NORTH SYDNEY GIRLS HIGH SCHOOL

YEAR 12 – TERM 2 ASSESSMENT

2006

MATHEMATICS EXTENSION 1

TIME ALLOWED: One Hour
Plus 2 minutes reading time

INSTRUCTIONS:

- Start each question on a new page
- Hand each question in separately, including a sheet for non-attempts
- Show all necessary working

This task is worth 20% of the HSC Assessment Mark

Question One – (10 marks)

Marks

a) Find the exact value of

(i) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ 1

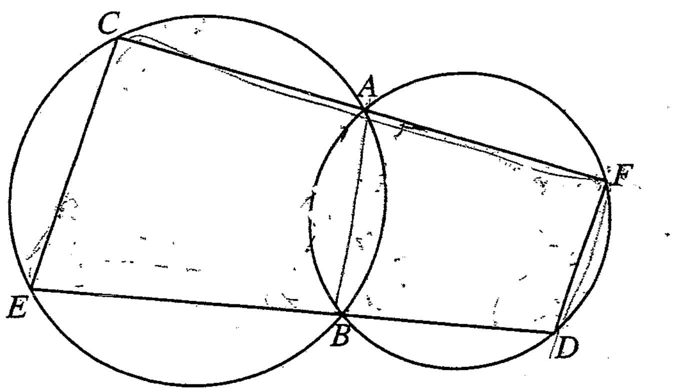
(ii) $\tan\left(\cos^{-1}\frac{1}{3}\right)$ 2

b) Find the following integrals

(i) $\int\frac{dx}{\sqrt{16-x^2}}$ 1

(ii) $\int\frac{dx}{4+3x^2}$ 2

c) Two circles intersect at A and B. CAF and EBD are straight lines.
Prove that CE is parallel to FD. 4



Question Two – (9 marks)

Point $P\ (2ap,\ ap^2)$ lies on the parabola $x^2 = 4ay$

(i) Show that the equation of the tangent to the curve at P is
$y = px - ap^2$. 2

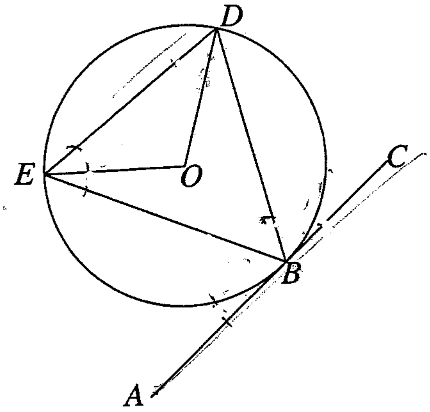
(ii) This tangent cuts the x axis at T.
Find the coordinates of T. 1

(iii) If S is the focus of the parabola prove that ST and PT are at right angles to each other. 3

(iv) Show that the locus of the centre of the circle that passes through P, S and T is the curve $2ay = a^2 + x^2$. 3

Question Three – (11 marks) **Marks**

a) ABC is a tangent at B to the circle centre O. $\angle ABE = 50°$ and $\angle BED = 65°$. Find the size of $\angle DOE$ giving reasons for your answer. **3**



b) Find the equation of the normal to the curve $y = \tan^{-1}(2x)$ at the point where $y = \frac{\pi}{4}$. **4**

c) Find the derivative of $\sin^{-1}(x-1)$ and hence evaluate **4**

$$\int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{x(2-x)}}$$

Question Four (10 marks)

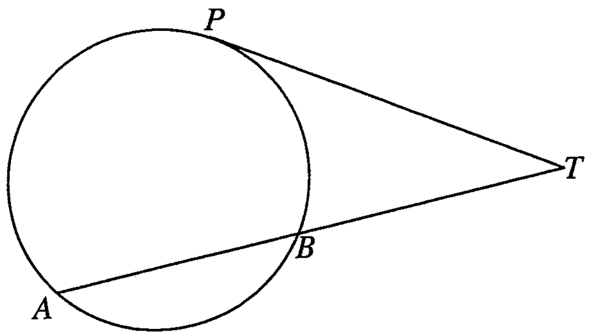
a) Solve the equation $\sin x + \cos x = 0$ for all real x. **2**

b) State the domain and range of $3y = \sin^{-1}\left(\frac{x}{2}\right)$ and sketch the curve. **3**

c) Prove, by Mathematical Induction, that $\frac{2^n - (-1)^n}{3}$ is odd for all positive integers n. **5**

Question Five (10 marks) **Marks**

a) PT is a tangent to the circle at P. $AB = 12\text{cm}$, $PT = 8\text{cm}$. Find the length of BT giving reasons for your answers. **3**



b) If $f(x) = \dfrac{x-4}{x-2}$, find $f^{-1}(x)$ and find its range. **4**

c) Show that $\tan^{-1}\left(\dfrac{1}{2}\right) - \tan^{-1}\left(\dfrac{1}{4}\right) = \tan^{-1}\left(\dfrac{2}{9}\right)$ **3**

Question Six (10 marks)

a) If $y = \dfrac{\cos^{-1}\left(\frac{x}{3}\right)}{x}$, find $\dfrac{dy}{dx}$. **3**

b) $P\ (2ap,\ ap^2)$ and $Q\ (2aq,\ aq^2)$ are two points on the parabola $x^2 = 4ay$.

i) Show that the equation of the chord PQ is $y - \left(\dfrac{p+q}{2}\right)x + apq = 0$. **2**

ii) If the chord PQ passes through the focus of the parabola show that $pq = -1$. **1**

iii) If M is the midpoint of the focal chord PQ, K is the foot of the perpendicular from M to the directrix and N is the midpoint of MK, find the equation of the locus of N. **4**

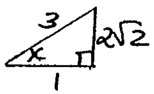
END OF TEST

SOLUTIONS

QUESTION 1 10 MARKS

a) (i) $\frac{\pi}{3}$

(ii) Let $x = \cos^{-1}\left(\frac{1}{3}\right) \Rightarrow \cos x = \frac{1}{3}$



$\tan x = 2\sqrt{2}$

b) (i) $\sin^{-1}\left(\frac{x}{4}\right) + c$

(ii) $\int \frac{dx}{4+3x^2} = \frac{1}{3}\int \frac{dx}{\frac{4}{3}+x^2}$

$= \frac{1}{3} \times \frac{\sqrt{3}}{2} \tan^{-1} \frac{x}{2/\sqrt{3}} + c$

$= \frac{\sqrt{3}}{6} \tan^{-1} \frac{\sqrt{3}x}{2} + c$

c) Join AB

$\therefore$ CABE, AFDB are both cyclic quadrilaterals

Let $\hat{ABD} = x^\circ$

$\therefore \hat{ECA} = x^\circ$ (exterior angle of cyclic quadrilateral equals interior opposite angle.)

$x^\circ + \hat{AFD} = 180^\circ$ (opposite angles of cyclic quadrilateral are supplementary)

$\therefore \hat{AFD} = (180 - x)^\circ$

$\hat{AFD} + \hat{ECA} = (180-x)^\circ + x^\circ$

$= 180^\circ$

$\therefore CE \parallel FD$ (supplementary cointerior angles).

QUESTION 2 9 MARKS

(i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

when $x = 2ap$, $\frac{dy}{dx} = p$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

(ii) $0 = px - ap^2$

$x = ap$

$T(ap, 0)$

(iii) $S(0, a)$

Slope $ST = \frac{a}{-ap}$

$= -\frac{1}{p}$

Slope $PT = \frac{ap^2}{2ap - ap}$

$= p$

$-\frac{1}{p} \times p = -1$

$\therefore ST \perp PT$

SOLUTIONS p 2

QUESTION 2 (cont)

(iv) Centre of circle is midpoint of PS

i.e. $\left(ap, \frac{ap^2+a}{2}\right)$

$x = ap \Rightarrow p = \frac{x}{a}$

$2y = a\frac{x^2}{a^2} + a$

$2ay = x^2 + a^2$

QUESTION 3 11 MARKS

a) $\hat{DBC} = \hat{BED}$ (angle between tangent and chord at point of contact equals angle in alternate segment)

$= 65^\circ$

$\hat{ABE} = 50^\circ$ (given)

$\hat{DBC} + \hat{ABE} + \hat{EBD} = 180^\circ$ (angles form straight angle)

$\therefore \hat{EBD} = 180^\circ - (50+65)^\circ$

$= 65^\circ$

$\hat{DOE} = 2 \times \hat{EBD}$ (angle at centre is twice angle at circumference standing on same arc)

$= 130^\circ$

b) $y = \tan^{-1} 2x$

$\frac{dy}{dx} = \frac{2}{1+4x^2}$

when $y = \frac{\pi}{4}$, $x = \frac{1}{2}$, $\frac{dy}{dx} = 1$

$\therefore$ Slope of normal is -1

$y - \frac{\pi}{4} = -1\left(x - \frac{1}{2}\right)$

$4y - \pi = -4x + 2$

$4x + 4y = \pi + 2$

c) $\frac{d}{dx} \sin^{-1}(x-1) = \frac{1}{\sqrt{1-(x-1)^2}}$

$= \frac{1}{\sqrt{1-x^2+2x-1}}$

$= \frac{1}{\sqrt{x(2-x)}}$

$\therefore \int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{x(2-x)}} = \left[\sin^{-1}(x-1)\right]_{1/2}^{1}$

$= \sin^{-1} 0 - \sin^{-1}\left(-\frac{1}{2}\right)$

$= 0 - \left(-\frac{\pi}{6}\right)$

$= \frac{\pi}{6}$

SOLUTIONS p3

QUESTION 4 10 MARKS

a) $\sin x = -\cos x$

$\tan x = -1$

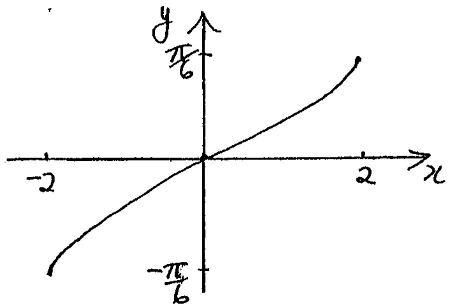
$x = n\pi + \tan^{-1}(-1)$

$= n\pi - \frac{\pi}{4}$ where n is an integer

b) Domain $-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$

Range $-\frac{\pi}{6} \le y \le \frac{\pi}{6}$



c) STEP 1 when $n = 1$,

$\frac{2^1 - (-1)^1}{3} = \frac{2+1}{3}$

$= 1$

which is odd.

STEP 2 Assume result is true for $n = k$

i.e. $\frac{2^k - (-1)^k}{3} = M$ where M is an odd integer

i.e. $2^k = 3M + (-1)^k$

Try to prove result is true for $n = k+1$

$\frac{2^{k+1} - (-1)^{k+1}}{3} = \frac{2.2^k - (-1)^k.(-1)}{3}$

$= \frac{2(3M + (-1)^k) - (-1)^k.(-1)}{3}$

$= \frac{6M + 2.(-1)^k + (-1)^k}{3}$

$= 2M + (-1)^k$

$= 2M - 1$ if k is odd

$= 2M + 1$ if k is even

and both these expressions are odd.

STEP 3: Since the result is true for $n = 1$, it is true for $n = 1+1 = 2$ and so on for all positive integers n.

SOLUTIONS p4

QUESTION 5 10 MARKS

a) $AT.TB = PT^2$ (square of length of tangent from external point equals product of intercepts of secant passing through this point)

Let $BT = x$

$(12+x)x = 64$

$x^2 + 12x - 64 = 0$

$(x-4)(x+16) = 0$

$x = 4 (-16)$

BT is 4 cm

b) Let $y = \frac{x-4}{x-2}$

Inverse $x = \frac{y-4}{y-2}$

$xy - 2x = y - 4$

$xy - y = 2x - 4$

$y = \frac{2x-4}{x-1}$

$f^{-1}(x) = \frac{2x-4}{x-1}$

Domain of $f(x)$: all real x, $x \ne 2$

$\therefore$ Range of $f^{-1}(x)$: all real y, $y \ne 2$

c) Let $x = \tan^{-1}\frac{1}{2} \Rightarrow \tan x = \frac{1}{2}$

$y = \tan^{-1}\frac{1}{4} \Rightarrow \tan y = \frac{1}{4}$

$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

$= \frac{\frac{1}{2} - \frac{1}{4}}{1 + \frac{1}{2}.\frac{1}{4}}$

$= \frac{1}{4} \times \frac{8}{9}$

$= \frac{2}{9}$

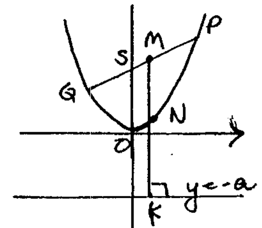
$\therefore \tan^{-1}(\frac{1}{2}) - \tan^{-1}(\frac{1}{4}) = \tan^{-1}(\frac{2}{9})$

QUESTION 6 10 MARKS

a) $\frac{dy}{dx} = \frac{x.\frac{-1}{3\sqrt{1-\frac{x^2}{9}}} - \cos^{-1}\frac{x}{3}}{x^2}$

$= \frac{-x - \sqrt{9-x^2}\cos^{-1}\frac{x}{3}}{x^2\sqrt{9-x^2}}$

b)



i) Slope $PQ = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p+q)(p-q)}{2a(p-q)}$

$= \frac{p+q}{2}$

Equation $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$y - ap^2 = (\frac{p+q}{2})x - ap^2 - apq$

$y - (\frac{p+q}{2})x + apq = 0$

ii) $(0, a)$ satisfies this equation

$a + apq = 0$

$pq = -1$

(iii) $M\left(a(p+q), \frac{ap^2+aq^2}{2}\right)$

$K\ (a(p+q), -a)$

$N\left(a(p+q), \frac{ap^2+aq^2-a}{2}\right)$

$x = a(p+q) \Rightarrow p+q = \frac{x}{a}$

$2y = \frac{ap^2+aq^2-2a}{2}$

$4y = a[(p+q)^2 - 2pq - 2]$

$= \frac{ax^2}{a^2}$ $\boxed{\text{i.e. } x^2 = 4ay}$