NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Assessment Task 3
Term 2, 2010

Name:____________________________ **Mathematics Class:**____________

Time Allowed: 60 minutes + 2 minutes reading time
Total Marks: 60

Instructions:

- Questions are of equal value.
- Start each question on a new page. Put your name on every page.
- Show all necessary working. Marks may be deducted for incomplete or poorly arranged work.
- Work down the page.
- Do not work in columns.
- Write on one side of each page only.
- Each question will be collected separately. If you do not attempt a question, submit a blank page with your name and the question number clearly displayed.

Question	1abcd	1efg	2ab	2c	3a	3b	4	5ab	5c	Total
H3	/6									/6
H5		/6	/7		/3		/12	/6		/34
H8				/5					/6	/11
H9						/9				/9
										/60

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (12 Marks) Marks

(a) Find the exact value of $\ln e^4 + \log_{125} 25$. 1

(b) Calculate the value of $\left(\frac{1}{\sqrt{e^3}} - 1\right)^2$ correct to 3 significant figures. 1

(c) If $\log_a x = 3$ and $\log_a y = 5$, evaluate $\log_a\left(\frac{a}{xy}\right)$ 2

(d) Write $4\log y + \log x - \frac{1}{2}\log m$ as a single log. 2

(e) Differentiate with respect to x:

(i) $y = 5e^{4x+1}$ 1

(ii) $y = \ln(4-2x)$ 1

(f) Find

(i) $\int e^{-5x}dx$ 1

(ii) $\int \frac{4}{e^{2x-1}}dx$ 1

(g) If \$5000 is placed in an investment account paying 7% p.a. compounded every six months, how much, to the nearest dollar, will the investment be worth after 3 years? 2

Question 2 (12 Marks) **Start a NEW page** Marks

(a) (i) Find $\int \frac{xe^x - 1}{x}dx$. 2

(ii) Hence, find the exact value of $\int_1^2 \frac{xe^x - 1}{x}dx$ 1

(b) Sue deposits \$1000 at the beginning of each year into a holiday fund for 6 years. The interest is compounded yearly at 8% p.a.

(i) To how much, to the nearest dollar, will the first \$1000 grow after 6 years? 1

(ii) Sue's planned holiday will cost \$7450. Will she have sufficient funds from the investment? Support your answer with working and calculations. 3

(c) Consider the section of the graph of $y = \log_e x$ from the point $\left(\frac{1}{e}, k\right)$ to $(e, 1)$.

(i) Find the value of k. 1

(ii) Sketch the graph of $y = \log_e x$ and indicate these two points. 1

(iii) Find the exact area between this part of the curve and the y-axis. 3

Question 3 (12 Marks) **Start a NEW page** Marks

(a) (i) Show that $\frac{1}{x-2} - \frac{3}{x+2} = \frac{8-2x}{x^2-4}$. 1

(ii) Hence, find $\int \frac{2x-8}{x^2-4}dx$. 2

(b) Consider the curve $y = (x-2)e^x$

(i) Show that $\frac{dy}{dx} = e^x(x-1)$ and $\frac{d^2y}{dx^2} = xe^x$. 2

(ii) Find any x-intercepts. 1

(iii) Show that there is one stationary point at $(1, -e)$ and determine its nature. 2

(iv) Find any points of inflexion. 1

(v) What happens to $(x-2)e^x$ as $x \to -\infty$. 1

(vi) Hence, sketch the curve showing the above features. 2

Question 4 (12 Marks) **Start a NEW page** **Marks**

(a) Differentiate $(\log_e x)^4$ **2**

(b) The curve $f(x)=\ln(x-1)$ between $x=e$ and $x=3e$ is rotated around the x-axis.

(i) Write the integral which gives the value of this volume. **1**
(Do not evaluate the volume)

(ii) Use Simpson's Rule with 3 function values to approximate the volume.
Give your answer correct to 2 decimal places. **3**

(c) Judy wishes to purchase a new lounge suite costing \$5432. The store offers to lend her the money under the following terms:

- No interest will be charged for the first 12 months.
- After 12 months she will be charged interest at 18% p.a., compounded monthly on any outstanding debt.
- The loan must be paid completely after 3years.
- Judy must pay a fixed amount, \$$M$, each month (including during the first non-interest period).

Let \$$A_n$ be the amount Judy owes after n months.

(i) Write an expression for the amount Judy owes after one year. **1**

(ii) Show that the expression for the amount Judy owes after 15 months is

$$A_{15}=1\cdot015^3(5432-12M)-M(1+1\cdot015+1\cdot015^2)$$ **2**

(iii) Hence, write an expression for the amount Judy owes after 3 years. **1**

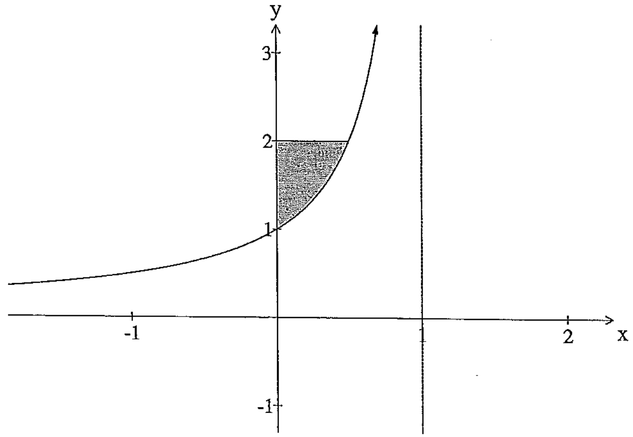
(iv) Find the amount \$$M$, to the nearest cents, of each monthly payment. **2**

Question 5 (12 Marks) **Start a NEW page** **Marks**

(a) Find the equation of the tangent to the curve $y=e^{x^2}-3x$ at the point where $x=1$.
Write your answer in $y=mx+b$ form. **3**

(b) Given $y=e^{4x}+e^{-3x}$ find the exact value of $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-4y$. **3**

(c) The shaded region bounded by the graph $y=\frac{1}{1-x}$ for $x<1$, the line $y=2$ and the y-axis is rotated about the y-axis to form a solid of revolution.



(i) Show that the volume of the solid is given by: **3**

$$V=\pi\int_1^2\left(1-\frac{2}{y}+\frac{1}{y^2}\right)dy$$

(ii) Find the exact volume V, of the solid formed. **3**

End of paper

Solutions

1 a) $\log_e e^4 + \log_{5^3} 5^2$

$= 4 + \frac{2}{3}$

$= 4\frac{2}{3}$

b) 0.604 (3 sig fig)

c) $\log_a a - \log_a x - \log_a y$

$= 1 - 3 - 5$

$= -7$

d) $\log y^4 + \log x - \log\sqrt{m}$

$= \log\left(\frac{xy^4}{\sqrt{m}}\right)$

e) (i) $\frac{dy}{dx} = 20e^{4x+1}$

(ii) $\frac{dy}{dx} = \frac{-2}{4-2x}$

$= \frac{1}{x-2}$

f) (i) $\int e^{-5x}\,dx = \frac{e^{-5x}}{-5} + C$

(ii) $\int \frac{4}{e^{2x-1}}\,dx = 4\int e^{1-2x}\,dx$

$= 4 \times \frac{e^{1-2x}}{-2} +$

$= -2e^{1-2x} + C$

g) $A = P(1+r)^n$

$r = 7\%$ pa $= 3.5\%$ per 6 month

$n = 3$ yrs $= 6$ periods

$\therefore A = 5000(1+0.035)^6$

$= 6146$

$\therefore \$6146$

2 a) (i) $\int \frac{xe^x - 1}{x}\,dx = \int \left(e^x - \frac{1}{x}\right)dx$

$= e^x - \ln x + C$

(ii) $\int_1^2 \frac{xe^x - 1}{x}\,dx = \left[e^x - \ln x\right]_1^2$

$= e^2 - \ln 2 - e^1 + \ln 1$

$= e^2 - e - \ln 2$

$\therefore \int_1^2 \frac{xe^x - 1}{x}\,dx = e^2 - e - \ln 2$

(b) $P = \$1000$, $r = 8\%$ p.a, $n = 6$ yrs

(i) Let A_n = amount invested after n yrs

$A_n = P(1+r)^n$

$\therefore A_1 = 1000(1+0.08)^6 = 1586.87$

$\therefore \$1586.87$

(ii) $A_2 = 1000 \times 1.08^5$

$A_3 = 1000 \times 1.08^4$

$\vdots$

$A_6 = 1000 \times 1.08^1$

Total $= 1000(1.08 + 1.08^2 + \cdots + 1.08^6)$

now, $(1.08 + 1.08^2 + \cdots + 1.08^6)$ is a GS with $S_n = \frac{a(r^n - 1)}{r-1}$

$a = 1.08$, $r = 1.08$, $n = 6$

$\therefore S_6 = \frac{1.08(1.08^6 - 1)}{0.08}$

Hence, Total $= 1000 \times \frac{1.08(1.08^6 - 1)}{0.08}$

$= 7922.80$

Now, $\$7922.80 > \$7450$ $\therefore$ She has enough for her holiday

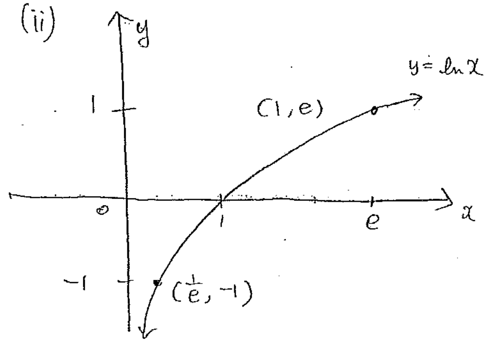
(c) (i) sub $x = \frac{1}{e}$

$y = \log_e e^{-1}$

$= -1$

$\therefore k = -1$

(ii)



(iii) Area $= \int_{-1}^{1} x\,dy$

now, $y = \log_e x$

$\therefore x = e^y$

$\therefore$ Area $= \int_{-1}^{1} e^y\,dy$

$= [e^y]_{-1}^{1}$

$= e - \frac{1}{e}$

Hence $\left(e - \frac{1}{e}\right)$ unit^2

③ (a) (i) $\dfrac{1}{x-2} - \dfrac{3}{x+2} = \dfrac{8-2x}{x^2-4}$

$$\text{LHS} = \frac{1}{x-2} - \frac{3}{x+2}$$
$$= \frac{x+2-3(x-2)}{(x-2)(x+2)}$$
$$= \frac{x+2-3x+6}{x^2-4}$$
$$= \frac{8-2x}{x^2-4}$$
$$= \text{RHS}$$

(ii)
$$\int \frac{2x-8}{x^2-4}\,dx = -\int \frac{8-2x}{x^2-4}\,dx$$
$$= -\int\left(\frac{1}{x-2} - \frac{3}{x+2}\right)dx$$
$$= \int\left(\frac{3}{x+2} - \frac{1}{x-2}\right)dx$$
$$= 3\ln(x+2) - \ln(x-2) + C$$

or $\ln\left(\dfrac{(x+2)^3}{x-2}\right) + C$

(b) $y = (x-2)e^x$

(i) $\dfrac{dy}{dx} = e^x + (x-2)e^x$
$= e^x(x-1)$

$\dfrac{d^2y}{dx^2} = e^x(x-1) + e^x$
$= e^x \times x$
$= xe^x$

(ii) to find x-int, let $y=0$
$0 = (x-2)e^x$
$\therefore x = 2$ (since $e^x \neq 0$)

(iii) to find a stationary point let $\frac{dy}{dx} = 0$
$e^x(x-1) = 0$
$\therefore x = 1$ ($e^x \neq 0$)
when $x = 1$,
$y = -e$ and $\frac{d^2y}{dx^2} = e^x > 0$
$\therefore (1, -e)$ is a min. turning point

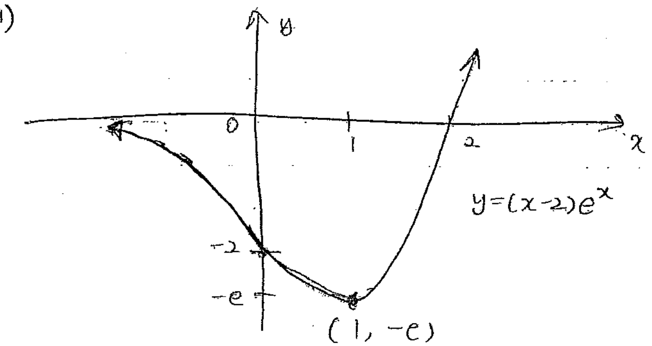
(iv) to find a POI, let $\frac{d^2y}{dx^2} = 0$
$xe^x = 0$
$\therefore x = 0$, $(e^x \neq 0)$
when $x=0$, $y = -2$

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	$-$	0	$+$

$\therefore (0, -2)$ is a POI

(v) as $x \to -\infty$, $e^x \to 0$ $\therefore (x-2)e^x \to 0^-$

(vi)



④ (a) $\frac{d}{dx}(\log_e x)^4 = 4(\log_e x)^3 \times \frac{1}{x}$

$= \frac{4(\log_e x)^3}{x}$

(b) $f(x) = \ln(x-1)$

(i) $V = \pi \int_a^b y^2 \, dx$

$\therefore V = \pi \int_e^{3e} (\ln(x-1))^2 \, dx$

(ii)

x	e	$2e$	$3e$
$\pi(f(x))^2$	0.29π or 0.92	2.22π or 6.97	3.87π or 12.16

$V \doteqdot \frac{h}{3}\left[f(e) + 4f(2e) + f(3e)\right]$

$= \frac{e}{3}(0.92 + 4 \times 6.97 + 12.16)$

$V \doteqdot 37.12$ unit^3

(c) (i) $A_1 = 5432 - M$

$A_2 = 5432 - 2M$

$\vdots$

$A_{12} = 5432 - 12M$

(ii) $A_{13} = (5432 - 12M) \times 1.015 - M$

$A_{14} = ((5432 - 12M) \times 1.015 - M) \times 1.015 - M$

$= (5432 - 12M) \times 1.015^2 - M(1 + 1.015)$

$\therefore A_{15} = (5432 - 12M) \times 1.015^3 - M(1 + 1.015 + 1.015^2)$

(iii) $A_{36} = (5432 - 12M) \times 1.015^{24} - M(1 + 1.015 + \cdots + 1.015^{23})$

(iv) Now $(1 + 1.015 + \cdots + 1.015^{23})$ is a GS with

$a = 1, \quad r = 1.015, \quad n = 24. \qquad S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{24} = \frac{1.015^{24} - 1}{1.015 - 1}$

, and $A_{36} = 0$ ($\because$ the loan must be paid completely after 36 months)

Hence

$A_{36} = (5432 - 12M) \times 1.015^{24} - M\left(\frac{1.015^{24} - 1}{0.015}\right)$

$0 = 5432 \times 1.015^{24} - 12M \times 1.015^{24} - M\left(\frac{1.015^{24} - 1}{0.015}\right)$

$M\left(12 \times 1.015^{24} + \frac{1.015^{24} - 1}{0.015}\right) = 5432 \times 1.015^{24}$

$\therefore \$M = \$169.59$

(5) (a) $y = e^{x^2} - 3x$

$y' = 2xe^{x^2} - 3$

now, at $x = 1$ $\quad y = e - 3$

and $y' = 2e - 3$

$\therefore \quad y - y_1 = m(x - x_1)$

$y - (e-3) = (2e-3)(x-1)$

$y - (e-3) = (2e-3)x - 2e + 3$

$y = (2e-3)x - 2e + 3 + e - 3$

$\therefore \quad y = (2e-3)x - e$

(b) $\dfrac{dy}{dx} = 4e^{4x} - 3e^{-3x}$

$\dfrac{d^2y}{dx^2} = 16e^{4x} + 9e^{-3x}$

$\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 4y = 16e^{4x} + 9e^{-3x} + 12e^{4x} - 9e^{-3x} - 4e^{4x} - 4e^{-3x}$

$= 24e^{4x} - 4e^{-3x}$

(c) (i) $y = \dfrac{1}{1-x}$

$1 - x = \dfrac{1}{y}$

$\therefore \quad x = 1 - \dfrac{1}{y}$

$V = \pi \displaystyle\int_1^2 x^2\,dy$

$= \pi \displaystyle\int_1^2 \left(1 - \frac{1}{y}\right)^2 dy$

$\therefore \quad V = \pi \displaystyle\int_1^2 \left(1 - \frac{2}{y} + \frac{1}{y^2}\right) dy$

(ii) $V = \pi \displaystyle\int_1^2 \left(1 - \frac{2}{y} + y^{-2}\right) dy$

$= \pi \left[y - 2\ln y - y^{-1} \right]_1^2$

$= \pi \left(\left(2 - 2\ln 2 - \frac{1}{2}\right) - (1 - 2\cancel{\ln 1} - 1) \right)$

$= \pi \left(\frac{3}{2} - \ln 4 \right)$

$\therefore \quad V = \left(\frac{3}{2} - \ln 4 \right) \pi$