NORTH SYDNEY GIRLS HIGH SCHOOL

Mathematics Extension1 Assessment Task 3
Term 2 2010

Name: ______________________ **Mathematics Class:** ______________

Time Allowed: 55 minutes + 2 minutes reading time

Available Marks: 49

Instructions:

- Start each question on a new page.
- Attempt all six questions.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Write on one side of the page only. Do not work in columns.
- Questions are not of equal value.

	1ab	1c	2ab	2cd	3a	3b	3c	4a	4b	5	6	Totals
H8					2 /2							/2
H3	5 /6											/6
H5		0 /2	2 /4				3 /5		/5			/16
HE4				3 /5				/2		/8		/15
HE6						3 /3						/3
HE7											/7	/7
Totals	/8		/9		/10			/7		/8	/7	/49

Question 1 (8marks) Start a new page. **Marks**

(a) If $\log_a b = 2.8$ and $\log_a c = 4.1$, find

(i) $\log_a \dfrac{1}{\sqrt{c}}$ **1**

(ii) $a^{5.6}$ in terms of b. **1**

(b) Consider the function $y = e^{kx}$ where k is a constant

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ **2**

(ii) Determine the value(s) of k for which $y = e^{kx}$ satisfies the equation **2**

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 15y = 0.$$

(c) Find $\displaystyle\int \tan^2\left(\frac{x}{2}\right) dx$. **2**

Question 2 (9 marks) Start a new page. **Marks**

(a) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$. **2**

(b) Evaluate $\displaystyle\lim_{h\to 0}\left(\frac{h+\tan 2h}{\sin 2h}\right)$. **2**

(c) Find the exact value of $\tan\left[\sin^{-1}\left(-\dfrac{2}{3}\right)\right]$ **2**

α

(d) Consider the function $y = 2\sin^{-1}\left(\dfrac{x}{3}\right)$

(i) State its domain and range. **2**

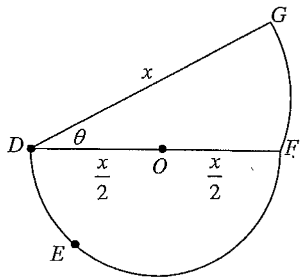
(ii) Sketch the curve. **1**

Question 3 (10 Marks). Start a new page. **Marks**

(a) Determine the value of m such that $\int_1^{2m} 3t^2\,dt = 215$ **2**

(b) Use the substitution $u = 5x + 4$ to evaluate $\int_0^1 \frac{5x}{\sqrt{5x+4}}\,dx$. **3**

(c) A solid has a uniform cross section as shown below.



The cross-section consists of a semi-circle DEF, centre O and radius $\frac{x}{2}$ and a sector of radius x, centre D and angle θ. All angles are measured in radians.

(i) What is the perimeter of $DEFG$ in terms of x and θ? **2**

(ii) If the area of the cross-section is given to be constant, then the perimeter P is found to be $P = \frac{2}{x} + x\left(1 + \frac{\pi}{4}\right)$. (Do NOT prove this) **3**

Show that the minimum perimeter of the cross-section occurs when

$$x^2 = \frac{8}{\pi + 4}$$

Question 4 (7 Marks) Start a new page **Marks**

(a) Find $\int \frac{dx}{9 + 4x^2}$ **2**

(b) Billy Bunter is 60 years only and plans to retire in 2014.
He joined a superannuation scheme at the beginning of 1974.
At the beginning of each year, he invested \$750 in the fund.
The investment earns 9% per annum compound interest, calculated yearly.

(i) Find, to the nearest dollar, the amount to which the first investment made in 1974 will have grown by the beginning of 2014 **1**

(ii) To what amount, to the nearest dollar, will the whole investment reach by the beginning of 2014? **2**

(iii) If the investment interest rate was halved after the first 20 years, what amount would the first investment of 1974 reach by the beginning 2014? **2**

Question 5 (8 Marks) Start a new page. **Marks**

Consider the function $f(x) = (1 + x)(3 - x)$

(a) Sketch the graph of this function using the same scale on both coordinate axes showing the maximum turning point and the intercepts on the x-axis. **2**

(b) The function $y = g(x)$ is defined by restricting the domain of $y = f(x)$ so that $y = g(x)$ has an inverse function. What is the largest domain containing the value $x = 0$, for which the function $y = g(x)$ has an inverse function $y = g^{-1}(x)$? **1**

(c) Sketch the graph of $y = g^{-1}(x)$ with the graph of $y = g(x)$ on a new set of axes. **3**

(d) Find the equation of the function $y = g^{-1}(x)$. **2**

Question 6 (7 Marks) Start a new page. **Marks**

Consider the function $f(x) = e^x\left(1-\frac{x}{10}\right)^{10}$ whose graph has only two turning points at $(0,1)$ and $(10,0)$.

(a) Explain why $f(x) \geq 0$ for all x. **1**

(b) Show that as x approaches negative infinity, $f(x)$ approaches zero. **1**

(c) Without using calculus, sketch the graph of $y = e^x\left(1-\frac{x}{10}\right)^{10}$. **2**

(d) Using your graph, state the range of $f(x)$ for $x < 10$. **1**

(e) From the graph, deduce that **2**

$$e^x \leq \left(1-\frac{x}{10}\right)^{-10} \quad \text{for } x < 10$$

END OF TEST

Question 1 Ext 1 - SOLUTIONS - June 2010

a) i) $\log_a \frac{1}{\sqrt{c}} = -\frac{1}{2}\log_a c$

$= -\frac{1}{2} \times 4.1$

$= -2.05$

ii) $a^{5.6} = (a^{2.8})^2$

$= 6^2$

b) i) $y = e^{kx}$

$\frac{dy}{dx} = ke^{kx}$

$\frac{d^2y}{dx^2} = k^2e^{kx}$

ii) $k^2e^{kx} + 8ke^{kx} + 15e^{kx} = 0$

$e^{kx}(k^2 + 8k + 15) = 0$

$e^{kx}(k+5)(k+3) = 0$

$k = -5$ or -3, $e^{kx} \neq 0$

c) $\int \tan^2\left(\frac{x}{2}\right) dx$

$= \int \sec^2\left(\frac{x}{2}\right) - 1 \, dx$

$= 2\tan\frac{x}{2} - x + C$

Question 2

a) $\int_0^{\pi/4} \cos^2 x \, dx$

$= \frac{1}{2}\int_0^{\pi/4} (\cos 2x + 1)\, dx$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^{\pi/4}$

$= \frac{1}{4} + \frac{\pi}{8}$ or $\frac{2+\pi}{8}$

b) $\lim_{h\to 0}\left(\frac{h}{\sin 2h} + \frac{\tan 2h}{\sin 2h}\right)$

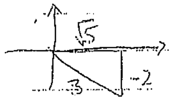
$= \lim_{h\to 0}\left(\frac{2h}{\sin 2h} \times \frac{1}{2} + \frac{1}{\cos 2h}\right)$

$= \frac{1}{2} + 1$

$= \frac{3}{2}$

c) $\tan\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$

$= -\frac{2}{\sqrt{5}}$ or $\frac{-2\sqrt{5}}{5}$



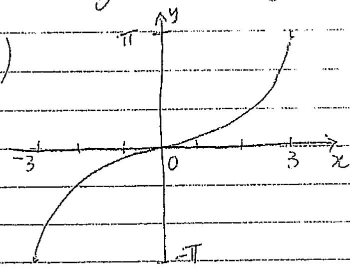
4th Quadrant angle

d) $y = 2\sin^{-1}\left(\frac{x}{3}\right)$

(i) Domain $-3 \le x \le 3$

Range $-\pi \le y \le \pi$

(ii)



Question 3

a) $\int_1^{2m} 3t^2\,dt$

$= \left[t^3\right]_1^{2m}$

$= 8m^3 - 1$

$8m^3 - 1 = 215$

$8m^3 = 216$

$m = 3$

b) $u = 5x+4$

$du = 5dx$

$\frac{1}{5}du = dx$

$x = 1, \quad u = 9$

$x = 0, \quad u = 4$

$\int_0^1 \frac{5x}{\sqrt{5x+4}}\,dx$

$= \frac{1}{5}\int_4^9 \frac{u-4}{u^{1/2}}\,du$

$= \frac{1}{5}\int_4^9 u^{1/2} - 4u^{-1/2}\,du$

$= \frac{1}{5}\left[\frac{2u^{3/2}}{3} - 8u^{1/2}\right]_4^9$

$= \frac{1}{5}\left(18 - 24 - \frac{16}{3} + 16\right)$

$= \frac{\cancel{13}14}{15}$

Q3 (cont)

c)(i) Perimeter $= \pi.\frac{x}{2} + x\theta + x$

$= x\left(\frac{\pi}{2} + \theta + 1\right)$

(ii) $P = \frac{2}{x} + x\left(1 + \frac{\pi}{4}\right)$

$\frac{dP}{dx} = \frac{-2}{x^2} + \left(1 + \frac{\pi}{4}\right)$

Minimum occurs when $\frac{dP}{dx} = 0$

$\frac{2}{x^2} = 1 + \frac{\pi}{4}$

$\frac{2}{x^2} = \frac{4+\pi}{4}$

$\frac{8}{4+\pi} = x^2$

Test for min or max ⇒

x^2	$\frac{7}{4+\pi}$	$\frac{8}{4+\pi}$	$\frac{9}{4+\pi}$
$\frac{dP}{dx}$	<0	0	>0

⇓ OR

$\frac{d^2P}{dx^2} = +\frac{4}{x^3}$

Since $x > 0$, (x is a measurement)

then $\frac{d^2P}{dx^2} > 0$

∴ a minimum occurs when

$x^2 = \frac{8}{4+\pi}$

Question 4

a) $\int \frac{dx}{9+4x^2}$

$= \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$

b) (i) $A_{40} = 750 \times 1.09^{40}$

$= \$23{,}557$

(ii) Total $= A_{40} + A_{39} + \ldots + A_1$

$= 750 \times 1.09^{40} + 750 \times 1.09^{39} + \ldots\ 750 \times 1.09^1$

$= 750\left(1.09 + 1.09^2 + \ldots\ 1.09^{40}\right)$

$= \frac{750 \times 1.09\left(1.09^{40} - 1\right)}{1.09 - 1}$

$= \$276{,}219$

$a = 1.09$

$r = 1.09^2$

$n = 40$

The above assumes a new investment of \$750 was not invested at the beginning of 2014.

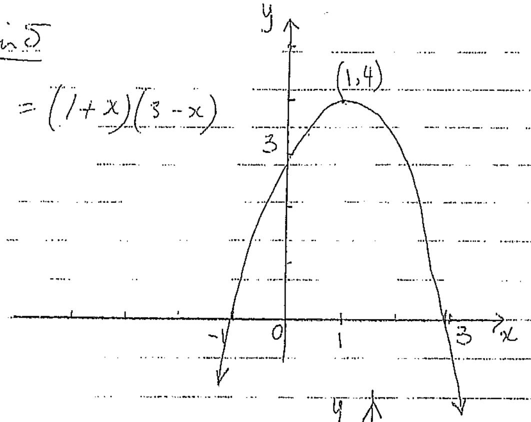
(iii) $A_{40} = \left(750 \times 1.09^{20}\right) \times 1.045^{20}$

$= \$10{,}137.18$

Question 5

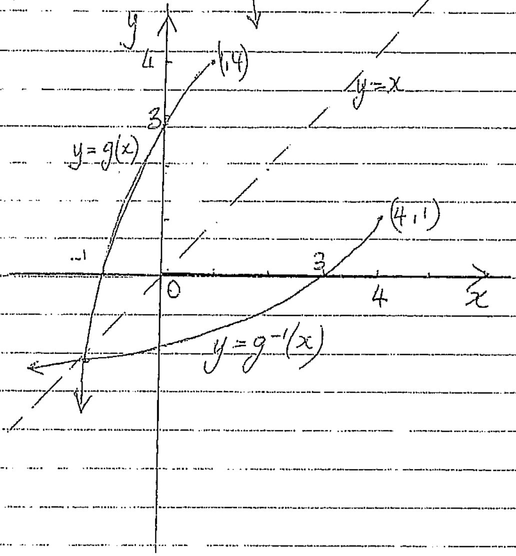
$f(x) = (1+x)(3-x)$

a)



b)i Domain : $x \leq 1$

c)



d) $y = (1+x)(3-x)$

Replace x with y

$x = (1+y)(3-y)$

$x = 3 + 2y - y^2$

$y^2 - 2y + x - 3 = 0$

$y = \dfrac{2 \pm \sqrt{4 - 4(x-3)}}{2}$

$y = 1 \pm \sqrt{1-(x-3)}$

$g^{-1}(x) = 1 - \sqrt{4-x}$

Question 6

$f(x) = e^x\left(1 - \frac{x}{10}\right)^{10}$

a) $e^x > 0$ for all x and $\left(1-\frac{x}{10}\right)^{10} \geq 0$

since $\left(1-\frac{x}{10}\right)$ is raised to an even power.

Note : $\left(1-\frac{x}{10}\right) = 0$ when $x = 10$

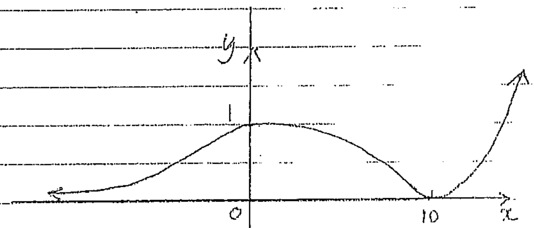
b) As $x \to -\infty$, $e^{-x} \to 0$

$\left(1-\frac{x}{10}\right)^{10} \to \infty$.

but e^{-x} dominates $\left(1-\frac{x}{10}\right)^{10}$

$\therefore f(x) \to 0$ as $x \to -\infty$

c)



d) Range : $0 \leq y \leq 1$

e) From d) we know

$e^x\left(1-\frac{x}{10}\right)^{10} \leq 1$

Now $\left(1-\frac{x}{10}\right)^{10} > 0$ for $x < 10$ * This statement worth 1 mark because it allows us to divide the inequality and retain $\leq$

$\therefore e^x \leq \dfrac{1}{\left(1-\frac{x}{10}\right)^{10}}$

ie $e^x \leq \left(1-\frac{x}{10}\right)^{-10}$ for $x < 10$