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NORTH SYDNEY GIRLS' HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE, 1990

MATHEMATICS 3U/4U COMMON PAPER

QUESTION 1

(a) Simplify $\dfrac{\tan 5x - \tan x}{1 + \tan 5x.\ \tan x}$.

(b) Find the acute angle between the lines $2y - x + 1 = 0$ and $y = 5x + 2$ (give answer correct to the nearest minute).

(c) If α, β, γ are the roots of the equation $2x^3 + 5x - 3 = 0$, find the value of $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$.

(d) Find the co-ordinates of the point that divides the interval PQ externally in the ratio 3 : 4 if P is the point (2, 5) and Q is the point (-1, 0).

(e) Find the limiting sum of

$$1 + \sin^2 x + \sin^4 x + \ldots\ldots \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

QUESTION 2

(a) A circular oil slick lies on the surface of calm water. Its area is increasing at the rate of 12 m^2/min. At what rate is the radius increasing at the time at which the radius is 3 metres?

(b) Use the substitution $u = \sin x$ to show that

$$\int_0^{\frac{\pi}{6}} \frac{\cos x.\ dx}{4\sin^2 x + 1} = \frac{\pi}{8}$$

(c) PQR is an equilateral triangle. QR is produced to T so that $RT = \frac{1}{3} QR$.

If $RT = x$ units, prove that $PT = x\sqrt{13}$ units.

(d) Find the derivative, with respect to x, of:

(i) $\log(x^3 + 1)$

(ii) $e^{x^2} \cos 4x$

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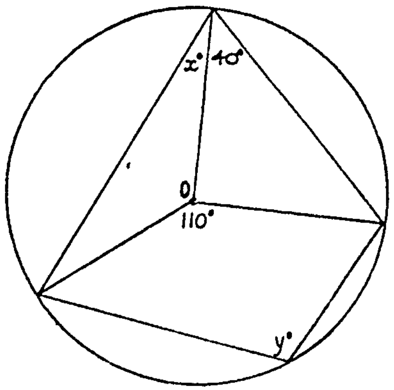
QUESTION 3

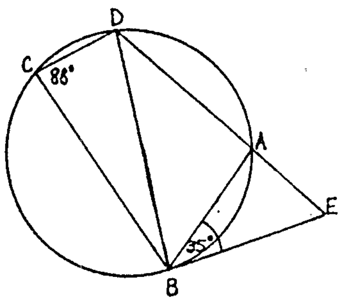
(a) (i) Find the value of the pronumerals x and y, giving reasons for your answers.

(O is the centre of the circle)



(ii) If $\angle BCD = 88^0$ and $\angle EBA = 35^0$, find $\angle BAE$ and $\angle BDE$, giving reasons for your answers.

(BE is a tangent)



(b) Find all values of θ in the range $0 \leqslant \theta \leqslant 360^0$ for which

$$3\cos\theta + \sqrt{3}\sin\theta = \sqrt{3}$$

(c) Solve the equation $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 4 = 0$

Expand and simplify your values of x, leave as surds.

QUESTION 4

(a) The equation $x^2 = 1 - x$ has approximately the solution $x = 0.5$. Use one application of Newton's Method to obtain a better approximation.

(b) Evaluate exactly $\displaystyle\int_0^1 \left(e^{-x} + \frac{1}{1+x} + \frac{1}{\sqrt{1-x^2}}\right) . dx$

(c) In how many ways can a train of ten carriages be arranged if four of the carriages

(i) are to be kept in a given order?

(ii) must be kept together but in any order?

(d) If $y = \left(\frac{e}{2}\right)^x$ show that $\frac{1}{y} \cdot \frac{dy}{dx} = 1 - \log_e 2$

(e) Find the limit of $\dfrac{\sin 4h}{\tan 5h}$ as h approaches 0.

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QUESTION 5

$0.6x^2$

$\frac{1}{2}$

$\frac{1 \pm \sqrt{1-4(0.8)(0.6)}}{2(0.5)}$

(a) Solve $\frac{x^2 + 6}{x} < 5$

$0.5x^2 + 0.5 - x = 0$

(b) Consider $f(x) = \frac{x}{x^2 + 1}$

$0.9x^2 + 0.9 - x = 0$

$\frac{1 \pm \sqrt{1-4(0.9 \times 0.9)}}{2(0.9)}$

(i) Prove that $f(x)$ is an odd function.

(ii) Find the co-ordinates and nature of its turning points.

(iii) Find the range of the function.

(iv) Hence or otherwise, sketch $y = f(x)$.

(v) Find the area enclosed by the curve, the x – axis and the lines $x = -1$ and $x = 1$.

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QUESTION 6

(a) The rate at which an object changes temperature is proportional to the difference between its temperature and that of the surrounding medium, that is:

$$\frac{dT}{dt} = -k\,(T - M)$$

where T is the temperature at any time t and M is the temperature of the surrounding medium (a constant).

(i) Show that the temperature, T, of the body at any time t is given by the formula

$$T = M + Ae^{-kt}$$

(ii) A metal bar has a temperature of 1230°C and cools to 1030°C in 10 minutes, when the surrounding temperature is 30°C. How long will it take to cool to 80°C?

(b) The tangent at P $(2ap, ap^2)$ on the parabola $x^2 = 4ay$ meets the x-axis in T. The normal at P meets the y-axis in N.

(i) Find the co-ordinates of M, the midpoint of TN.

(ii) Show that the locus of M is the parabola $x^2 = \frac{a}{2}(y - a)$

QUESTION 7

(a) The coefficient of x in the expansion of $\left(x + \frac{1}{ax^2}\right)^7$ is $\frac{7}{3}$.

Find all the possible values of 'a'.

(b) Sketch the graph of $y = 4\sin^{-1}(2x + 1)$, stating its largest possible domain and range.

(c) Prove by Mathematic Induction that

$$3^n \geqslant 1 + 2n \qquad \text{for } n \geqslant 0$$

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NSGHS Trial HSC 1990 3U Maths

Question 1

(a) $\dfrac{\tan 5x - \tan x}{1+\tan 5x \tan x}$ let $A = 5x$, $B = x$

Now $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$ ✓

$\therefore \dfrac{\tan 5x - \tan x}{1+\tan 5x \tan x} = \tan(5x - x)$

$= \tan 4x$ ✓

(b) $2y - x + 1 = 0$ $\quad y = 5x+2$

$y = \frac{x-1}{2}$ $\quad \frac{dy}{dx} = 5$ ✓

$\therefore \frac{dy}{dx} = \frac{1}{2}$ ✓

$\tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1m_2}\right|$

$= \left|\dfrac{\frac{1}{2}-5}{1+\frac{1}{2}\times 5}\right|$

$\theta = 52°8'$ ✓

(c) $2x^3+5x-3=0$ $\quad \alpha, \beta, \gamma$ are roots

let $y = \frac{1}{x}$ $\therefore x = \frac{1}{y}$ sub in eq^n

$\frac{2}{y^3} + \frac{5}{y} - 3 = 0$ multiply by y^3 ✓

$\therefore 2 + 5y^2 - 3y^3 = 0$ ✓

$\therefore$ req'd eq^n is $3x^3 - 5x^2 - 2 = 0$

(when reverted back to x)

(d) $x = \left(\dfrac{mx_2+nx_1}{m+n}\right) = \dfrac{3(1)+4(2)}{3+4}$ ← question says

$= \frac{5}{7}$

$y = \dfrac{my_2+ny_1}{m+n} = \dfrac{3(0)+4(5)}{3+4}$ EXTERNAL division

$= 2\frac{6}{7}$ $\quad \therefore P\left(\frac{5}{7}, \frac{20}{7}\right)$

(e) $1 + \sin^2 x + \sin^4 x + \ldots$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$a = 1 \quad r = \sin^2 x$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\sin^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$ ✓

Question 2

(a) $\frac{dA}{dt} = 12\text{m}^2/\text{min}$ ✓

Find $\frac{dr}{dt}$ when $r = 3$m

circular oil slick = sphere? If so,

$A = \frac{4}{3}\pi r^2$ $\quad \frac{dA}{dr} = \frac{8}{3}\pi r$

$\frac{dr}{dt} = \frac{dA}{dt} \cdot \frac{dr}{dA}$ $\quad \frac{dr}{dA} = \frac{1}{8\pi}$

$= 12 \times \frac{1}{8\pi} = \frac{3}{2\pi}$ m/min.

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos x\,dx}{4\sin^2 x + 1} = \frac{\pi}{8}$ $\quad u = \sin x$, $du = \cos x\,dx$ ✓

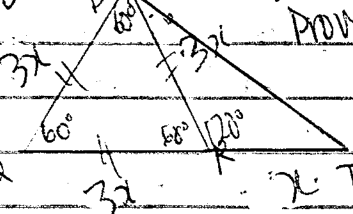
$\displaystyle\int_0^{0.5} \frac{du}{4u^2+1}$ ✓

$\displaystyle\frac{1}{4}\int_0^{0.5} \frac{du}{u^2+\frac{1}{4}}$

$= \frac{1}{4}\times 2\left[\tan^{-1} 2u\right]_0^{\frac{1}{2}}$ ✓

$= \frac{1}{2}\left[\frac{\pi}{4}\right] = \frac{\pi}{8}$ as req^{id}

(c) Prove $PT = x\sqrt{13}$



$(PT)^2 = (3x)^2 + x^2 - 2(3x)(x)\cos 120°$

$(PT)^2 = 10x^2 - 6x^2\cos 120°$

$= 10x^2 - 6x^2\left(-\frac{1}{2}\right)$ ✓

$= 13x^2$ ✓

$\therefore PT = \sqrt{13}\,x$ units as req'd

(d)(i) $\log(x^3+1)$ $\quad$ let $u = x^3+1$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ $\quad \frac{du}{dx} = 3x^2$

$= \frac{1}{u}\times 3x^2$

$= \frac{3x^2}{x^3+1}$ ✓

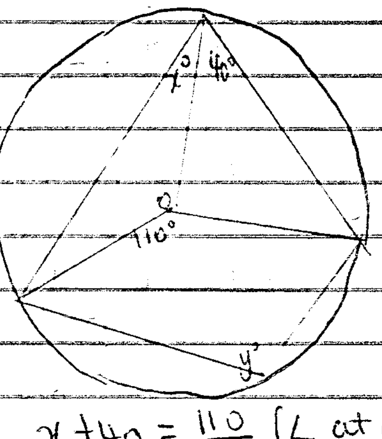
(ii) $e^{x^2}\cos 4x$

$= e^{x^2}\frac{\sin 4x}{4} + \cos 4x(2xe^{x^2})$ ✓

$= e^{x^2}\left[\frac{\sin 4x}{4} + 2x\cos 4x\right]$ ✓

Question 3

(a)(i)



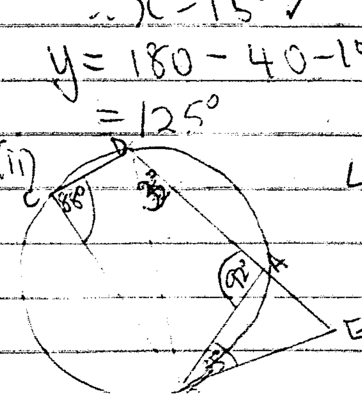
$x + 40 = \frac{110}{2}$ (∠ at centre ✓ twice angle at circumference)

$\therefore x = 15°$ ✓

$y = 180 - 40 - 15$ (opp ∠s of cyclic quad are supplementary)

$= 125°$

(ii)



∠BDE = 35° (∠ in alt. segment)

∠DAB = 180 − 88 = 92° (opp ∠ of cyclic quad = 180°)

∠BAE = 88° (sup

for surds coeff. choose $R\cos(\theta-\alpha)$ method is easier $\Rightarrow 2\sqrt{3}\cos(\theta-\frac{\pi}{6})=\sqrt{3}$

$\cos(\theta-\frac{\pi}{6})=\frac{1}{2}$ for $-30^\circ \le \theta-30^\circ \le 330^\circ$

(b) $0 \le x \le 360^\circ$, $3\cos\theta+\sqrt{3}\sin\theta=\sqrt{3}$

let $\tan\frac{\theta}{2}=t$, $\sin\theta=\frac{2t}{1+t^2}$, $\cos\theta=\frac{1-t^2}{1+t^2}$

Too long!

$3\left(\frac{1-t^2}{1+t^2}\right)+\sqrt{3}\left(\frac{2t}{1+t^2}\right)=\sqrt{3}$ ✓

$3-3t^2+2\sqrt{3}t=\sqrt{3}+\sqrt{3}t^2$

$(\sqrt{3}+3)t^2-2\sqrt{3}t+\sqrt{3}-3=0$

$t=\frac{2\sqrt{3}\pm\sqrt{12-4(\sqrt{3}+3)(\sqrt{3}-3)}}{2(\sqrt{3}+3)}$

$=\frac{2\sqrt{3}\pm\sqrt{12-4(3-9)}}{2\sqrt{3}+6}$

$=\frac{2\sqrt{3}\pm 6}{2(\sqrt{3}+3)}=\frac{\sqrt{3}\pm 3}{\sqrt{3}+3}=1$ or $\frac{\sqrt{3}-3}{\sqrt{3}+3}$

Now $\frac{\sqrt{3}-3}{\sqrt{3}+3}\times\frac{\sqrt{3}-3}{\sqrt{3}-3}=\frac{3-6\sqrt{3}+9}{3-9}$

$=\frac{12-6\sqrt{3}}{-6}=-2+\sqrt{3}$

$\therefore \tan\frac{\theta}{2}=\sqrt{3}-2$ or 1

Note $0 \le \theta \le 360^\circ$

$\frac{\theta}{2}=n\pi+\tan^{-1}(1)$ or

$\frac{\theta}{2}=n\pi+\tan^{-1}(\sqrt{3}-2)$

$\theta=\frac{\pi}{2}$, ✓ $\frac{11\pi}{6}$ ✓ $90^\circ, 330^\circ$

(c) $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-4=0$

let $u=x^{\frac{1}{3}}$

$\therefore u^2-2u-4=0$

$u=\frac{2\pm\sqrt{4-4(-4)}}{2}=\frac{2\pm2\sqrt{5}}{2}$

$=1\pm\sqrt{5}$ ✓

$\therefore x^{\frac{1}{3}}=1\pm\sqrt{5}$ ✓

$x=16+8\sqrt{5}$ or $16-8\sqrt{5}$

Question 4.

(a) $x^2=1-x$ root near $x_1=0.5$

$x^2+x-1=0$

$\therefore x_2=x_1-\frac{f(x_1)}{f'(x_1)}$ ✓

$f(0.5)=-\frac{1}{4}$ ✓

$f'(x)=2x+1$

$f'(0.5)=2$

$x_2=0.5-\frac{-0.25}{2}$ ✓

$\therefore x_2=0.625$ ✓

(b) $\int_0^1\left(e^{-x}+\frac{1}{1+x}+\frac{1}{\sqrt{1-x^2}}\right)dx$

$=\left[-e^{-x}+\ln(1+x)+\sin^{-1}x\right]_0^1$ ✓

$=\left[-\frac{1}{e}+\ln 2+\frac{\pi}{2}\right]-\left[-1+0+0\right]$ ✓

$=1-\frac{1}{e}+\ln 2+\frac{\pi}{2}$ ✓

(c)(ii) $7!\times 4!=20160$ ✓

(i) $6!$

(d) $y=\left(\frac{e}{2}\right)^x$ show $\frac{1}{y}\frac{dy}{dx}=1-\log_e 2$

$y=\frac{e^x}{2^x}$

$\frac{dy}{dx}=\frac{2^xe^x-e^x2^x\ln 2}{2^{2x}}$

$=\frac{2^xe^x(1-\ln 2)}{2^{2x}}$ ✓ $=\frac{e^x(1-\ln 2)}{2^x}$ ✓

Now LHS $=\frac{1}{y}\frac{dy}{dx}$ ✓

$=\frac{2^x}{e^x}\times\frac{e^x}{2^x}(1-\ln 2)=1-\ln 2=$ RHS

(e) Find the limit of $\frac{\sin 4h}{\tan 5h}$ as $h\to 0$

$\lim\frac{f(x)}{g(x)}=\frac{L}{M}$ where L is limit of $f(x)$ & M is limit of $g(x)$

$\therefore \lim_{h\to0}\sin 4h=4\frac{4}{5}\times\lim_{h\to0}\frac{\sin 4h}{4}\times\frac{5}{\tan 5h}$

$\lim_{h\to0}\tan 5h=5$

$\therefore \lim_{h\to0}\frac{\sin 4h}{\tan 5h}=\frac{4}{5}$ ✓

Question 5:

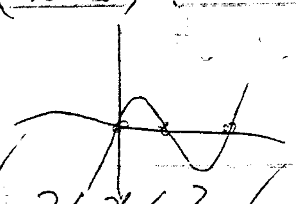
(a) $\frac{x^2+6}{x}<5$ multiply by x^2

$x(x^2+6)<5x^2$

$x^3+6x-5x^2<0$

$x(x^2-5x+6)<0$

$x(x-2)(x-3)<0$ ✓



$x<0$, $2<x<3$ ✓

(b)(i) $f(x)=\frac{x}{x^2+1}$

$f(-x)=\frac{-x}{x^2+1}=-f(x)$ ✓

$\therefore$ it's an odd fn.

(ii) $f'(x)=\frac{x^2+1-2x^2}{(x^2+1)^2}=\frac{-x^2+1}{(x^2+1)^2}$ ✓

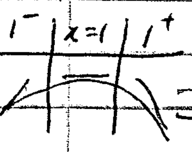
$=\frac{1}{x^2+1}-\frac{2x^2}{(x^2+1)^2}$

(ii) $f'(x)=0$ at ~~x=0~~ $x^2=1 \Rightarrow x=\pm 1$.

$-x^2+1=0$ $\therefore$ turning point at $(1, \frac{1}{2})$

~~$\frac{d^2y}{dx^2} = \frac{(x^2+1)^2(-2x)-(-x^2+1)(2[x^2+1])}{(x^2+1)^4}$~~

If $\frac{d^2y}{dx^2}$ is difficult to find, use



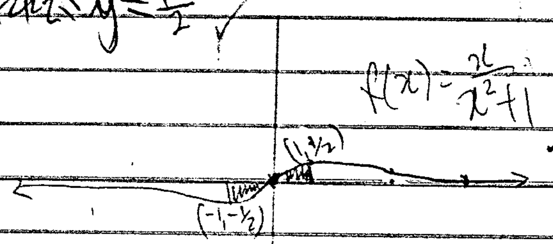
$= \frac{-2x(x^2+1)^2-4x(x^2+1)(-x^2+1)}{(x^2+1)^4}$

$= \frac{-2x}{(x^2+1)^2} - \frac{4x(-x^2+1)}{(x^2+1)^3}$

$\therefore$ concave down.

(iii) [illegible] $\le y \le \frac{1}{2}$ ✓

(iv)



✓

(v) $A=\int_{-1}^{1} \frac{x}{x^2+1}\,dx$ let $u=x^2+1$ ✓

$du = 2x\,dx$

$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\left[\ln(x^2+1)\right]_{-1}^{1}$

$= \frac{1}{2}\ln 2 + \frac{1}{2}\ln 2 = \ln 2$

Question 6.

(a) i) $T = M + Ae^{-kt}$ $\therefore Ae^{-kt} = T-M$ ✓

$\frac{dT}{dt} = -kAe^{-kt} = -k(T-M)$ as req$^{\text{ed}}$

(ii) $M=30$ when $t=0$ $T=1230$

$1230 = 30 + A$ $\therefore A=1200$ ✓

when $t=10$ $T=1030$

$1030 = 30+1200e^{-k10}$

$\therefore k = \frac{\ln\frac{5}{6}}{-10} = 0.018232155$ ✓

When $T=80$ $t=?$

$80 = 30+1200e^{-kt}$ ✓

$t = \frac{\ln\frac{1}{24}}{-0.018232155} = 174$ min 19 seconds

(b) i) $x^2=4ay$ $P(2ap, ap^2)$ should draw a diagram.

$\frac{dy}{dx} = \frac{x}{2a} = p$

$\therefore$ eq^n of tangent at P

$(y-ap^2) = p(x-2ap)$

$y = px - ap^2$ $T(x, 0)$

$\therefore x = ap$ $\therefore T(ap, 0)$ ✓

$m_{normal} = -\frac{1}{p}$

eq^n of normal $(y-ap^2) = -\frac{1}{p}(x-2ap)$

$x+py = 2ap+ap^3$

$N(0,y)$ $\therefore y = 2a+ap^2$ ✓

$\therefore$ Midpoint TN $x = \frac{ap+0}{2} = \frac{ap}{2}$

$y = \frac{a}{2}(2+p^2)$ ✓

$M\left(\frac{ap}{2}, \frac{a}{2}(p^2+2)\right)$

(ii) $x = \frac{ap}{2}$ $p = \frac{2x}{a}$ ①

$y = \frac{a}{2}(p^2+2)$ ② sub ① in ②

$y = \frac{a}{2}\left(\frac{4x^2}{a^2}+2\right)$ ✓

$= \frac{2x^2}{a} + a$

$(y-a)\frac{a}{2} = x^2$

$\therefore$ locus of M is parabola

$x^2 = \frac{a}{2}(y-a)$ ✓

Question 7

(a) —

(b) $y = 4\sin^{-1}(2x+1)$

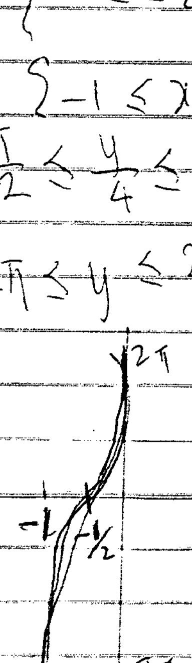
$D_x = \{-1 \le 2x+1 \le 1\}$

$= \{-2 \le 2x \le 0\}$

$= \{-1 \le x \le 0\}$ ✓

$R_y: \left\{-\frac{\pi}{2} \le \frac{y}{4} \le \frac{\pi}{2}\right\}$

$\{-2\pi \le y \le 2\pi\}$ ✓



(c) Step 1: Prove true for $n=1$

$3 \geq 1+2$ true for $n=1$

Step 2: Assume true for $n=k$

ie $3^k \geq 1+2k$

Step 3: Prove true for $n=k+1$

ie RTP $3^{k+1} \geq 1+2(k+1) = 3+2k$

$3 \cdot 3^k \geq 3(1+2k)$ (from assumption)

Now since $3+6k \geq 3+2k$

$\therefore 3^{k+1} \geq 1+2(k+1)$

Step 4: since true for $n=1$

& assumed true for $n=k$

& proven true for $n=k+1$

$\therefore$ true for $n=1+1=2$ &

so on for all positive integers, n

Test true for inequality $n=2$

$\therefore$ LHS $= 3^2 = 9$

RHS $= 5$

$\therefore 3^2 > 1+2\times 2$