NORTH SYDNEY BOYS HIGH SCHOOL

2007

ASSESSMENT TASK 2

Mathematics Extension 1

Examiner: S. Ireland

General Instructions

- Working time – 60 minutes
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)
- ☑ Mr Lowe
- ○ Mr Rezcallah
- ○ Mr Barrett
- ○ Mr Trenwith
- ○ Mr Weiss
- ○ Mr Ee

Student Number:

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	Total	Total
Mark	/12	/11	/4	/9	/5	/6	/9	/56	/100

Question 1 (12 marks)

a) Find a primitive of each of the following **6**

(i) $2+\dfrac{x^2}{4}$ (ii) $\dfrac{5x^2-2\sqrt{x}}{\sqrt{x}}$ (iii) $\left(t+\dfrac{4}{t}\right)^2$

b) Evaluate this integral, giving the answer in exact form: **3**

$$\int_0^5 \frac{3}{2\sqrt{x+1}}dx$$

c) Solve the inequality $\dfrac{4x+3}{x-4}\geq 1$ **3**

Question 2 (11 marks)

a) Differentiate $\dfrac{2x^2+1}{3x^2-4}$ and hence evaluate $\displaystyle\int_0^1 \frac{x}{(3x^2-4)^2}dx$ **4**

b) At any point on a curve $y=f(x)$, $\dfrac{d^2y}{dx^2}=6x-8$.

If $\dfrac{dy}{dx}=7$ and $y=1$ when $x=0$, find y in terms of x. **4**

c) Given that **3**

$$\int_2^k (x-5)dx=0$$

find the value of k.

Question 3 (4 marks)

Using mathematical induction, prove that **4**

$$1^2+3^2+....+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1) \quad \text{for } n\geq 1$$

Question 4 (9 marks)

a) The chances that golfers Don, Jon and Ron will qualify for a certain tournament are $\frac{1}{7}$, $\frac{1}{10}$ and $\frac{1}{4}$ respectively. Find the probability that

(i) none qualifies 2

(ii) at least one qualifies. 2

b) In a group of 35 students, 16 play cricket, 10 play futsal, and 14 play neither sport. If a student is selected at random, what is the probability he plays only futsal? 2

c) The probability that a particular type of missile will hit a certain target is $\frac{1}{4}$. How many such missiles need to be fired in order to be more than 90% certain of hitting the target? 3

Question 5 (5 marks)

The function $f(x) = \sqrt[4]{4-x^2}$ is defined in the domain $0 \le x \le 2$

a) Draw up a table of values of $f(x)$ correct to 2 decimal places for 2

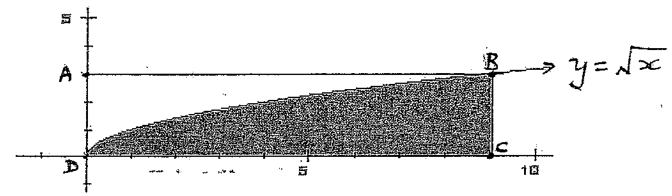
$x = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2.$

b) Use Simpson's rule with five function values to estimate the area between the curve $y = f(x)$, the x-axis and the y-axis. Give your answer correct to 2 decimal places. 3

Question 6 (6 marks)

a) Find the volume of the solid obtained when the region between the curves $y = x^2$ and $y = x^3$, from $x = 0$ to $x = 1$, is rotated about the x-axis. 3

b) A point is selected at random within rectangle ABCD in the diagram below. What is the probability that it lies within the shaded region? 3



Question 7 (9 marks)

a) Find the x-intercept(s) and y-intercept(s) of the curve $y = 4 - \sqrt{2x}$ 2

b) Find the exact area of the region enclosed by the curve $y = 4 - \sqrt{2x}$, the x-axis and the y-axis. 3

c) The region enclosed by the curve $y = 4 - \sqrt{2x}$, the x-axis and the y-axis is rotated about the y-axis. Find the exact volume of the solid of rotation so formed. 4

Solution of Ext. 1 Task 2 − 2007

(b) $A \doteqdot \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

apply Simpson's rule twice :−

$\therefore A \doteqdot \frac{1}{6}\left(1{\cdot}41 + 4(1{\cdot}39) + 1{\cdot}32\right) + \frac{1}{6}\left(1{\cdot}32 + 4(1{\cdot}15) + 0\right)$ ✓✓ correct algorithm

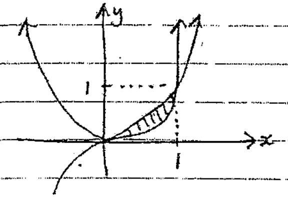
$= \frac{14{\cdot}21}{6}$

$= 2{\cdot}368\dot{3}$

$= 2{\cdot}37$ units^2 (to 2 dec. places) ✓ correct answer to 2 decimal places.

Question 6 :

(a)



$V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^1 x^4\,dx - \pi\int_0^1 x^6\,dx$ ✓

$= \pi\int_0^1 (x^4 - x^6)\,dx$

$= \pi\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1$ ✓

$= \pi\left(\frac{1}{5} - \frac{1}{7}\right)$

$= \frac{2\pi}{35}$ units^2 ✓

(b)

Area of shaded region $= \int_0^9 x^{\frac{1}{2}}\,dx$

$= \left[\frac{2}{3}x^{3/2}\right]_0^9$ ✓

$= \frac{2}{3}\cdot 9^{3/2}$

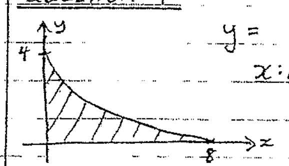
$= 18$ ✓

Area ABCD $= 9\times 3 = 27$

$\therefore$ P(point in shaded region) $= \frac{18}{27}$

$= \frac{2}{3}$ ✓

Question 7 :



$y = 4 - \sqrt{2x}$

x:int : $y=0 \therefore 4 = \sqrt{2x}$

$\therefore x = 8$

y.int : $x=0$

$\therefore y = 4$

Solution of Ext. 1 Task 2 − 2007

(b) $A = \int_0^8 \left(4 - \sqrt{2}\cdot x^{\frac{1}{2}}\right)dx$ ✓

$= \left[4x - \frac{2\sqrt{2}}{3}x^{\frac{3}{2}}\right]_0^8$

$= 32 - \frac{2\sqrt{2}}{3}\cdot 8^{\frac{3}{2}}$ ✓

$= 32 - \frac{2\sqrt{2}}{3}\cdot(2\sqrt{2})^3$

$= 32 - \frac{16(4)}{3}$

$= 10\frac{2}{3}$ units^2 ✓

OR. $\sqrt{2}x = 4 - y \Rightarrow x = \frac{(4-y)^2}{2}$

$A = \int_0^4 x\,dy$

$= \frac{1}{2}\int_0^4 (4-y)^2\,dy$

$= \frac{1}{2}\left[\frac{(4-y)^3}{-3}\right]_0^4$

$= -\frac{1}{6}\left[(4-y)^3\right]_0^4$

$= -\frac{1}{6}\left[0 - 4^3\right]$

$= \frac{32}{3}$

Note : $\int \sqrt{2x} \neq \frac{2(2x)^{3/2}}{3}$ (one mark is lost as this yields an answer of $\left|-10\frac{2}{3}\right| = 10\frac{2}{3}$

(c) $V = \pi\int_0^4 x^2\,dy$

since $\sqrt{2x} = 4 - y$

$\therefore 2x = (4-y)^2$

$x^2 = \frac{(4-y)^4}{4}$ ✓

$\therefore V = \frac{\pi}{4}\int_0^4 (4-y)^4\,dy$

$= \frac{\pi}{4}\left[\frac{(4-y)^5}{(-1)(5)}\right]_0^4$ ✓

$\therefore V = \frac{-\pi}{20}\left[(4-4)^5 - (4-0)^5\right]$

$= \frac{256\pi}{5}$ units^3 ✓ exact answer.

2nd method : Longer

or : $V = \frac{\pi}{4}\int_0^4 (4-y)^4\,dy$

$= \pi\int_0^4 \left(\frac{y^4}{4} - 4y^3 + 24y^2 - 64y + 64\right)dy$

$= \pi\left[\frac{y^5}{5} - y^4 + 8y^3 - 32y^2 + 64y\right]_0^4$

$= \pi\,(10\cdot24) = \frac{256\pi}{5}$.

N.B : volume about the x axis no marks. Unless the student did perfect working to arrive at the correct answer of $\frac{64\pi}{3}$ unit^3. That is only awarded −

Solution of Ext 1 Task 2 - 2007

Question 1:

(a) (i) $\int\left(2+\frac{x^2}{4}\right)dx = 2x + \frac{x^3}{12} + C$ — ✓✓ for $2x+\frac{x^3}{12}$

(ii) $\int \frac{5x^2 - 2\sqrt{x}}{\sqrt{x}}\,dx = \int\left(5x^{3/2} - 2\right)dx$ — ✓ for division

$= \frac{5x^{\frac{5}{2}}}{5/2} - 2x + C$

$= 2x^{\frac{5}{2}} - 2x + C$ — ✓ for correct answer

(iii) $\int\left(t+\frac{4}{t}\right)^2 dt = \int\left(t^2 + 8 + \frac{16}{t^2}\right)dt$ — ✓ for expansion

$= \frac{t^3}{3} + 8t - \frac{16}{t} + C$ — ✓ correct answer

(b) $\int_0^5 \frac{3}{2\sqrt{x+1}}\,dx = \int_0^5 \frac{3}{2}(x+1)^{-\frac{1}{2}}dx$ — ✓

$= \left[3(x+1)^{\frac{1}{2}}\right]_0^5$ — ✓ correct integration

$= 3(5+1)^{\frac{1}{2}} - 3(0+1)^{\frac{1}{2}}$

$= 3\sqrt{6} - 3$ — ✓ correct answer (in exact form)

(c) $\frac{4x+3}{x-4} \geqslant 1$

$\therefore \frac{4x+3}{x-4} - 1 \geqslant 0$, $x \neq 4$

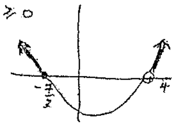
$\therefore \frac{4x+3-x+4}{x-4} \geqslant 0$

$\therefore \frac{3x+7}{x-4} \geqslant 0$

$\therefore (x-4)^2 \times \frac{3x+7}{x-4} \geqslant (x-4)^2 \times 0$ — ✓ for correct method

$(x-4)(3x+7) \geqslant 0$

$\therefore x \leqslant -\frac{7}{3}$ or $x > 4$.



✓✓ for correct answer [$x \geqslant 4$ loses 1 mark]

Solution of Ext. 1 Task 2 - 2007

Question 2:

(a) $\frac{dy}{dx} = \frac{v\,du - u\,dv}{v^2}$ where $u = 2x^2+1$

$\therefore du = 4x$

& $v = 3x^2 - 4$

$\therefore dv = 6x$

$\therefore \frac{dy}{dx} = \frac{(3x^2-4)(4x) - (2x^2+1)(6x)}{(3x^2-4)^2}$ — ✓ correct use of quotient rule

$= \frac{12x^3 - 16x - 12x^3 - 6x}{(3x^2-4)^2}$

$= \frac{-22x}{(3x^2-4)^2}$ — ✓ correct derivative

$\therefore \int_0^1 \frac{x\,dx}{(3x^2-4)^2} = \frac{-1}{22}\int \frac{-22x\,dx}{(3x^2-4)^2}$

$= -\frac{1}{22}\left[\frac{2x^2+1}{3x^2-4}\right]_0^1$ — ✓ correct definite integral

[~~Note:~~ question says "hence"...]

$= \frac{-1}{22}\left[\frac{3}{-1} - \left(-\frac{1}{4}\right)\right]$

$= \frac{-1}{22} \cdot \frac{-11}{4}$

$= \frac{1}{8}$ — ✓ correct answer

(b) $\frac{d^2y}{dx^2} = 6x - 8$

$\therefore \frac{dy}{dx} = 3x^2 - 8x + C$ — ✓

If $x=0$, $\frac{dy}{dx} = 7$, $\therefore 7 = 0+0+C$

$\therefore C = 7$ — ✓

$\frac{dy}{dx} = 3x^2 - 8x + 7$

$\therefore y = x^3 - 4x^2 + 7x + C_1$ — ✓

if $x=0$, $y=1$, $\therefore 1 = 0-0+0+C_1$

$\therefore C_1 = 1$

$\therefore y = x^3 - 4x^2 + 7x + 1.$ — ✓

Solutions of Ext.1 Task.2 - 2007

(c) $\int_2^k (x-5)\,dx = \left[\frac{x^2}{2} - 5x\right]_2^k$

$= \left(\frac{k^2}{2} - 5k\right) - (2-10)$ — ✓

$= \frac{k^2}{2} - 5k + 8$

$= 0$ (according to the question)

$\therefore k^2 - 10k + 16 = 0$

$(k-8)(k-2) = 0$

$\therefore k = 2, 8$ — ✓✓

Question 3:

To prove: $1^2+3^2+\cdots+(2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$ for $n \geq 1$

Step 1: Prove true for $n=1$.

$LHS = 1^2 = 1$

$RHS = \frac{1}{3}\cdot 1\cdot(2-1)(2+1) = 1$

$LHS = RHS$ $\therefore$ true for $n=1$ — ✓ for first step.

Step 2:- Assume true for $n=k$.

$S_k = 1^2+3^2+\cdots+(2k-1)^2 = \frac{1}{3}\cdot k(2k-1)(2k+1)$

Aim: to prove true for $n=k+1$.

ie that $1^2+3^2+\cdots+[2(k+1)-1]^2 = \frac{1}{3}\cdot(k+1)[2(k+1)-1][2(k+1)+1]$

ie that $1^2+3^2+\cdots+[2(k+1)-1]^2 = \frac{1}{3}(k+1)(2k+1)(2k+3)$

Proof: $S_{k+1} = S_k + T_{k+1}$

$\therefore 1^2+3^2+\cdots+(2k-1)^2+[2(k+1)-1]^2$

$= \frac{1}{3}\cdot k(2k-1)(2k+1) + [2(k+1)-1]^2$ by assumption — ✓ for correct start

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$

$= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)]$

$= \frac{1}{3}(2k+1)[2k^2 - k + 6k + 3]$ — ✓ for correct manipulation

$= \frac{1}{3}(2k+1)(2k^2+5k+3)$

$= \frac{1}{3}(2k+1)(2k+3)(k+1)$

$= \frac{1}{3}(k+1)(2k+1)(2k+3)$

So S_{k+1} is true if S_k is true.

Step 3: If it's true for $n=1$, it's true for $n=2$, and so on, for all $n \geq 1$ — ✓ for conclusion after correct proof.

Solution of Ext. 1 Task 2 - 2007

Question 4:

(a) (i) $P(\bar{D}) = \frac{6}{7}$, $P(\bar{J}) = \frac{9}{10}$, $P(\bar{R}) = \frac{3}{4}$

$\therefore P(\text{none qualifies}) = \frac{6}{7}\cdot\frac{9}{10}\cdot\frac{3}{4}$ — ✓

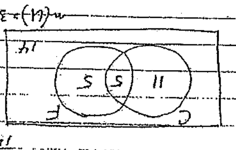
$= \frac{81}{140}$ — ✓

(ii) $P(\text{at least one qualifies}) = 1 - P(\text{none qualifies})$

$= 1 - \frac{81}{140}$ — ✓

$= \frac{59}{140}$ — ✓

(b)



5 students play only futsal — ✓

$\therefore P(\text{student plays only futsal}) = \frac{5}{35}$

$= \frac{1}{7}$ — ✓

(c) $P(\text{a missile hits}) = \frac{1}{4}$

$\therefore P(\text{a missile misses}) = \frac{3}{4} = 0.75$

for n missiles, $P(\text{all miss}) = (0.75)^n$

$\therefore P(\text{at least 1 hit}) = 1 - (0.75)^n$ — ✓

We want $P(\text{at least 1 hit}) > 0.9$

$\therefore 1 - (0.75)^n > 0.9$

$\therefore (0.75)^n < 0.1$ — ✓

$n \log 0.75 < \log 0.1$

$\therefore n > \frac{\log 0.1}{\log 0.75}$

$\therefore n > 8.003922$

$\therefore$ need 9 missiles. — ✓

Question 5:

$f(x) = \sqrt[4]{4-x^2}$, $0 \leq x \leq 2$

x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f(x)$	1.41	1.39	1.32	1.15	0

✓✓ (deduct 1 for each 2 errors).