac{a^2 - b^2}{a^2}x_1, 0\right)$ or $(e^2 x_1, 0)$ (from $a^2(1 - e^2) = b^2$).

so, $QS = |e^2 x_1 - ae|$

$= e|ex_1 - a|$ ✓

Also, $PM = \dfrac{a}{e} - x_1$

so $e^2 PM = e^2\left(\dfrac{a}{e} - x_1\right)$

$= e(a - ex_1)$ ✓

Hence $QS = e^2 PM$. ✓

QUESTION 5.

(a) (i) $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 = x+\frac{1}{x}-2$ ✓

≥ 0, since a square can never be negative.

So $x+\frac{1}{x} \geq 2$ ✓

Hence the sum of a positive number and its reciprocal is never less than 2.

If $x = 1$ then $x+\frac{1}{x}=2$ ✓

(ii) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{a}{b}+\frac{b}{a}+2$

Let $A=\frac{a}{b}$. Then $\frac{1}{A}=\frac{b}{a}$, so by part (i), $A+\frac{1}{A}\geq 2$.

Hence $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$ ✓

The minimum value of 4 occurs when $a=b$. ✓

(b) Let $\angle APB = \alpha$.

Then $\therefore\ \alpha = \phi - \theta$ (exterior angle of ΔAPB).

But $\frac{d\theta}{dt}=\frac{d\phi}{dt}$ (given). ✓

So $\frac{d\alpha}{dt}=0$, and hence α is constant.

Hence by the converse of the angle in a semicircle theorem,
the locus of P is an arc of a circle through AB. ✓

(c) (i) $xy=c^2$

$$y=\frac{c^2}{x}$$

$$y'=-\frac{c^2}{x^2}$$

When $x=ct$, $y'=-\frac{1}{t^2}$ ✓

So the tangent is $y-\frac{c}{t}=-\frac{1}{t^2}(x-ct)$

$$yt^2-ct=-x+ct$$

$$x+yt^2=2ct \quad ✓$$

(ii) Solve simultaneously $x+yt_1^2=2ct_1$ –(1)

$$x+yt_2^2=2ct_2 \text{ –(2)} \quad ✓$$

Substituting, $y(t_1^2-t_2^2)=2c(t_1-t_2)$

$$y=\frac{2c(t_1-t_2)}{t_1^2-t_2^2}=\frac{2c}{t_1+t_2} \quad ✓$$

Multiplying (1) by t_2^2 and (2) by t_1^2,

$$xt_2^2+yt_1^2t_2^2=2ct_1t_2^2$$

$$xt_1^2+yt_1^2t_2^2=2ct_2t_1^2$$

Substituting, $x(t_2^2-t_1^2)=2ct_1t_2(t_2-t_1)$

$$x=\frac{2ct_1t_2}{t_2+t_1}$$

Hence they intersect at $M\left(\frac{2ct_1t_2}{t_2+t_1},\frac{2c}{t_1+t_2}\right)$ ✓

(iii) The locus of M is $x=\frac{2ct_1t_2}{t_1+t_2}$, $y=\frac{2c}{t_1+t_2}$. ✓

Then $t_1+t_2=\frac{2ck}{x}$ and $t_1+t_2=\frac{2c}{y}$

i.e. $yk=x$

so the locus of M is $y=\frac{1}{k}x$ $(k\neq 0)$, a line passing through the origin. ✓

QUESTION 6.

(a) (i) L.H.S. $= \sin(A+B) + \sin(A-B)$

$= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$

$= 2\sin A\cos B$

$=$ R.H.S. ✓

(ii) L.H.S. $= 2\sin 2\theta\cos 3\theta + \sin\theta$

$= \sin(2\theta + 3\theta) + \sin(2\theta - 3\theta) + \sin\theta$

$= \sin 5\theta + \sin(-\theta) + \sin\theta$

$= \sin 5\theta - \sin\theta + \sin\theta$ $\quad (\sin(-\theta) = -\sin(\theta),$ as $\sin\theta$ is an odd function$)$

$= \sin 5\theta$

$=$ R.H.S. ✓

(iii) R.H.S. $= \dfrac{\sin 5\theta}{\sin\theta}$

$= \dfrac{2\sin 2\theta\cos 3\theta + \sin\theta}{\sin\theta}$

$= \dfrac{4\sin\theta\cos\theta\cos 3\theta + \sin\theta}{\sin\theta}$

$= 4\cos\theta\cos 3\theta + 1$

$=$ L.H.S. ✓

(iv) $\cos\theta\cos 3\theta = -\dfrac{1}{2}$

$\dfrac{1}{4}\left(\dfrac{\sin 5\theta}{\sin\theta} - 1\right) = -\dfrac{1}{2}$ from (iii).

$\dfrac{\sin 5\theta}{\sin\theta} = -1 \quad (\sin\theta \neq 0)$

$\sin 5\theta = -\sin\theta$

$\sin 5\theta = \sin(-\theta)$. ✓

$5\theta = n\pi + (-1)^n(-\theta)$

$5\theta = n\pi + (-1)^{n+1}\theta$

Hence if n is even, $\theta = \dfrac{n\pi}{6}$ and if n is odd, $\theta = \dfrac{n\pi}{4}$ $\quad n \in J \quad (\neq 0)$ ✓

(b) (i) Arrangements of JESSICA if placed in a line $\dfrac{7!}{2!} = 2520$. ✓

(ii) The probability that S's are on the ends is $\dfrac{5!}{2520} = \dfrac{1}{21}$. ✓

(iii) The probability that at least two other letters separate the S's is given by

Probability $= \dfrac{\text{total no. of ways} - (\text{no. with S's together} + \text{no. with one letter separating the S's})}{\text{total no. of ways}}$

$= \dfrac{\dfrac{7!}{2!} - (6! + 5 \times 5!)}{\dfrac{7!}{2!}}$

$= \dfrac{7! - 2 \times 11 \times 5!}{7!}$

$= 1 - \dfrac{22}{42}$

$= \dfrac{10}{21}$ ✓✓

(c) (i) Rachel must travel 3 blocks east and 2 north in any order, i.e. EEENN or EENEN etc.

i.e. $\dfrac{5!}{3!2!}$ $\left(\text{or } \binom{5}{3} = \binom{5}{2}\right) = 10$ ways. ✓

(ii) To reach (p, q) there are $\underbrace{EEE\ldots E}_{p \text{ times}}\ \underbrace{NNN\ldots N}_{q \text{ times}}$

i.e. $\dfrac{(p+q)!}{p!q!}$ ✓✓ $\left(\text{or } \binom{p+q}{q} = \binom{p+q}{p}\right)$

(d) $\cos^{-1}\sqrt{1-x^2} = \sin^{-1}(2x-1)$

$\sin(\cos^{-1}\sqrt{1-x^2}) = \sin(\sin^{-1}(2x-1))$ ✓(taking sine of both sides)

$x = 2x - 1$

Hence $x = 1$ ✓

Check: LHS $= \cos^{-1} 0 = \dfrac{\pi}{2}$

RHS $= \sin^{-1} 1 = \dfrac{\pi}{2}$ ✓

So $x = 1$ is the only solution.

QUESTION 7.

(a) (i) $(x-2)^2+y^2=1$

$y^2=1-(x-2)^2$

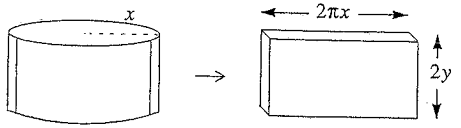
$y=\pm\sqrt{1-(x-2)^2}$ ✓

So the points are $A(x,\sqrt{1-(x-2)^2})$ and $A'(x,-\sqrt{1-(x-2)^2})$

Hence $AA'=\sqrt{1-(x-2)^2}-(-\sqrt{1-(x-2)^2})$

$=2\sqrt{1-(x-2)^2}$ ✓

(ii)



$\Delta V \doteqdot 2\pi x(2\sqrt{1-(x-2)^2})\Delta x$

$=4\pi x\sqrt{1-(x-2)^2}\ \Delta x$ ✓

(iii) $V=\int_1^3 4\pi x\sqrt{1-(x-2)^2}\ dx$ ✓

$=4\pi\int_1^3 x\sqrt{1-(x-2)^2}\ dx$

Let $u=x-2$. $\therefore du=dx$ and $x=u+2$.

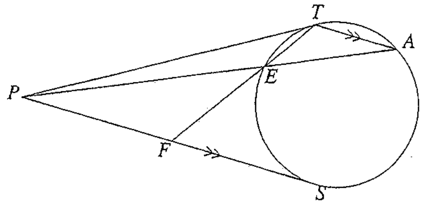
Hence $V=4\pi\int_{-1}^{1}(u+2)\sqrt{1-u^2}\ du$

$$=4\pi\left[\int_{-1}^{1}\underbrace{u\sqrt{1-u^2}}_{\text{odd}}du+\int_{-1}^{1}\underbrace{\sqrt{1-u^2}}_{\text{semicircle}}dx\right]$$

$=4\pi[0+\pi]$

$=4\pi^2$ cubic units ✓

(b) (i)



In the triangles EFP and PFT,

$\angle PFT=\angle EFP$ (common)

$\angle PTF=\angle TAE$ (alternate segment with PT tangent)

$=\angle EPF$ (alternate angles with $TA \parallel PS$) ✓

Hencè $\Delta PFT\ |||\ \Delta EFP$ (AA) ✓

(ii) Since $\Delta EFP\ |||\ \Delta PFT$ (vertices written in correct order)

then $\dfrac{PF}{FT}=\dfrac{EF}{PF}$ (sides in same ratio) ✓

Hence $PF^2=EF\times FT$

(iii) $FS^2=FT\times FE$ (property of tangents) ✓

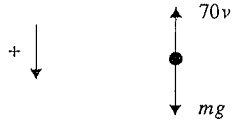
but $PF^2=EF\times FT$ (from part (ii))

$FS^2=PF^2$

Hence $FS=PF$ (taking the positive square root)

So F is the midpoint of PS. ✓

(c) (i) At $t=0, y=0, \dot{y}=v=30\text{ m s}^{-1}$.



The equation of motion is $m\ddot{y}=mg-70v$.

$\ddot{y}=10-2v$ ✓

$\dfrac{dv}{dt}=10-2v$

$\dfrac{dt}{dv}=\dfrac{1}{10-2v}$

$t=\displaystyle\int\frac{1}{10-2v}dv$

$t=-\dfrac{1}{2}\ln(10-2v)+C$

When $t=0$, $v=30$, so

$t=-\dfrac{1}{2}\ln\left(\dfrac{10-2v}{50}\right)$

$-2t=\ln\left(\dfrac{10-2v}{50}\right)$

$\dfrac{2v-10}{50}=e^{-2t}$

$2v-10=50e^{-2t}$

$2v=10+50e^{-2t}$

$v=5+25e^{-2t}$ ✓

(ii) As $t\to\infty$, $v\to 5\text{ m s}^{-1}$. ✓

(iii) $y=5t-\dfrac{25}{2}e^{-2t}+C'$

At $t=0$, $y=0$, so $y=5t-\dfrac{25}{2}e^{-2t}+\dfrac{25}{2}$ ✓

After 1 minute ($t=60$), $y=5\times 60+\dfrac{25}{2}e^{-120}+\dfrac{25}{2}$

$\doteqdot 312.5$ m

The mass will be $400-312.5=87.5$ metres above the ground. ✓

QUESTION 8.

(a) (i) Let $\tan^{-1}(n+1) = \alpha$ and $\tan^{-1}(n-1) = \beta$. Then $\tan\alpha = n+1$ and $\tan\beta = n-1$.

Now
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$= \frac{n+1-(n-1)}{1+(n+1)(n-1)} \quad ✓$$
$$= \frac{2}{1+n^2-1} = \frac{2}{n^2}$$

Hence $\alpha - \beta = \tan^{-1}\frac{2}{n^2}$, since $\alpha - \beta$ and $\tan^{-1}\frac{2}{n^2}$ are both acute.

i.e., $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\frac{2}{n^2}$ ✓

(ii)
$$\sum_{r=1}^{n} \tan^{-1}\frac{2}{r^2} = \tan^{-1}\frac{2}{1^2} + \tan^{-1}\frac{2}{2^2} + \tan^{-1}\frac{2}{3^2} + \ldots + \tan^{-1}\frac{2}{(n-2)^2} + \tan^{-1}\frac{2}{(n-1)^2} + \tan^{-1}\frac{2}{n^2}$$
$$= (\tan^{-1}2 - \tan^{-1}0) + (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}4 - \tan^{-1}2) + \ldots$$
$$+ (\tan^{-1}(n-1) - \tan^{-1}(n-3)) + (\tan^{-1}n - \tan^{-1}(n-2))$$
$$+ (\tan^{-1}(n+1) - \tan^{-1}(n-1))$$
$$= -\tan^{-1}0 - \tan^{-1}1 + \tan^{-1}(n+1) + \tan^{-1}n \quad ✓$$
$$= -0 - \frac{\pi}{4} + \pi + \tan^{-1}\left(\frac{n+1+n}{1-(n+1)n}\right)$$
$$= \frac{3\pi}{4} + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) \quad ✓$$

(iii) As $n \to \infty$, $\frac{2n+1}{1-n-n^2} \to 0$
$$\tan^{-1}\frac{2n+1}{1-n-n^2} \to 0$$
$$\text{so } \sum_{r=1}^{\infty} \tan^{-1}\frac{2}{r^2} = \frac{3\pi}{4} \quad ✓$$

(b) (i) When $n = 4$, $3^4 = 81$ and $4^3 = 64$. So $3^n > n^3$ is true for $n = 4$. ✓

Assume the statement is true for $n = k$, i.e. assume $3^k > k^3$.

We must prove that the statement is true for $n = k+1$, i.e. that $3^{k+1} > (k+1)^3$.

Consider $3^{k+1} - (k+1)^3$.
$$3^{k+1} - (k+1)^3 = 3 \times 3^k - k^3 - 3k^2 - 3k - 1$$
$$= 3(3^k - k^3) + 2k^3 - 3k^2 - 3k - 1$$
$$= 3(3^k - k^3) + (k^3 - 3k^2 + 3k - 1) + (k^3 - 6k)$$
$$= 3(3^k - k^3) + (k-1)^3 + k(k^2 - 6) \quad ✓✓$$

Now $3^k - k^3$ is positive, by assumption,

and $(k-1)^3$ is positive since $k > 3$,

and $k(k^2 - 6)$ is positive since $k > 3$.

Thus $3^{k+1} - (k+1)^3 > 0$, i.e. $3^{k+1} > (k+1)^3$. ✓

Therefore by the principle of mathematical induction, $3^n > n^3$ for all $n > 3$.

(ii)
$$3^n > n^3$$
$$(3^n)^{\frac{1}{3n}} > (n^3)^{\frac{1}{3n}} \quad \text{(taking the } 3n\text{th root of both sides)}$$
$$\text{i.e. } 3^{\frac{1}{3}} > n^{\frac{1}{n}}$$
$$\sqrt[3]{3} > \sqrt[n]{n} \quad ✓$$

(c) (i)
$$S = (\sqrt{50}+7)^{2000} + (\sqrt{50}-7)^{2000}$$
$$= (\sqrt{50})^{2000} + \binom{2000}{1}(\sqrt{50})^{1999} \times 7^1 + \binom{2000}{2}(\sqrt{50})^{1998} \times 7^2 + \ldots + 7^{2000}$$
$$+ (\sqrt{50})^{2000} - \binom{2000}{1}(\sqrt{50})^{1999} \times 7^1 + \binom{2000}{2}(\sqrt{50})^{1998} \times 7^2 - \ldots + 7^{2000}$$
$$= 2\left[(\sqrt{50})^{2000} + \binom{2000}{2}(\sqrt{50})^{1998} \times 7^2 + \binom{2000}{4}(\sqrt{50})^{1996} \times 7^4 + \ldots + 7^{2000}\right]$$
$$= 2\left[50^{1000} + \binom{2000}{2}50^{999} \times 7^2 + \binom{2000}{4}50^{998} \times 7^4 + \ldots + 7^{2000}\right]$$

So S is even. ✓✓

(ii)
$$50 > 49$$
$$\sqrt{50} > 7$$
$$\sqrt{50} - 7 > 0$$

Also, $50 < 7.1^2$
$$\sqrt{50} < 7.1$$
$$\sqrt{50} - 7 > 0.1$$

Hence $0 < \sqrt{50} - 7 < 0.1$ ✓

(iii) $(\sqrt{50}+7)^{2000} = S - (\sqrt{50}-7)^{2000}$

$= \underset{\substack{\uparrow \\ \text{even} \\ \text{whole number}}}{S} - (\underset{\substack{\uparrow \\ \text{at least two} \\ \text{thousand zeros}}}{0.000\ldots})$, because $(\sqrt{50}-7)^{2000} < (10^{-1})^{2000}$

i.e. $(\sqrt{50}-7)^{2000} < 10^{-2000}$.

Hence at least the first two thousand digits after the decimal point are 9s. ✓✓