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HSC Trial Examination 2001

Mathematics Extension 1

This paper must be kept under strict security and may only be used on or after the afternoon of Thursday 9 August, 2001, as specified in the NEAP Examination Timetable

General Instructions

Reading time 5 minutes

Working time 2 hours

Write using blue or black pen.

Board-approved calculators may be used.

A table of standard integrals is provided on page 10.

All necessary working should be shown in every question.

Examination structure

Total marks 84

Attempt all questions.

All questions are of equal value.

QUESTION 1. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of $\int_{1}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$. 3

(b) Solve $\frac{3}{x^2-1} \geq 0$. 3

(c) Use the substitution $u^2 = 4 + x$ to evaluate $\int_{0}^{5} \frac{x}{\sqrt{4+x}}dx$. 4

(d) Simplify $\frac{{}^nP_{r+1}}{{}^nP_r}$. 2

QUESTION 2. (12 marks) Use a SEPARATE writing booklet. Marks

(a) (i) Differentiate $x\tan^{-1}x$. 1

(ii) Hence evaluate $\int_0^1 \tan^{-1}x\,dx$. 3

(b) (i) Sketch the graph of $y=\cos x, -\pi \le x \le \pi$ and use this graph to show that $\cos x + x = 0$ has only one solution. 2

(ii) Use Newton's method with a first approximation of $x=-1$ to find a second approximation to the root of $\cos x + x = 0$. 2

(c) There are 12 videotapes arranged in a row on a shelf in a video shop. There are 3 identical copies of *Gone with the Wind*, 4 of *Tootsie* and 5 of *Gladiator.*

(i) How many different arrangements of the videotapes are there? 1

(ii) How many different arrangements are there if videos with the same title are grouped together? 1

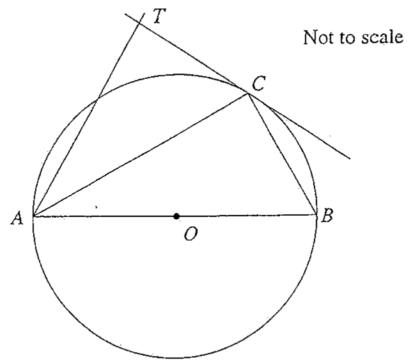
(iii) The 12 videotapes are arranged at random in a row on the shelf. Find the probability that the arrangement has a copy of *Gone with the Wind,* at one end of the row, and a copy of *Gladiator* at the other end. 2

QUESTION 3. (12 marks) Use a SEPARATE writing booklet. Marks

(a) (i) Write down an expression for $\tan(\alpha+\beta)$. 1

(ii) Hence evaluate $\alpha+\beta$, for $0 \le \alpha+\beta \le 2\pi$, if $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$. 2

(b) 3



In the diagram AOB is the diameter of a circle centre O, and C is the point of contact of the tangent TC such that AC bisects $\angle TAB$.

Prove that AT is perpendicular to TC.

(c) The acceleration of a particle moving in a straight line is given by $\frac{d^2x}{dt^2} = -e^{-2x}$, where x is the displacement, in metres, from the origin O and t is time elapsed in seconds. Initially the particle moves from the origin with velocity +2 metres per second. 3

Find the velocity of the particle at $x=1$, correct to 2 decimal places.

(d) The coefficients of x^3 and x^4 in the expansion of $(1+bx)^{12}$, $b \neq 0$, are equal. 3

Find the value of b.

QUESTION 4. (12 marks) Use a SEPARATE writing booklet. Marks

(a) Consider the expression $P(x) = x^3 + ax^2 + bx + 6$, where a and b are constants.

When $P(x)$ is divided by $(x-2)(x+1)$, the remainder is $4 - 4x$, and the quotient is $x + k$.

(i) By using the division transformation, or otherwise, find the values of a and b. 4

(ii) Find the value k for these values of a and b 1

(b) (i) Write down an expression for $\cos 2A$ in terms of $\sin A$. 1

(ii) Hence show that $\sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x)$. 1

(iii) Sketch $y = \sin^2\frac{x}{2}$ for $0 \le x \le 2\pi$. 2

(iv) State the amplitude and period of $y = \sin^2\frac{x}{2}$. 1

(v) Find the exact area of the region bounded by the curve $y = \sin^2\frac{x}{2}$ and the x-axis between $x = 0$ and $x = \frac{\pi}{3}$. 2

QUESTION 5. (12 marks) Use a SEPARATE writing booklet. Marks

(a) Use mathematical induction to prove that 4

$$\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots + \log\left(\frac{n+1}{n}\right) = \log(n+1) \text{ for } n \ge 1.$$

(b) Newton's law of cooling states that the rate of change of the temperature T of a body at any time t is proportional to the difference in the temperature of the body and the temperature M of the surrounding medium,

i.e. $\frac{dT}{dt} = k(T - M)$, where k is a constant.

(i) Show that $T = M + Ae^{kt}$, where A is a constant, satisfies this equation. 1

(ii) A freezer is maintained at a constant temperature of –8 °C. When water at 25°C is placed in the freezer, the temperature of the water falls to 15°C in 10 minutes. Find the temperature of the water after 10 more minutes, correct to the nearest degree. 3

(c) Tony and Rosa each throw an unbiased coin five times.

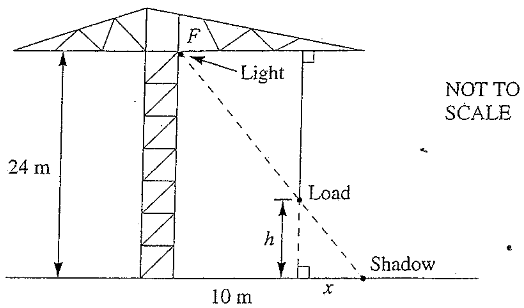
(i) Find the probability they both throw exactly three heads. 2

(ii) Assuming they both throw at least one head, find the probability they both throw the same number of heads. 2

QUESTION 6. (12 marks) Use a SEPARATE writing booklet. Marks

(a) A particle moves with simple harmonic motion and has a speed of 5 centimetres per second when passing through the centre O of its path. The period is π seconds. Find the speed of the particle when it is 1.5 centimetres from O. **2**

(b)



The diagram shows a crane that can lift loads vertically up to a height of 24 metres. When work is being done at night, a floodlight at point F casts a shadow of each load onto the level ground.

When loads are being lifted, their height above ground level, h metres, after t seconds, is given by $h = 2t$, where $0 \le t \le 12$. As shown in the diagram, each load is lifted from a starting point which is on the ground and which is 10 metres from the base of the crane. The shadow is cast x metres away from the load's starting point.

(i) Show that $x = \dfrac{10h}{24 - h}$. **1**

(ii) Find the position of the shadow when $t = 4$. **1**

(iii) Find the rate at which the shadow is moving when $t = 4$. **3**

(c) Consider the function $f(x) = \dfrac{1}{x^2} - 1, x > 0$.

(i) Write down the equations of the horizontal and vertical asymptotes for $y = f(x)$. **1**

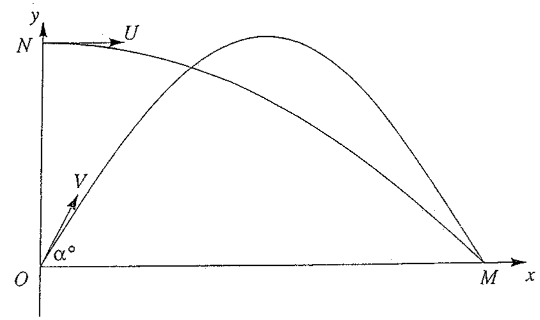
(ii) Show that $y = f(x)$ has no stationary points. **1**

(iii) Show that the curve of $y = f(x)$ is always concave up. **1**

(iv) Determine the inverse function, $f^{-1}(x)$. **1**

(v) Sketch $y = f^{-1}(x)$. **1**

QUESTION 7. (12 marks) Use a SEPARATE writing booklet. Marks



The diagram shows the path of two projectiles in the same plane.

Particle A is fired from a point O, with speed V metres per second, at an angle of elevation α radians to the horizontal. It reaches a maximum height, H metres, above O and lands at point M on the same horizontal level as O.

Particle B is fired from point N, H metres above O, with speed U metres per second in a horizontal direction, and also lands at point M.

With the above axes, you may assume that the velocity and the position of particle A, at time t seconds after it is fired, are given by

$$\dot{x} = V\cos\alpha \qquad \dot{y} = V\sin\alpha - gt$$
$$x = Vt\cos\alpha \qquad y = Vt\sin\alpha - \frac{1}{2}gt^2$$

where g is the acceleration due to gravity.

Also with the above axes, you may assume that the velocity and the position of particle B are given by

$$\dot{x} = U \qquad \dot{y} = -gt$$
$$x = Ut \qquad y = -\frac{1}{2}gt^2 + H$$

(a) Show that the greatest height reached by a particle A is given by $H = \dfrac{V^2\sin^2\alpha}{2g}$. **1**

(b) Show that the range of particle A is given by $R = \dfrac{V^2\sin 2\alpha}{g}$. **2**

(c) Show that the range of particle B is given by $R = \dfrac{UV\sin\alpha}{g}$. **2**

QUESTION 7. (Cont.) Marks

(d) Show that the relationship between U and V can be given by $U = 2V\cos\alpha$. 1

(e) Show that the trajectory of particle B can be expressed as $y = \frac{V^2\sin^2\alpha}{2g} - \frac{g}{2U^2}x^2$. 1

(f) If particles A and B land at point M at the same instant, and if θ is the difference in angles of impact of the two particles, show that $\tan\theta = \frac{\tan\alpha}{2 + \tan^2\alpha}$, where θ is acute. 3

(g) By expressing the equation in part (f) as a quadratic in terms of $\tan\alpha$, or otherwise, find the range of values that θ can take, correct to the nearest minute. 2

Standard integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right).$$

Note: $\ln x = \log_e x, \quad x > 0$

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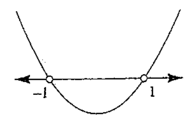
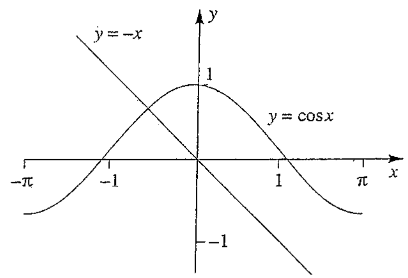
Mathematics Extension 1

Solutions and suggested marking scheme

QUESTION 1

Sample Answer	Outcome listing and mark guide
(a) $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{3}}$ $= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$ $= \frac{\pi}{3} - \frac{\pi}{6}$ $= \frac{\pi}{6}$	HE4 • Gives correct answer . . . . 3 • Gives correct integral and gives at least one correct $\sin^{-1}$ evaluation . . . . 2 • Gives correct integral 1
(b) $\frac{3}{x^2-1} \geq 0$ $x^2-1 \neq 0$ i.e. $x \neq 1$ or $x \neq -1$ $(x^2-1) > 0$ $(x-1)(x+1) > 0$ $x < -1$ or $x > 1$ 	PE3 • Gives correct solution . . . . 3 • Gives solution of $x \leq -1, x \geq 1$ . . . . 2 • Gives correct factorisation 1
(c) $u^2 = 4 + x$ $x = u^2 - 4$ $dx = 2u\,du$ If $x = 5$, $u^2 = 9$ $\therefore u = 3$ If $x = 0$, $u^2 = 4$ $\therefore u = 2$ $\int_0^5 \frac{x}{\sqrt{4+x}}dx = \int_2^3 \frac{u^2-4}{\sqrt{u^2}}2u\,du$ $= 2\int_2^3 (u^2-4)du$ $= 2\left[\frac{1}{3}u^3 - 4u\right]_2^3$ $= 2\left\{[9-12] - \left[\frac{8}{3} - 8\right]\right\}$ $= \frac{14}{3}$ or $4\frac{2}{3}$ or 4.67 (2 d.p.)	HE6 • Gives correct solution . . . . 4 • Finds correct integral. . . . . 3 • Gives correct expression to be integrated . 2 • Shows significant progress in finding the correct expression to be integrated 1
(d) $\frac{^nP_{r+1}}{^nP_r} = \frac{n!}{(n-(r+1))!} \div \frac{n!}{(n-r)!}$ $= \frac{n!}{(n-r-1)!} \times \frac{(n-r)(n-r-1)!}{n!}$ $= n - r$	PE3 • Gives correct answer . . . . 2 • Gives correct expression for $^nP_{n+1}$ and ^nP_r 1

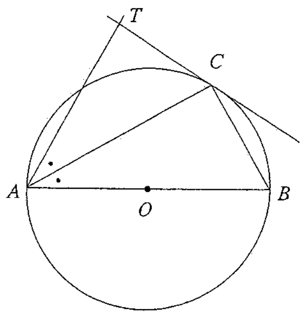
QUESTION 2

Sample Answer	Outcome listing and mark guide
(a) (i) $\frac{d}{dx}x\tan^{-1}x = \tan^{-1}x + \frac{x}{1+x^2}$	HE4, H5 • Gives correct derivative. 1
(ii) $\int_0^1 \tan^{-1}x\,dx = \int_0^1 \frac{d}{dx}x\tan^{-1}x\,dx - \int_0^1 \frac{x\,dx}{1+x^2}$ $= \left[x\tan^{-1}x\right]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ $= [(1)\tan(1) - 0]$ $- \left[\frac{1}{2}\ln(1+1) - \frac{1}{2}\ln(1+0)\right]$ $= \left[\frac{\pi}{4} - 0\right] - \left[\frac{1}{2}\ln 2 - 0\right]$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$	HE4, H5 • Gives correct answer. . . . . 3 • Gives the two correct integrals or • Gives the correct answer to one of the integrals. . . . . 2 • Gives correct expression to integrate 1
(b) (i)  To solve: $\cos x + x = 0$ $\cos x = -x$ $\therefore$ draw $y = \cos x$ and $y = -x$, $-\pi \le x \le \pi$ There is only one point of intersection. $\therefore \cos x + x = 0$ has only one solution.	P4 • One mark for drawing two correct graphs. • One mark for giving a satisfactory explanation 2
(ii) Newton's method Let $f(x) = \cos x + x$ $f(-1) = \cos(-1) - 1$ $= -0.459697\ldots$ $f'(x) = -\sin x + 1$ $f'(-1) = -\sin(-1) + 1$ $= 1.84147\ldots$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $= -1 - \frac{-0.459697\ldots}{1.84147\ldots}$ $= -0.75036\ldots$ $= -0.75$ (to 2 decimal places)	PE3, H5 • Gives correct answer. . . . . 2 • Finds correct values of $f(-1)$ and $f'(-1)$ and quotes correct formula 1

QUESTION 2 (Continued)

Sample Answer	Outcome listing and mark guide
(c) (i) Number of arrangements $= \frac{12!}{3!4!5!}$ $= 27\ 720$	PE3 • Gives correct answer. 1
(ii) Number of arrangements $= 3!$ $= 6$	PE3 • Gives correct answer. 1
(iii) Number of arrangements with *Gone with the Wind* at one end and *Gladiator* at the other end $= 2! \times \frac{10!}{2! \times 4! \times 4!}$ $= 6300$ $P(\text{above arrangement}) = \frac{6300}{27\ 720}$ $= \frac{5}{22}$	PE3, H5 • Gives correct answer. . . . . 2 • Gives correct answer for number of arrangements. 1

QUESTION 3

<table>
<tr><th>Sample Answer</th><th>Outcome listing and mark guide</th></tr>
<tr><td>(a) (i) $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$</td><td>H5
• Gives correct expression . . . . 1</td></tr>
<tr><td>(ii) $\tan(\alpha+\beta)=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}$
$=\dfrac{\frac{5}{6}}{\frac{5}{6}}$
$=1$
$\therefore \alpha+\beta=\frac{\pi}{4}$ or $\alpha+\beta=\frac{5\pi}{4}$</td><td>H5
• One mark for finding $\tan(\alpha+\beta)=1$
• One mark for the two answer . . . . 2</td></tr>
<tr><td>(b)

In ΔACT and ΔABC
$\angle TAC=\angle BAC$ (AC bisects $\angle TAB$)
$\angle TCA=\angle CBA$ (alternate segment theorem)
$\Delta ACT \;|||\; \Delta ABC$ (two pairs of corresponding angles equal)
$\angle ATC=\angle ACB$ (corresponding angles of similar triangles)
$\angle ACB=90°$ (angle of a semi-circle)
$\therefore \angle ATC=90°$
$\therefore AT\perp TC$</td><td>PE2, PE3
• Gives correct proof with reasons. 3

• Gives partially correct proof with reasons
or
• Gives correct proof without reasons 2

• Gives correct initial step to a solution, e.g. correctly uses alternate segment theorem . . . . 1</td></tr>
</table>

QUESTION 3 (Continued)

<table>
<tr><th>Sample Answer</th><th>Outcome listing and mark guide</th></tr>
<tr><td>(c) $\dfrac{d^2x}{dt^2}=-e^{-2x}$
$\frac{1}{2}v^2=\int -e^{-2x}dx$
$=\frac{1}{2}e^{-2x}+c$
At $x=0$, $v=+2$.
$\frac{1}{2}(4)=\frac{1}{2}e^{\circ}+c$
$c=\frac{3}{2}$
$\therefore v^2=e^{-2x}+3$
at $x=1$, $v^2=e^{-2}+3$
$v^2=3.13533\ldots$
$v=1.77068\ldots$
$\therefore$ velocity is 1.77 m/s (to 2 decimal places).</td><td>HE5
• One mark for correct primitive
• One mark for correct expression for $v^2=e^{-2x}+3$
• One mark for correct answer. . . . . 3</td></tr>
<tr><td>(d) For $(1+bx)^{12}$, $T_4={}^{12}C_3(bx)^3$; $T_5={}^{12}C_4(bx)^4$
$\therefore$ coefficient of x^3 is ${}^{12}C_3b^3$
and coefficient of x^4 is ${}^{12}C_4b^4$
$\therefore {}^{12}C_3b^3={}^{12}C_4b^4$
$\dfrac{b^4}{b^3}=\dfrac{{}^{12}C_3}{{}^{12}C_4}$
$\therefore b=\dfrac{\frac{12!}{3!9!}}{\frac{12!}{4!8!}}$
$=\dfrac{4!8!}{3!9!}$
$\therefore b=\frac{4}{9}$</td><td>PE3, HE6, HE7
• Gives correct answer 3

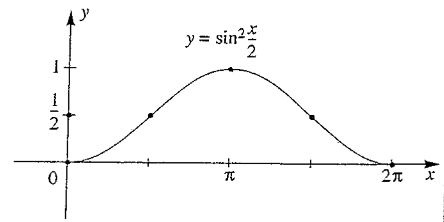
• Gives correct expression for b 2

• Gives correct expressions for coefficients of x^3 and x^4 . . . . 1</td></tr>
</table>

QUESTION 4

Sample Answer	Outcome listing and mark guide
(a) (i) For polynomial $P(x)$, $P(x)=A(x)Q(x)+R(x)$ $A(x)=$ divisor $Q(x)=$ quotient $R(x)=$ remainder and degree $R(x)<$ degree $A(x)$ Here x^3+ax^2+bx+6 $=(x-2)(x+1)(x+k)+(4-4x)$ Let $x=2$ $8+4a+2b+6=0+(-4)$ $4a+2b=-18$ $2a+b=-9$ [Eq. 1] Let $x=-1$ $-1+a-b+6=0+8$ $a-b=3$ [Eq. 2] On adding Eq. 1 and Eq. 2 $3a+0=-6$ $a=-2$ Then from Eq. 2, $b=-5$ $\therefore a=-2,\ b=-5$	PE3 • Writes the division transformation or uses the remainder theorem • Determines the 2 equations needed to find a and b • Gives the values for a and b .... 4 • Writes the division transformation or uses the remainder theorem • Determines the 2 equations in a and b • Attempts to solve equations or • Writes the division transformation, or uses the remainder theorem • Determines one correct equation in a and b • Finds a and b correctly from their equations .... 3 • Writes the division transformation or uses the remainder theorem • Determines one correct equation in a and b and attempts to find the other .... 2 • Writes the division transformation or uses the remainder theorem and attempts to find the two equations 1
(ii) Now x^3-2x^2-5x+6 $=(x-2)(x+1)(x+k)+(4-4k)$ Let $x=0$ $6=-2\times1\times k+4$ $2=-2k$ $k=-1$	PE3 • Finds the value of k which is correct for their values of a and b. 1
(b) (i) $\cos 2A=\cos^2 A-\sin^2 A$ $=1-2\sin^2 A$	H5 • Gives correct expression 1
(ii) $\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$ $=\left(1-\sin^2\frac{x}{2}\right)-\sin^2\frac{x}{2}$ $=1-2\sin^2\frac{x}{2}$ $2\sin^2\frac{x}{2}=1-\cos x$ $\sin^2\frac{x}{2}=\frac{1}{2}(1-\cos x)$	H5 • Uses a correct method.......... 1

QUESTION 4 (Continued)

Sample Answer	Outcome listing and mark guide
(iii)  (Note that the curve is more easily drawn by thinking of it as $y=\frac{1}{2}(1-\cos x)$ rather than as $y=\sin^2\frac{x}{2}$.)	P5, H5 • One mark for correct shape • One mark for correct values on x and y axes 2
(iv) Amplitude $=\frac{1}{2}$ Period $=2\pi$	H5 • Gives both correct values 1
(v) Area $=\int_0^{\frac{\pi}{3}}\sin^2\frac{x}{2}\,dx$ $=\int_0^{\frac{\pi}{3}}\frac{1}{2}(1-\cos x)\,dx$ $=\frac{1}{2}\Big[x-\sin x\Big]_0^{\frac{\pi}{3}}$ $=\frac{1}{2}\left[\left(\frac{\pi}{3}-\sin\frac{\pi}{3}\right)-(0-0)\right]$ $=\frac{1}{2}\left[\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right]$ $=\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ $\therefore$ Area is $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ unit^2	H8 • One mark for writing correct integrated expression $\frac{1}{2}\Big[x-\sin x\Big]_0^{\frac{\pi}{3}}$ • One mark for correct answer, in any form.......... 2

QUESTION 5

Sample Answer

(a) To be proved:

$$\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) \ldots + \log\left(\frac{n+1}{n}\right)$$

$= \log(n+1),\ n \geq 1$

Step 1

Let $n = 1$.

The statement becomes $\log 2 = \log 2$ which is true.

Step 2

Assume the statement is true for some positive integer k. That is, assume

$$\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots + \log\left(\frac{k+1}{k}\right)$$

$= \log(k+1)$ [statement 1]

Now it has to be shown that the following statement, where $n = k+1$, is true.

$$\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots + \log\left(\frac{k+1}{k}\right) + \log\left(\frac{k+2}{k+1}\right)$$

$= \log(k+2)$ [statement 2]

Now $\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots + \log\left(\frac{k+1}{k}\right) + \log\left(\frac{k+2}{k+1}\right)$

$= \log(k+1) + \log\left(\frac{k+2}{k+1}\right)$ (by statement 1 above)

$= \log(k+1) + \log(k+2) - \log(k+1)$ (by logarithm laws)

$= \log(k+2)$, as required.

Hence if statement 1 is true, then so is statement 2.

Now, if the original statement is true when $n = k$, where k is a positive integer, then it is also true when $n = k+1$. But the statement is true when $n = 1$. Therefore, the statement is true for all positive integers n.

Outcome listing and mark guide

PE2, HE2

- One mark for showing statement is true for $n = 1$
- One mark for correctly using statement 1 to attempt to prove statement 2
- One mark for correctly proving the statement for $n = k+1$
- One mark for correctly stating conclusion 4

Sample Answer

(b) (i) $T = M + Ae^{kt}$

$\therefore \frac{dT}{dt} = Ake^{kt}$

Now consider the equation

$\frac{dT}{dt} = k(T - M)$

$\frac{dT}{dt} = Ake^{kt}$, from above

$k(T - M) = k(M + Ae^{kt} - M)$, from above

$= kAe^{kt}$

$\therefore T = M + Ae^{kt}$ satisfies the equation.

Outcome listing and mark guide

H2, H3, H5

- Gives correct derivative of T, and correctly substitutes expressions for T and $\frac{dT}{dt}$ into the given equation 1

QUESTION 5 (Continued)

Sample Answer

(ii) $T = M + Ae^{kt}$

Here, $M = -8$, and at $t = 0, T = 25$

$25 = -8 + Ae^{0}$

$\therefore A = 33$

$\therefore T = -8 + 33e^{kt}$

Now at $t = 10, T = 15$

$15 = -8 + 33e^{10k}$

$23 = 33e^{10k}$

$e^{10k} = \frac{23}{33}$

$10k = \log_e \frac{23}{33}$

$k = \frac{1}{10}\log_e \frac{23}{33}$

$= -0.036101\ldots$

$= -0.0361$ (correct to 4 decimal places)

$\therefore T = -8 + 33e^{-0.0361t}$

Now let $t = 20$

$T = -8 + 33e^{-0.0361 \times 20}$

$= 8.030$

$\therefore$ after 10 more minutes, the temperature of the water is 8°C.

Outcome listing and mark guide

H3, HE3

- One mark for correctly finding the value of A
- One mark for correctly finding the value of k, given their value of A
- One mark for correctly finding the temperature after 10 more minutes, given their values of A and k 3

Sample Answer

(c) (i) Probability of one person throwing exactly 3 heads

$= {}^5C_3[P(T)]^2[P(H)]^3$

$= {}^5C_3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$

$= {}^5C_3\left(\frac{1}{2}\right)^5$

$\therefore$ probability of *both* Tony and Rosa throwing 3 heads.

$= {}^5C_3\left(\frac{1}{2}\right)^5 \times {}^5C_3\left(\frac{1}{2}\right)^5$

$= \left({}^5C_3\left(\frac{1}{2}\right)^5\right)^2$

$= \left(10 \times \left(\frac{1}{2}\right)^5\right)^2$

$= \frac{100}{1024}$

$= \frac{25}{256}$

Outcome listing and mark guide

HE3

- Finds the correct probability, in exact or decimal form 2

- Finds correct probability of one person throwing exactly 3 heads, i.e. ${}^5C_3\left(\frac{1}{2}\right)^5$ . . 1

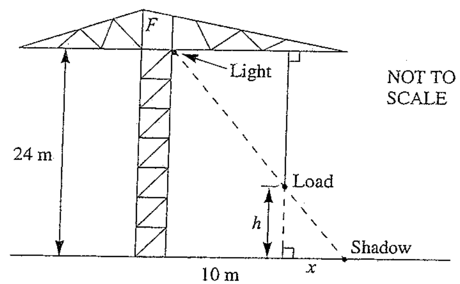
QUESTION 5 (Continued)

Sample Answer	Outcome listing and mark guide
(ii) Probability of Tony and Rosa throwing same number of heads $= [P(1H)]^2 + [P(2H)]^2 + [P(3H)]^2 + [P(4H)]^2 + [P(5H)]^2$ $= \left({}^5C_1\left(\frac{1}{2}\right)^5\right)^2 + \left({}^5C_2\left(\frac{1}{2}\right)^5\right)^2 + \left({}^5C_3\left(\frac{1}{2}\right)^5\right)^2 + \left({}^5C_4\left(\frac{1}{2}\right)^5\right)^2 + \left({}^5C_5\left(\frac{1}{2}\right)^5\right)^2$ $= [({}^5C_1)^2 + ({}^5C_2)^2 + ({}^5C_3)^2 + ({}^5C_4)^2 + ({}^5C_5)^2] \times \frac{1}{2^{10}}$ $= [5^2 + 10^2 + 10^2 + 5^2 + 1^2] \times \frac{1}{1024}$ $= \frac{251}{1024}$	HE3 • Shows that the probability is given by sum of the series with terms such as $\left({}^5C_1\left(\frac{1}{2}\right)^5\right)^2$ • Finds the correct probability in any form . . . . 2 • Attempts to find the probability by finding the sum of the series with terms such as $\left({}^5C_1\left(\frac{1}{2}\right)^5\right)^2$ 1

QUESTION 6

Sample Answer	Outcome listing and mark guide
(a) For simple harmonic motion, $v^2 = n^2(a^2 - x^2)$ Period $= \pi$ sec, given $\therefore \frac{2\pi}{n} = \pi$ $n = 2$ At $x = 0,\ v = 5$ $\therefore 5^2 = 2^2(a^2 - 0^2)$ $25 = 4a^2$ $a = \frac{5}{2}$ $= 2.5$ Now let $x = 1.5$ $v^2 = 2^2(2.5^2 - 1.5^2)$ $v^2 = 4 \times 4$ $v = \pm 4$ $\therefore$ speed of particle when it is 1.5 cm from 0 is 4 cm/s.	HE3 • Finds the correct speed . . . . 2 • Shows substantial progress towards finding the correct speed 1

(b)

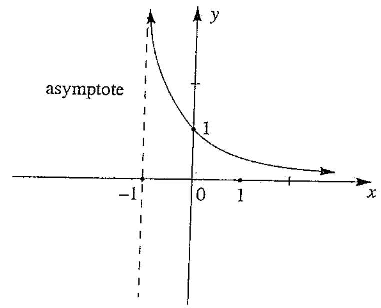


Sample Answer	Outcome listing and mark guide
(i) From similar triangles in diagram: $\frac{24}{10 + x} = \frac{h}{x}$ $24x = 10h + hx$ $24x - hx = 10h$ $x(24 - h) = 10h$ $x = \frac{10h}{24 - h}$	H5 • Shows appropriate ratios, based on similar triangles, as being equal 1
(ii) $h = 2t$ (given) $\therefore$ when $t = 4,\ h = 8$ $\therefore x = \frac{10 \times 8}{24 - 8}$ $= 5$ $\therefore$ shadow is 5 metres to the right of load's starting point.	P4, H9 • Correctly states position of shadow 1

QUESTION 6 (Continued)

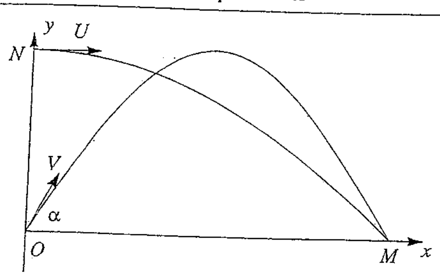
Sample Answer	Outcome listing and mark guide
(iii) $\frac{dx}{dt}=\frac{dx}{dh}\times\frac{dh}{dt}$ $\frac{dx}{dh}=\frac{(24-h)10-10h\times(-1)}{(24-h)^2}$ $=\frac{240-10h+10h}{(24-h)^2}$ $=\frac{240}{(24-h)^2}$ $h=2t$ $\frac{dh}{dt}=2$ $\therefore\frac{dx}{dt}=\frac{240}{(24-h)^2}\times 2$ $=\frac{480}{(24-h)^2}$ At $t=4$, $h=8$ $\therefore\frac{dx}{dt}=\frac{480}{(24-8)^2}$ $=1.875$ $\therefore$ shadow is moving at a rate of 1.875 m/s to the right.	HE5 • One mark for correctly finding the expression for $\frac{dx}{dh}$ • One mark for correctly finding the expression for $\frac{dx}{dt}$ in terms of h • One mark for correctly stating the speed of the shadow 3
(c) (i) $f(x)=\frac{1}{x^2}-1, x>0$ To find horizontal asymptote, let $x\to\infty$ Then $f(x)\to -1$ $\therefore$ horizontal asymptote is $y=-1$ Vertical asymptote is $x=0$	P5 • Correctly states the horizontal and vertical asymptotes........................ 1
(ii) $f'(x)=-2x^{-3}$ $=\frac{-2}{x^3}$ There is no value of x for which $\frac{-2}{x^3}=0$ $\therefore y=f(x)$ has no stationary points.	H6 • Uses the derivative to show the result.... 1
(iii) $f''(x)=6x^{-4}$ $=\frac{6}{x^4}$ $\frac{6}{x^4}>0$, for all x, except $x=0$ $\therefore y=f(x)$ is always concave up.	H6 • Uses the second derivative to show the result 1

QUESTION 6 (Continued)

Sample Answer	Outcome listing and mark guide
(iv) $y=\frac{1}{x^2}-1$, $x>0$, $y>-1$ Interchange x, y $x=\frac{1}{y^2}-1$, $y>0$, $x>-1$ $y^2x=1-y^2$ $y^2(x+1)=1$ $y^2=\frac{1}{x+1}$ $y=\frac{1}{\sqrt{x+1}}$, since $x>-1$, $y>0$	HE4 • Determines the inverse function correctly 1
(v) 	P5, HE4 • Draws the correct graph for their inverse function 1

QUESTION 7

Sample Answer | **Outcome listing and mark guide**



(a)

$y = Vt\sin\alpha - \frac{1}{2}gt^2$

$\dot{y} = V\sin\alpha - gt$

Height is greatest when $\dot{y} = 0$

i.e. when $t = \frac{V\sin\alpha}{g}$

$\therefore y = V\sin\alpha \times \frac{V\sin\alpha}{g} - \frac{1}{2} \times g \times \frac{V^2\sin^2\alpha}{g^2}$

$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$

$= \frac{V^2\sin^2\alpha}{2g}$

$\therefore H = \frac{V^2\sin^2\alpha}{2g}$

HE3
- Shows result by setting $\dot{y} = 0$, finding expression for t, and substituting into expression for y. 1

(b)

$x = Vt\cos\alpha$

Range is obtained by letting $y = 0$

$Vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t\left(V\sin\alpha - \frac{1}{2}gt\right) = 0$

$t = 0$ or $t = \frac{2V\sin\alpha}{g}$ (ignore $t = 0$)

$x = V\cos\alpha \times \frac{2V\sin\alpha}{g}$

$= \frac{V^2 \times 2\sin\alpha\cos\alpha}{g}$

$\therefore R = \frac{V^2\sin 2\alpha}{g}$

HE3
- One mark for finding correct expression for t
- One mark for correctly deriving the expression for R. 2

QUESTION 7 (Continued)

Sample Answer | **Outcome listing and mark guide**

(c)

For particle B, $x = Ut$

Range is obtained by letting $y = 0$.

$y = -\frac{1}{2}gt^2 + H$

$-\frac{1}{2}gt^2 + H = 0$

$\therefore t = \sqrt{\frac{2H}{g}}$

$\therefore R = U\sqrt{\frac{2H}{g}}$

$= U\sqrt{\frac{2}{g} \times \frac{V^2\sin^2\alpha}{2g}}$

$= U\sqrt{\frac{V^2\sin^2\alpha}{g^2}}$

$= \frac{UV\sin\alpha}{g}$

HE3
- One mark for finding correct expression for t
- One mark for correctly deriving the expression for R. 2

(d)

To show relationship between U and V, equate the two different expressions for R

From (b), $R = \frac{V^2\sin 2\alpha}{g}$ [Eq. 1]

From (c), $R = U\frac{V\sin\alpha}{g}$ [Eq. 2]

Let Eq. 1 = Eq. 2

$\frac{V^2\sin 2\alpha}{g} = \frac{UV\sin\alpha}{g}$

$2\sin\alpha\cos\alpha V = U\sin\alpha$

$\therefore U = 2V\cos\alpha$

HE3
- Correctly equates the 2 different expressions for R and simplifies 1

(e)

For B, $x = Ut$

$y = \frac{1}{2}gt^2 + H$

Eliminate t

$t = \frac{x}{U}$

$y = -\frac{1}{2}g\left(\frac{x}{U}\right)^2 + H$

$= -\frac{1}{2}g\left(\frac{x}{U}\right)^2 + \frac{V^2\sin^2\alpha}{2g}$ from (a)

$\therefore y = \frac{V^2\sin^2\alpha}{2g} - \frac{g}{2U^2}x^2$

HE3
- Eliminates t to find equation of trajectory 1

QUESTION 7 (Continued)

Sample Answer	Outcome listing and mark guide
(f) Similarly for A, $y = x\tan\alpha - \dfrac{gx^2}{2V^2\cos^2\alpha}$ For A: $\dfrac{dy}{dx} = \tan\alpha - \dfrac{gx}{V^2\cos^2\alpha}$ At M: $\dfrac{dy}{dx} = \tan\alpha - \dfrac{g\left(\dfrac{V^2 2\sin\alpha\cos\alpha}{g}\right)}{V^2\cos^2\alpha}$ $= \tan\alpha - 2\tan\alpha = -\tan\alpha$ For B: $\dfrac{dy}{dx} = -\dfrac{g}{U^2}x$ At M: $\dfrac{dy}{dx} = -\dfrac{g}{U^2} \times \dfrac{V^2\sin 2\alpha}{g}$ $= -\dfrac{g}{4V^2\cos^2\alpha} \times \dfrac{V^2 2\sin\alpha\cos\alpha}{g}$ $= -\dfrac{1}{2}\tan\alpha$ Now $\tan\theta = \left\|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right\|$ $= \left\|\dfrac{-\frac{1}{2}\tan\alpha - (-\tan\alpha)}{1 + (-\tan\alpha)\left(-\frac{1}{2}\tan\alpha\right)}\right\|$ $= \dfrac{\frac{1}{2}\tan\alpha}{1 + \frac{1}{2}\tan^2\alpha}$ $= \dfrac{\tan\alpha}{2 + \tan^2\alpha}$ $\therefore \tan\theta = \dfrac{\tan\alpha}{2 + \tan^2\alpha}$	HE3 • One mark for finding gradient of direction of impact of A • One mark for finding gradient of direction of impact of B • One mark for substitution into formula for $\tan\theta$, and for correct simplifications 3
(g) $\tan\theta \times (2 + \tan^2\alpha) = \tan\alpha$ $2\tan\theta + \tan\theta\tan^2\alpha = \tan\alpha$ $\tan\theta\tan^2\alpha - \tan\alpha + 2\tan\theta = 0$ $\tan\alpha = \dfrac{1 \pm \sqrt{1 - 8\tan^2\theta}}{2\tan\theta}$ For $\tan\alpha$ to be real, $\Delta \geq 0$. $\Delta = b^2 - 4ac$ $= 1 - 8\tan^2\theta \geq 0$ $8\tan^2\theta \leq 1$ $\tan^2\theta \leq \dfrac{1}{8}$ $0 \leq \tan\theta \leq \dfrac{1}{\sqrt{8}}$ (since θ is an acute angle) $\therefore 0° \leq \theta \leq 19°28'$ (correct to the nearest minute)	HE3 • One mark for forming the equation, and for showing $1 - 8\tan^2\theta \geq 0$ • One mark for deducing the range of values for θ 2