NORTH SYDNEY GIRLS HIGH SCHOOL

2006

TRIAL HSC EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120

Attempt Questions 1–10

All questions are of equal value

NAME:__________________ **TEACHER:**__________________

QUESTION	MARK
1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
8	/12
9	/12
10	/12
TOTAL	/120

Question 1 (12 marks) **Marks**

a) Find a value for e^2 correct to 3 significant figures. **2**

b) Solve the equation $x^2 = \frac{x}{10}$. **2**

c) Express $(2\sqrt{3}+5)^2$ in the form $a+\sqrt{b}$. **2**

d) Find a primitive of $x + \frac{1}{x}$. **2**

e) A DVD player is marked for sale at $91.30 including 10% GST. The player is on special with 20% off the marked price. What should you pay for the player after discount if you are exempt from the GST? **2**

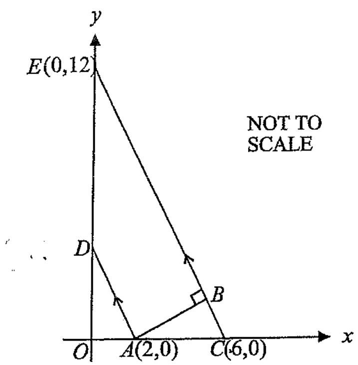
f) Solve $|x-2| \geq 3$ **2**

Question 2: Begin a new booklet. (12 marks) Marks

a) Given $f(x)=\begin{cases}-2-x & \text{for } x<2,\\ (x-2)^2 & \text{for } x\geq 2\end{cases}$

evaluate $3f(5)-f(-1)$ 2

b)



In the diagram above, A, C and E are the points $(2,0)$, $(6,0)$ and $(0,12)$ respectively. The line AD is parallel to the line CE and the line AB is perpendicular to the lines AD and CE.

i) Show that the equation of the line CE is $y=-2x+12$. 2

ii) Find coordinates of the point D. 2

iii) Show that the perpendicular distance from A to the line CE is $\frac{8\sqrt{5}}{5}$. 2

iv) Find the lengths of AD and CE. 2

v) Hence or otherwise, find the area of the trapezium $ACED$. 2

Question 3: Begin a new booklet. (12 marks) Marks

a) Differentiate with respect to x:

i) $x\ln x$ 2

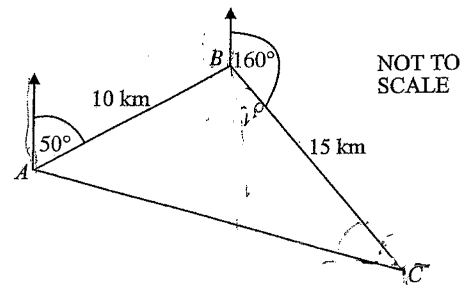
ii) $\sin e^{2x}$ 2

b) Find:

i) $\int_{\sqrt{13}}^{5}\frac{dx}{\sqrt{x^2-9}}$ using the table of standard integrals. 2

ii) $\int\frac{\sec^2 2x}{1+\tan 2x}dx$ 2

c) A marathon runner runs 10 km from point A to point B on the bearing of 050° in relation to point A. He then runs a further 15 km to point C on the bearing of 160° in relation to point B.



i) Show $\angle ABC=70^\circ$. 1

ii) Show $AC^2=25(13-12\cos 70^\circ)$. 1

iii) Find the bearing of A from C. Express your answer to the nearest whole degree. 2

Question 4: Begin a new booklet. (12 marks) Marks

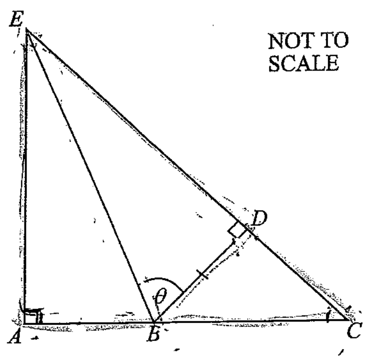
a) i) Find all the values of θ for which

$$4\cos\theta = 3 \text{ where } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

Express your answer(s) in radian measure correct to two decimal places. **2**

ii) Hence solve $4\cos^2\theta + \cos\theta = 3$ for $0 \le \theta \le 2\pi$ **2**

b) In the diagram, ACE is a right-angled triangle. The point B lies on AC and the point D lies on CE. Also $\angle BDE = 90^\circ$, $AB = BD$ and $\angle DBE = \theta^\circ$.



i) Show that $\Delta ABE \equiv \Delta DBE$. **2**

ii) Show that $\angle ACE = 2\theta - 90^\circ$. **1**

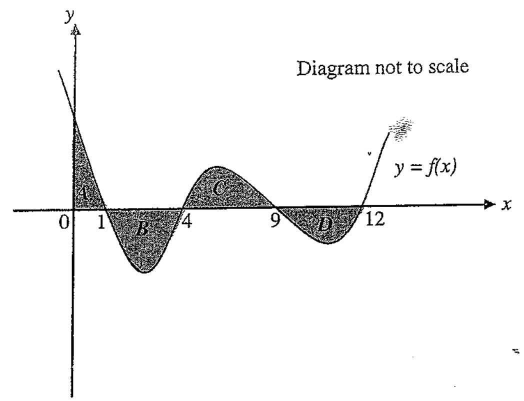
iii) Show that $\Delta ACE \parallel\!| \Delta DCB$. **2**

iv) Hence show that $EA : AB = CE : CB$ **1**

Question 4 continues on the next page.

Question 4 (cont'd) Marks

c)



The graph of the function $y = f(x)$ is shown in the diagram above.
The area of shaded region B is twice the area of the shaded region A. The areas of shaded regions C and D are equal.

Write an alternative, equivalent expression for $\int_0^{12} f(x)\,dx$ in terms of one integral of $f(x)$. **2**

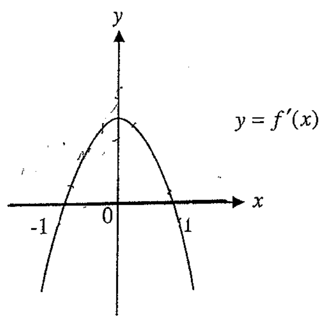
Question 5: Begin a new booklet. (12 marks) Marks

a) Solve $\log_8(x+2)-\log_8(x-1)=\frac{4}{3}$ **4**

b) If $\log_a 2 = x$ and $\log_a 3 = y$, express $\log_a\sqrt{12}$ in terms of x and y in simplest form. **3**

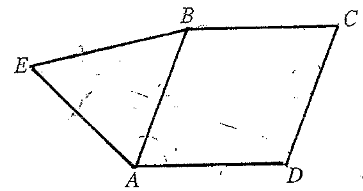
c) Find the values of k for which the equation $kx^2-(k+1)x=-1$ has two distinct roots. **2**

d) By considering the graph of $y=f'(x)$ below, sketch $y=f(x)$ given that it passes through the origin. Show clearly any turning points or points of inflexion. **3**



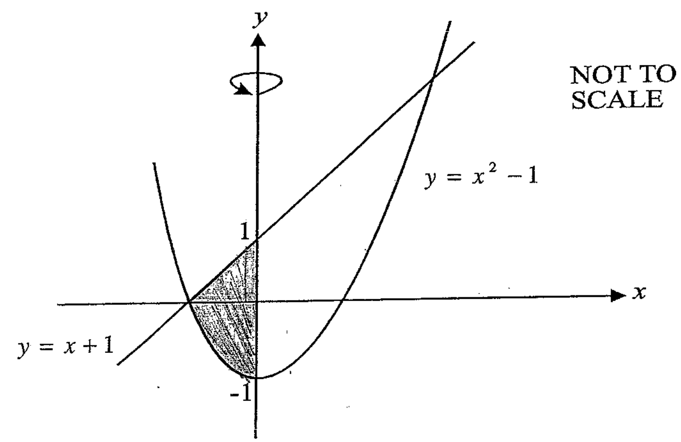
Question 6: Begin a new booklet. (12 marks) Marks

a)



$ABCD$ is a rhombus with $\angle BCD = 36^\circ$. ΔABE is equilateral. Find $\angle EDA$ giving reasons for your answer. **2**

b)



The graph of the line $y=x+1$ and the curve $y=x^2-1$ are shown on the diagram above.

i) Find the coordinates of the points of intersection of the two graphs. **1**

ii) The shaded region shown on the graph is rotated about the y-axis. By considering two integrals or otherwise find the volume of the solid formed. **3**

Question 6 continues on the next page.

Question 6 (cont'd) **Marks**

c) In a game, two coins are used. One of the coins is a large, fair coin and the other is a small, biased coin. The probability of a tail appearing on the small, biased coin is two thirds. Two players take it in turns to throw the two coins simultaneously. Throwing a head scores 1 point and throwing a tail scores 0 points. A game involves two turns by each player.

i) What is the probability that a player scores 4 points in a game? **2**

ii) What is the probability that a player scores at least 1 point in a game? **2**

iii) What is the probability that a player scores 3 points in a game? **2**

Question 7: Begin a new booklet. (12 marks) **Marks**

a) Consider the geometric series

$$1+\cos^2 x+\cos^4 x+\ldots, \text{ for } 0<x<\frac{\pi}{4}$$

i) Explain why this series has a limiting sum. **2**

ii) Find the limiting sum as $x\to\frac{\pi}{4}$. **2**

b) Given the series $\log_e 3+\log_e 9+\log_e 27+\ldots$.
Show that it is ARITHMETIC and find the sum of its first six terms as an exact value. **3**

c) The displacement of a certain particle is given by $x=5+2\sin\pi t$ where the displacement x is in metres and time t is in seconds.

i) Find an expression for the velocity of the particle at any time t. **1**

ii) At what time is the particle **first** at rest? **2**

iii) Find where the particle is when its acceleration is first $2\pi^2\text{m/s}^2$. **2**

Question 8: Begin a new booklet. (12 marks) **Marks**

a) Without using Calculus, sketch the graph of $y=2-\frac{1}{e^x}$, $x>0$ and state its range. **2**

b) Let $f(x)=\sqrt{16-x^2}$

i) Calculate $f(-4)$, $f(-2)$, $f(0)$, $f(2)$ and $f(4)$ and use your results to find $\int_0^4 f(x)dx$ using Simpson's rule. **2**

ii) Hence explain whether or not Simpson's rule gives an approximation or an exact answer for this particular function over the given interval. **1**

c) Kim is appointed manager of a bird sanctuary of 20 000 birds. The number of birds is increased by 2·5% per quarter. Just before this increase is made, B birds are sold to other establishments. This occurs each quarter.
Let A_n be the number of birds remaining at the end of the nth quarter. [Ignore changes in numbers due to natural causes.]

i) Show that $A_3=20000\times1\cdot025^3-B(1+1\cdot025+1\cdot025^2)$. **1**

ii) Show that $A_n=20000\times1\cdot025^n-40B(1\cdot025^n-1)$. **2**

iii) If there are no birds remaining after 7 years, what would be the value of B? **2**

iv) If Kim was to sell 1283 birds per quarter, how long would it take to sell all birds from the time of his appointment? **2**

Question 9: Begin a new booklet. (12 marks) Marks

a) The function $F(x)=\dfrac{e^{x}-e^{-x}}{2}$ is defined for all real values of x.

i) Show $F(x)$ is an ODD function. Find any stationary or inflexion point(s) and determine their nature. Hence sketch the curve of $y=F(x)$. **3**

ii) Find the equation of the tangent to the curve at the origin. **2**

iii) Find the area of the region bounded by the curve and the line $y=2x$ from $x=0$ to $x=1$. (Leave your answer in terms of e.) **2**

b) The number N of a certain species is falling according to $N=N_0e^{-0\cdot03t}$ where t is in days and N_0 is the initial number of species present.

i) Show that $N=N_0e^{-0\cdot03t}$ is a solution to the differential equation $\dfrac{dN}{dt}=-0\cdot03N$. **1**

ii) How long, to the nearest day, will it take for the number of species to halve? **1**

iii) Find, in terms of N_0, the rate of change at the time when the number of species has halved. **1**

iv) Find the number of days, to the nearest whole number, for the number of species to fall to just below 5% of the initial number. **2**

Question 10: Begin a new booklet. (12 marks) Marks

a) An underground wine cellar is in the shape of a rectangular prism with a floor area of $12\,\text{m}^2$ and a ceiling height of 2 m.
At 2 pm one Saturday, water begins to enter the cellar. The rate at which the volume, V, of water in the cellar changes over time t hours, is given by

$$\frac{dV}{dt}=\frac{24t}{t^2+15}$$

where $t=0$ represents 2 pm on Saturday and V is measured in cubic metres.

The cellar is initially dry.

i) Show that the volume of water in the cellar at time t is given by
$$V=12\ln\left(\frac{t^2+15}{15}\right),\ t>0$$ **2**

ii) Find the time when the cellar will be completely filled with water if the water continues to enter the cellar at the given rate. Express your answer to the nearest minute. **2**

iii) The owners return to the house and manage to simultaneously stop the water entering the cellar and start the pump in the cellar. This occurs at 6 pm on Saturday.

The rate at which the water is pumped out of the cellar is given by

$\dfrac{dV}{dt}=\dfrac{t^2}{k}$ where k is a constant.

At exactly 8 pm the cellar is emptied of water.

Find the value of k. Answer correct to 4 significant figures. **2**

Question 10 continues on the next page.

Question 10 (cont'd) Marks

b) AOB is a sector of a circle with centre at O and radius r such that $\angle AOB = \frac{\pi}{3}$. $CDEF$ is a rectangle drawn in the sector and $\angle EOF = \alpha$ as shown in the diagram.

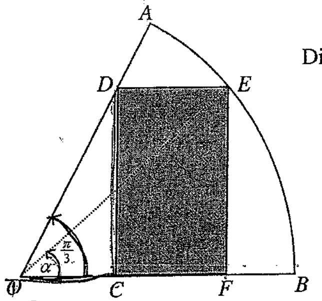


Diagram not to scale

i) Show that $\sin\alpha = \frac{DC}{r}$. **1**

ii) Use result i) and the fact that $\tan\frac{\pi}{3} = \sqrt{3}$ to show that

$$CF = r\cos\alpha - \frac{r\sin\alpha}{\sqrt{3}}$$ **1**

iii) Given that $\frac{1}{2}\sin 2\alpha = \sin\alpha\cos\alpha$, show that the area of rectangle $BDEF$ can be expressed as

$$A = r^2\left(\frac{1}{2}\sin 2\alpha - \frac{\sqrt{3}}{3}\sin^2\alpha\right)$$ **2**

iv) Find the value for α that will produce the rectangle of maximum area. **2**

END OF TEST

Q1 a) 7·39

b) $x^2 = \frac{x}{10}$
$10x^2 - x = 0$
$x(10x-1) = 0$
$x = 0$ or $\frac{1}{10}$

c) $(2\sqrt{3}+5)^2 = 12 + 20\sqrt{3} + 25$
$= 37 + \sqrt{1200}$
$a = 37$
$b = 1200$

d) $\int\left(x + \frac{1}{x}\right)dx = \frac{1}{2}x^2 + \ln x \; (+c)$

e) Sales price = \$73·04
Price without GST = \$66·40

f) $x - 2 \le -3$ OR $x - 2 \ge 3$
$x \le -1$ OR $x \ge 5$

Q2 a) $f(5) = (5-2)^2$
$= 9$
$f(-1) = -1$
$3f(5) - f(-1) = 27 + 1$
$= 28$

b) (i) $m_{CE} = \frac{0-12}{6-0}$
$= -2$

$y - 0 = -2(x-6)$
$y = -2x + 12$

(ii) $m_{AB} = -2$
$y - 0 = -2(x-2)$
$y = -2x + 4$
y intercept is 4
$A(0,4)$

(iii) $d = \frac{|2\times 2 + 1\times 0 - 12|}{\sqrt{2^2+1^2}}$
$= \frac{|-8|}{\sqrt{5}}$
$= \frac{8}{\sqrt{5}} \left(\frac{8\sqrt{5}}{5}\right)$ units

(iv) $AD = \sqrt{2^2+4^2}$ $CE = \sqrt{6^2+12^2}$
$= \sqrt{20}\;(2\sqrt{5})$ $= \sqrt{180}\;(6\sqrt{5})$

(v) $A = \frac{1}{2}\,\frac{8\sqrt{5}}{5}(2\sqrt{5}+6\sqrt{5})$
$= \frac{1}{2}\,\frac{8\sqrt{5}}{5}\cdot 8\sqrt{5}$
$= 32$ 32 square units

Q3

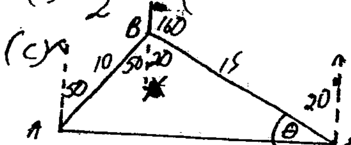
(a)(i) $1\times\ln x + x\times\frac{1}{x} = \ln x + 1$

(ii) $2e^{2x}\cos e^{2x}$

b) (i) $\left[\ln(x+\sqrt{x^2-9})\right]_{\sqrt{13}}^{5} = [\ln 9 - \ln(\sqrt{13}+2)]$
$= \ln\left(\frac{9}{\sqrt{13}+2}\right)$

(ii) $\frac{1}{2}\ln(1+\tan 2x) + C$

(c)



(i) $\angle ABC = \angle ABX + \angle XBC$ [alternate angles parallel lines / Adjacent complementary angles]
$= 50° + 20°$
$= 70°$ [Reasons not necessary]

(ii) $AC^2 = 15^2 + 10^2 - 2\times 10\times 15\cos 70°$
$= 325 - 300\cos 70°$
$= 25(13 - 12\cos 70°)$

(iii) Find θ
$\frac{\sin\theta}{10} = \frac{\sin 70°}{AC}$ ie $\frac{\sin\theta}{10} = \frac{\sin 70°}{\sqrt{25(13-12\cos 70°)}}$
$\theta = 39°4'$

Q4

a) (i)

$$\cos\theta = \frac{3}{4}$$

$$\theta \doteqdot 0.72, -0.72$$

(ii) $4\cos^2\theta + \cos\theta - 3 = 0$

$$(4\cos\theta - 3)(\cos\theta + 1) = 0$$

$$\cos\theta = \frac{3}{4}, -1$$

$$\theta = 0.72, 5.56, \pi$$

c)

$$I = \int_0^2 f(x)\,dx$$

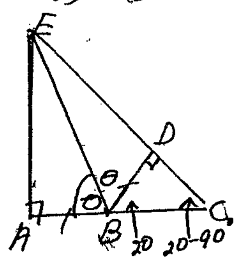
$$= A - B + C - D$$

$$= A - 2A$$

$$= -A$$

$$= -\int_0^1 f(x)\,dx$$

b) (i) In $\triangle ABE$, $\triangle DBE$



$\angle EAB = \angle EDB = 90^\circ$ [GIVEN]

EB is COMMON

AB = DB [GIVEN]

$\therefore \triangle ABE \equiv \triangle DBE$ [RHS CONGRUENCE TEST]

(ii) $\angle ABE = \theta$ [CORRESPONDING SIDES OF CONGRUENT TRIANGLES]

Since $\angle DBA = \angle BDC + \angle DCB$ [EXTERIOR ANGLE OF TRIANGLE EQUALS SUM OF INTERIOR OPPOSITES]

ie $2\theta^\circ = 90^\circ + \angle DCB$

$\therefore \angle DCB = 2\theta - 90$

ie $\angle ACE = 2\theta - 90$ [Since $\angle DCB = \angle ACE$]

(iii) In $\triangle DBC$, $\angle DBC = 2\theta^\circ$ [Result (ii) and angle sum of a triangle]

In $\triangle ACE$, $\triangle DCB$

$\angle EAC = \angle BDC = 90^\circ$ [Given]

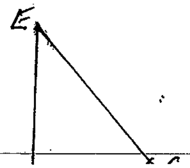
$\angle DBC = \angle AEC = 2\theta$ [Angle sum of triangles as before]

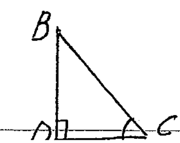
$\therefore \triangle ACE \parallel\!\mid \triangle DCB$ [Equiangular]

(iv) SINCE $\triangle ACE \parallel\!\mid \triangle DCB$

$$\frac{EA}{BD} = \frac{CE}{CB} \Rightarrow \frac{EA}{AB} = \frac{CE}{CB} \quad \left[\begin{matrix}AB = BD\\ \text{GIVEN}\end{matrix}\right]$$

$\therefore EA : AB = CE : CB$





Q5

a) $\log_8(x+2) - \log_8(x-1) = \frac{4}{3}$

$$\log_8 \frac{x+2}{x-1} = \frac{4}{3}$$

$$\frac{x+2}{x-1} = 8^{\frac{4}{3}}$$

$$\frac{x+2}{x-1} = 16 \quad (3m)$$

$$16(x-1) = x+2$$

$$16x - 16 = x + 2$$

$$15x = 18$$

$$x = \frac{6}{5} \quad (1m)$$

b) $\log_a \sqrt{12} = \frac{1}{2}\log_a 12$

$$= \frac{1}{2}\log_a [4 \times 3]$$

$$= \frac{1}{2}\log_a 4 + \frac{1}{2}\log_a 3 \quad (2m)$$

$$= x + \frac{1}{2}y \quad (1m)$$

c) $kx^2 - (k+1)x + 1 = 0$

$$\Delta = (k+1)^2 - 4.k.1$$

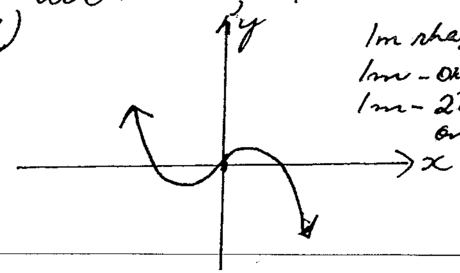
$$= k^2 + 2k + 1 - 4k$$

$$= k^2 - 2k + 1$$

$$= (k-1)^2 > 0 \text{ for } k > 1, \ k < 1$$

Since $\Delta > 0 \Rightarrow$ 2 real distinct roots for all k real, $k \neq 1$

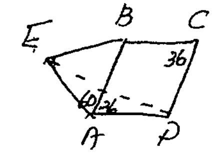
d)



1m shape
1m - origin
1m - 2 intercepts on x axis

Q6

a)



1m $\begin{cases} \angle BAD = 36^\circ & [\text{opposite angles of rhombus}] \\ \angle EAB = 60^\circ & [\triangle EAB \text{ equilateral}] \end{cases}$

$\therefore \angle EDA = \frac{180 - 96}{2}$ [$\triangle EAD$ is isosceles]

1m $= \frac{84^\circ}{2}$

$= 42^\circ$

b) (i) $y = x + 1$ ①

$y = x^2 - 1$ ②

$$x^2 - 1 = x + 1$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = -1 \text{ or } 2$$

$$(-1, 0), (2, 3)$$

(ii)

$$V = \pi\int_0^1 (y-1)^2\,dy + \pi\int_{-1}^0 (y+1)\,dy$$

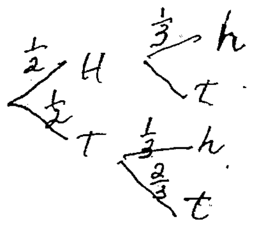
$$= \pi\left\{\left[\frac{(y-1)^3}{3}\right]_0^1 + \left[\frac{y^2}{2} + y\right]_{-1}^0\right\}$$

$$= \pi\left\{\left[0 + \frac{1}{3}\right] + \left[0 + \frac{1}{2}\right]\right\}$$

$$= \frac{5\pi}{6}$$

ie $\frac{5\pi}{6}$ CUBIC UNITS

(1) 6 c)



$P(4 \text{ points}) = P(Hh).P(Hh)$
$= \frac{1}{6} \times \frac{1}{6}$
$= \frac{1}{36}$

(ii) $P(X \geq 1) = 1 - P(X=0)$
$= 1 - P[Tt]P[Tt]$
$= 1 - \frac{1}{3} \times \frac{1}{3}$
$= \frac{8}{9}$

(iii)
$P(3 \text{ head}) = P(HhHt) + P(HhTh) + P(HtHh) + P(ThHt)$
$= \frac{1}{2}\times\frac{1}{3}\times\frac{1}{2}\times\frac{2}{3} + \frac{1}{2}\times\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{2}{3}\times\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3}$
$= \frac{1}{18} + \frac{1}{36} + \frac{1}{18} + \frac{1}{36}$
$= \frac{1}{6}$

Q7

a) (i) $r = \cos x$
Since $-1 \leq \cos x < 1$ for $0 < x < \frac{\pi}{4}$ (1m) ie
then $-1 < r \leq 1$ (1m)

(ii) $S = \frac{a}{1-r}$
$= \frac{1}{1-\cos^2 x}$ (1m)
$= \frac{1}{\sin^2 x}$
$= \operatorname{cosec}^2 x$ } (1m)
$= 2$ (at $x = \pi/4$)

b) $S = \ln 3 + 2\ln 3 + 3\ln 3$
which is arithmetic as
$T_3 - T_2 = T_2 - T_1 = \ln 3$ (1m)
$S_6 = \frac{6}{2}(\ln 3 + 6\ln 3)$
$= 3(7\ln 3)$
$= 21 \ln 3$

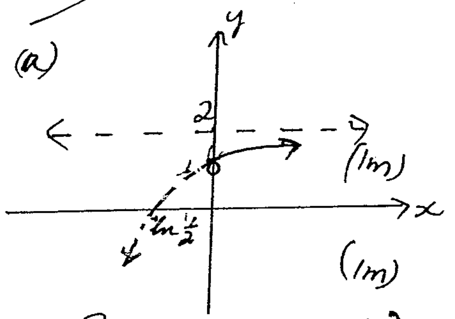
c)
(i) $x = 5 + 2\sin \pi t$
$v = 2\pi \cos \pi t$

(ii) Put $v = 0$
$2\pi \cos \pi t = 0$
$\cos \pi t = 0$
$\pi t = \pi/2$
$t = \frac{1}{2}$
ie $\frac{1}{2}$ s

(iii) $\ddot{x} = -2\pi^2 \sin \pi t$
ie $-2\pi^2 \sin \pi t = 2\pi^2$
$\sin \pi t = -1$
$\pi t = \frac{3\pi}{2}$
ie $t = \frac{3}{2}$ s
$\therefore x = 3$

Q.8

(a)



Range is $1 < y < 2$

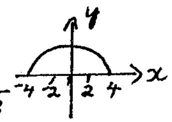
(b) (i) $f(-4) = 0$

$f(-2) = 2\sqrt{3}$

$f(0) = 4$

$f(2) = 2\sqrt{3}$

$f(4) = 0$



$$\int_0^4 f(x)\,dx \doteq \frac{4-0}{6}\{4 + 4\times 2\sqrt{3} + 0\}$$
$$= \frac{2}{3}(4 + 8\sqrt{3})$$
$$= \frac{8}{3}(1 + 2\sqrt{3}) \quad (\doteq 11.9)$$

(ii) IN THIS CASE, $f(x)$ IS NEITHER QUADRATIC (NOR CUBIC) FOR WHICH SIMPSONS RULE GIVES AN EXACT VALUE

OR

THE REGION HAS AN EXACT AREA OF $\frac{1}{4}\pi 4^2$ ie 4π AS IT IS A QUADRANT OF A CIRCLE

HENCE THE RATIONAL FUNCTION VALUES WOULD NOT PRODUCE A RESULT OF 4π.

c) $A_1 = 20000(1.025) - B$

$A_2 = [20000(1.025) - B]1.025 - B$

$= 20000(1.025)^2 - B(1 + 1.025)$

$A_3 = A_2 \times (1.025) - B$

$= [20000(1.025)^2 - B(1+1.025)]1.025 - B$

$= 20000(1.025)^3 - B(1 + 1.025 + 1.025^2)$

$A_n = 20000(1.025)^n - B(1 + 1.025 + \dots + 1.025^{n-1})$

$= 20000(1.025) - B\left(\frac{1.025^n - 1}{0.025}\right)$

$= 20000(1.025) - 40B(1.025^n - 1)$

(iii) Let $n = 28$ and $A_{28} = 0$

$0 = 20000 \times 1.025^{28} - 40B(1.025^{28} - 1)$

$B \doteq 1001$ (2)

(iv)

$0 = 20000(1.025)^n - 40\times 1283(1.025^n - 1)$

$20000(1.025)^n = 51320(1.025^n - 1)$

$31320 \times (1.025)^n = 51320$

$(1.025)^n = 1.638$

$n \ln 1.025 = \ln 1.638$

$n = 19.984$

ie 20 quarters 5 years

[TRIAL + ERROR SOLUTIONS IS ALSO ACCEPTABLE]

(b)

Question 9

a) (i) $F(x) = \dfrac{e^x - e^{-x}}{2}$

$F(-a) = \dfrac{e^{-a} - e^{-(-a)}}{2}$

$= \dfrac{e^{-a} - e^{a}}{2}$

$= -\dfrac{e^{a} - e^{-a}}{2}$

$= -F(a)$

(ii) $F'(x) = \dfrac{e^x + e^{-x}}{2}$

Put $F'(x) = 0$

$\dfrac{e^x + e^{-x}}{2} = 0$

$\left(e^x = -\dfrac{1}{e^x}\right)$ No Solution

$\therefore$ No stationary point

$F''(x) = \dfrac{e^x - e^{-x}}{2}$

$F''(x) = 0$

$\dfrac{e^x - e^{-x}}{2} = 0$

$e^x - e^{-x} = 0$

$e^x = \dfrac{1}{e^x}$

$e^{2x} = 1$

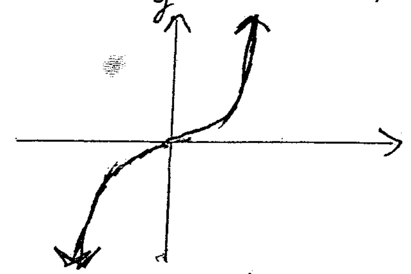
$x = 0$ ie $(0, 0)$

$F''(-1) = \dfrac{\left(\frac{1}{e} - e\right)}{2} < 0$

$F''(1) = \dfrac{e - \frac{1}{e}}{2} > 0$ $\Rightarrow$ CHANGE OF CONCAVITY

$(0,0)$ is a point of inflexion

$F'(x) > 0 \Rightarrow$ increasing function



(ii) at $(0,0)$

$F'(0) = 1$

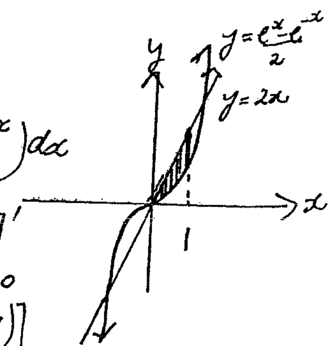
$y = x$

(iii) $A = \displaystyle\int_0^1 \left(2x - \frac{e^x - e^{-x}}{2}\right)dx$

$= \left[x^2 - \dfrac{e^x + e^{-x}}{2}\right]_0^1$

$= \left[\left(1 - \dfrac{e + e^{-1}}{2}\right) - (-1)\right]$

$= 2 - \dfrac{e + e^{-1}}{2} = 2 - \dfrac{e^2 + 1}{2e}$ sq units



b) (i) $\dfrac{dN}{dt} = -0.03 N_0 e^{-0.03t}$

$= -0.03 N$

(ii) $\frac{1}{2}N_0 = N_0 e^{-0.03t}$

$e^{0.03t} = 2$

$t = \dfrac{\ln 2}{0.03} \doteq 23$ DAYS

(iii) $\dfrac{dN}{dt} = -0.03\left(\frac{1}{2}N_0\right) = -0.0151$

(iv) OVER ONLY

iv)

$N < 0\cdot05N_0$

$N_0e^{-0.03t} < 0\cdot05N_0$

$e^{-0.03t} < \frac{1}{20}$

$\frac{1}{e^{0.03t}} < \frac{1}{20}$

$e^{0.03t} > 20$

$t > \frac{\ln 20}{0\cdot03}$

$t > 99\cdot857$

$t \doteqdot 100$ days

Question 10

a) (i) $V = 12\int \frac{2t}{t^2+15}\,dt$

$= 12\ln(t^2+15)+C$

$t=0,\ V=0$

$C = -12\ln 15$

$V = 12\ln\left(\frac{t^2+15}{15}\right)$

(ii) Volume is $24m^3$

$12\ln\left(\frac{t^2+15}{15}\right) = 24$

$\ln\left(\frac{t^2+15}{15}\right) = 2$

$t \doteqdot 9\cdot7895$

(iii) 6pm Sat, $t=4$

$\therefore V = 12\ln\left(\frac{16+15}{15}\right)$

$\doteqdot 8\cdot711244$

Then $\frac{dV}{dt} = \frac{t^2}{k}$

$V = \frac{1}{k}\int t^2\,dt$

$V = \frac{t^3}{3k} + C$

When $t=0$, $V = 8\cdot711244$

$\therefore C = 8\cdot711244\ldots$

$V = \frac{t^3}{3k} + 8\cdot711244$

at 8pm, $t=2$, $V=0$

$0 = \frac{8}{3k} + 8\cdot711244$

$k \doteqdot -0\cdot3061$

b) (i) $\sin\alpha = \frac{EF}{r}$ in $\triangle OEF$

$= \frac{DC}{r}$ [opposite sides of a rectangle]

(ii) $\cos\alpha = \frac{OF}{r}$ in $\triangle OEF$

$OF = r\cos\alpha$

Now $CF = OF - OC$ *

$= r\cos\alpha - \frac{r\sin\alpha}{\sqrt3}$

*OC: In $\triangle ODC$

$\tan\frac{\pi}{3} = \frac{DC}{OC}$

$\sqrt3 = \frac{r\sin\alpha}{OC}$

$OC = \frac{r\sin\alpha}{\sqrt3}$

(iii)

$A = CF \times EF$

$= \left(r\cos\alpha - \frac{r\sin\alpha}{\sqrt3}\right)r\sin\alpha$

$= r^2\sin\alpha\cos\alpha - \frac{r^2\sin^2\alpha}{\sqrt3}$

$= \frac{1}{2}r^2\sin 2\alpha - \frac{\sqrt3 r^2\sin^2\alpha}{3}$

$= r^2\left(\frac{1}{2}\sin 2\alpha - \frac{\sqrt3}{3}\sin^2\alpha\right)$

(iv) $\frac{dA}{d\alpha} = r^2\left(\cos 2\alpha - \frac{\sqrt3}{3}\sin 2\alpha\right)$

$\frac{dA}{d\alpha} = 0 \Rightarrow \tan 2\alpha = \sqrt3$

$\therefore \alpha = \frac{\pi}{6}$

NO TEST REQUIRED ...