NORTH SYDNEY GIRLS HIGH SCHOOL

2012
TRIAL HSC EXAMINATION

Mathematics

General Instructions
- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 100
Section 1 – Q 1-10 worth 10 marks
Section 2 – Q 11 – 16 each worth 15 marks

At the end of the examination, place your solution booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

Student Number:________________ **Teacher:**________________

Student Name:________________

QUESTION	MARKS
1-10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

STANDARD INTEGRALS

$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x} dx = \ln x, \quad x > 0$

$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

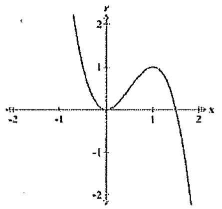
Total Marks – 100

SECTION 1

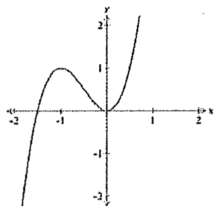
Ten questions worth 10 marks
Answer on the answer sheet provided

1. Which of the following is the graph of $f(x) = 2x^3 - 3x^2$?

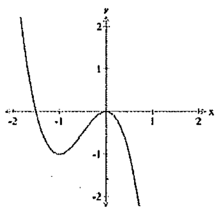
(A)



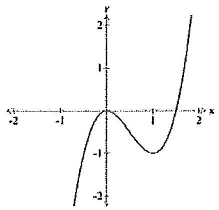
(B)



(C)



(D)



2. For what values of x is the curve $f(x) = 2x^3 + x^2$ concave down?

(A) $x < -\frac{1}{6}$ (B) $x > -\frac{1}{6}$ (C) $x < -6$ (D) $x > 6$

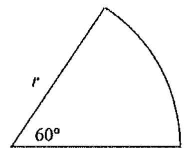
3. What is the greatest value of the function $y = 4 - 2\cos 2x$?

(A) 2 (B) 4 (C) 6 (D) 8

4. If $4^{x-1} = 32$, then the value of x is

(A) 10 (B) 3.5 (C) 3 (D) 6

5. The sector below has an area of 10π square units.



Not to scale

What is the value of r?

(A) $\sqrt{60}$ (B) $\sqrt{60\pi}$ (C) $\sqrt{\frac{\pi}{3}}$ (D) $\sqrt{\frac{1}{3}}$

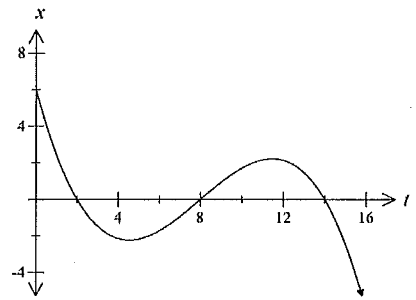
6. The chance of a fisherman catching a legal length fish is 4 in 5. If three fish are caught at random, what is the probability that exactly one is of legal length?

(A) $\frac{4}{125}$ (B) $\frac{12}{125}$ (C) $\frac{16}{125}$ (D) $\frac{48}{125}$

7. An infinite geometric series has a first term of 8 and a limiting sum of 12. What is the common ratio?

(A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

8. The displacement, x metres, from the origin of a particle moving in a straight line at any time (t seconds) is shown in the graph.



When was the particle at rest?

(A) $t = 2$, $t = 8$ and $t = 14$ (B) $t = 0$

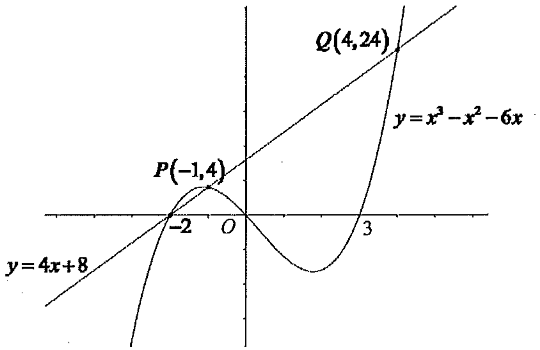
(C) $t = 4.5$ and $t = 11.5$ (D) Never

9. If $\tan 2x=\sqrt{3}$ in the domain $-\pi \le x \le \pi$, the value of x is:

(A) $\frac{\pi}{6},\frac{7\pi}{6}$ (B) $-\frac{5\pi}{6},-\frac{11\pi}{6}$

(C) A and B (D) None of the above

10. Consider the graphs of $y=x^3-x^2-6x$ and $y=4x+8$ as illustrated below.



The area enclosed between the curves is given by

(A) $A=\int_{-2}^{4}(4x+8)\,dx-\int_{-2}^{4}(x^3-x^2-6x)\,dx$

(B) $A=\int_{-2}^{4}(x^3-x^2-6x)\,dx-\int_{-2}^{4}(4x+8)\,dx$

(C) $A=\left|\int_{-2}^{-1}\left[(4x+8)-(x^3-x^2-6x)\right]dx\right|+\int_{-1}^{4}\left[(4x+8)-(x^3-x^2-6x)\right]dx$

(D) $A=\int_{-2}^{-1}\left[(4x+8)-(x^3-x^2-6x)\right]dx+\left|\int_{-1}^{4}\left[(4x+8)-(x^3-x^2-6x)\right]dx\right|$

End of Section 1

SECTION 2

Attempt Questions 11–16.
All questions are of equal value (15 marks each).

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks)

(a) Evaluate $\sqrt[5]{\frac{1.8+4.2}{3.1-1.6}}$ correct to four significant figures. **2**

(b) Factorise $8p^3+1$. **2**

(c) At a fun fair, Paula and Priscilla each played the same number of rounds of a game of Shoot the Balloons. **2**

Paula hit the target 30% of the time which was lower than the average by 10 shots.
Priscilla hit the target 40% of the time which was more than the average by 15 shots.

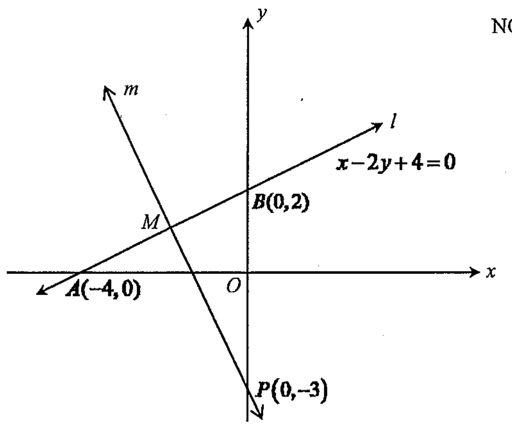
What was the average number of successful shots for the game?

Question 11 continues on Page 7

Question 11 (continued)

(d) In the diagram below, the equation of the line l is given by $x-2y+4=0$.

NOT TO SCALE



(i) Find the exact length of the interval AB. **1**

(ii) Find the gradient of AB. **1**

(iii) The equation of line m is $2x+y+3=0$. **1**
Show that the line m is perpendicular to l.

(iv) Show that m intersects l at the midpoint of segment AB. **2**

(v) Explain without any calculations why the lengths PA and PB are equal. **1**

(vi) The point $Q(-5,7)$ lies on the line m. Find the length of PQ. **1**

(vii) Explain why $APBQ$ is a kite. **1**

(viii) Find the area of quadrilateral $APBQ$. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $|3-2x| \le 5$ and graph the solution on a number line. **3**

(b) Solve the equation $2\log(x-5) = \log(2x-7)$ **2**

(c) Evaluate $\sum_{k=1}^{10}(10-3k)$. **2**

(d) If x, 4 and y are successive terms in an arithmetic sequence and x, 3 and y are successive terms in a geometric sequence, calculate $\frac{1}{x}+\frac{1}{y}$. **2**

(e) A bag contains one green, four blue and six red marbles.

(i) Two marbles are drawn from the bag with replacement. Find the probability that two blue marbles are drawn. **1**

(ii) What is the probability that at least one of the marbles is red or green? **2**

(iii) A single marble is now removed from the bag without noting its colour and it is replaced with a green marble.
A marble is now drawn from the bag. What is the probability that it is green? **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Differentiate:

(i) $\log_e(1-x^2)$ **1**

(ii) x^3e^{3x} **2**

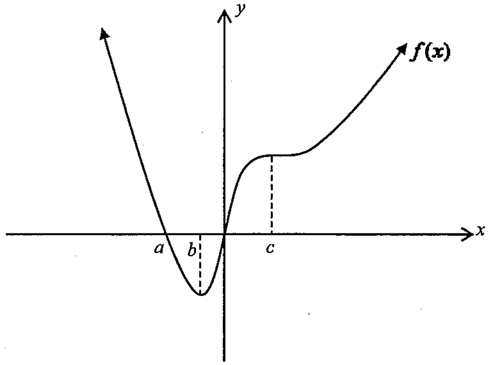
(b) Find:

(i) $\displaystyle\int \frac{dx}{\sqrt{3x-2}}$ **2**

(ii) $\displaystyle\int \frac{x^2-3x}{x^3}\,dx$ **2**

(c) Evaluate $\displaystyle\int_{-1}^{1} x(x-3)\,dx$. **2**

(d) The graph of $f(x)$ is shown below. Draw the graph of $f'(x)$ and mark the x-values a, b and c on your graph. **2**



(e) Consider the function $f(x)=\cos^2 x-\sin x$ in the domain $\pi \le x \le \frac{3\pi}{2}$

(i) Find $f'(x)$. **1**

(ii) Find the x-coordinates of the stationary points of $y=f(x)$ and determine their nature. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

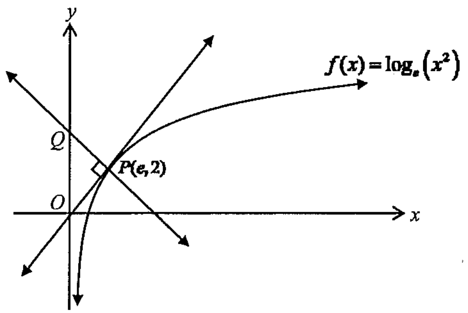
(a) Find the equation of the parabola with vertex $(-1,3)$ and directrix $y=-1$. **2**

(b) Consider the function $y=\log_e(x^2)$ for $x>0$

(i) Show that the tangent to the curve at the point $P(e,2)$, passes through the origin, O. **2**

(ii) Find the equation of the normal to the curve at P and find the point Q where the normal meets the y-axis. **2**

(iii) Show that the area of triangle OPQ is $\frac{4e+e^3}{4}$ square units. **1**



(c) (i) Draw a neat sketch of the curve $y=1+\cos 2x$ for $0\le x\le \pi$. **2**

(ii) On the same diagram, sketch $y=\frac{1}{2}x$ for $0\le x\le\pi$. **1**

(iii) Explain why the x-coordinates of the intersection points of the two graphs represent the solutions to the equation $x-2\cos 2x=2$. **1**

(iv) Using the graph, determine the number of solutions to the equation $x-2\cos 2x=2$ which exist in the given domain. **1**

(d) The population of a rare species of beetle can be modeled by $N=N_0e^{kt}$.
An environmental stress reduced the population from 5000 to 4000 in two days.

(i) How many beetles will there be seven days after the initial count of 5000? **2**

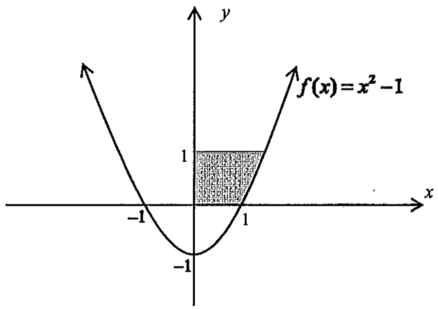
(ii) How many days will it take for the beetle population to fall below 250? **1**

BLANK PAGE

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of $f(x) = x^2 - 1$ is shown. The shaded region in the diagram is the area bounded by the curve, the positive x-axis, the y-axis and the line $y = 1$.

Find the volume of the solid of revolution formed when the shaded region is rotated around the y-axis. **2**



(b) Consider the parabola $y = x^2 + 1$.

(i) Find the y-coordinate of a point P on the parabola whose x-coordinate is p. **1**

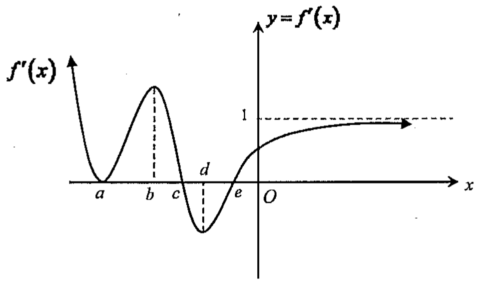
(ii) Find an expression for the perpendicular distance of the point P from the line $2x + 5y = 4$ in terms of p. **1**

(iii) Hence, or otherwise, show that the line $2x + 5y = 4$ does not intersect the parabola $y = x^2 + 1$. **2**

Question 15 continues on Page 13

Question 15 (continued)

(c) The graph below represents the gradient function $f'(x)$.
Specific x-values a, b, c, d and e are as indicated in the diagram.



(i) Justify why $f(x)$ has a maximum stationary point at $x=c$. **1**

(ii) For what value(s) of x is the graph of $y=f(x)$ increasing and concave up? **2**

(iii) What feature of the graph $y=f(x)$ exists at $x=a$? **1**

(d) The acceleration of a particle moving in a straight line is given by $a=\dfrac{2}{(t+1)^2}$.
Initially, the particle is 2 m to the right of the origin travelling towards the origin at a speed of 1 m/s.

(i) Find an expression for the velocity of the particle at time t? **2**

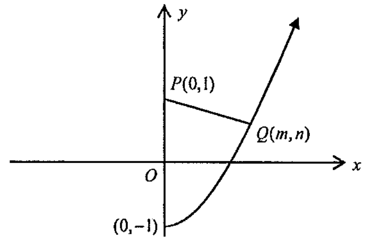
(ii) Show that the particle is at rest at $t=1$. **1**

(iii) What is the distance travelled by the particle in the first two seconds? **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the curve $y=x^2-1,\ x>0$. P is the point $(0,1)$ and $Q(m,n)$ is a point on the curve.



(i) Show that the length of the interval PQ is given by **1**

$$L=\sqrt{m^4-3m^2+4}$$

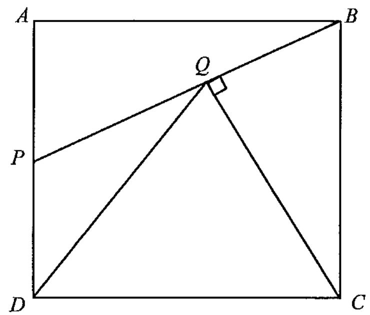
(ii) Hence, find the coordinates of the point Q on the curve that is closest to P. **4**

(b) $ABCD$ is a square of side length 2 units. P is the midpoint of AD. CQ is drawn perpendicular to PB.

(i) Prove $\angle APB=\angle QBC$. **1**

(ii) Hence or otherwise, show that $QC=\dfrac{4}{\sqrt{5}}$ units. **2**

(iii) Show that $QD=CD$. **2**

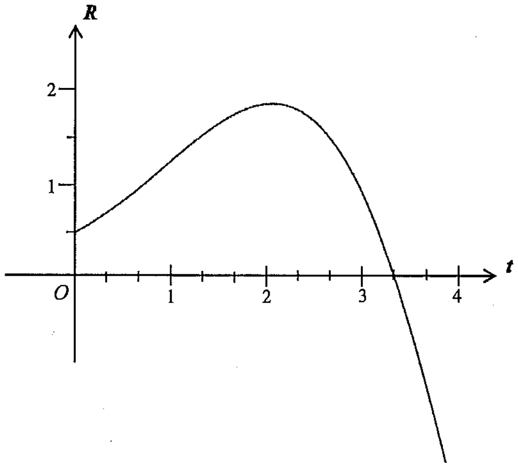


Question 16 continues on Page 15

Question 16 (continued)

(c) During a recent flood, the level of water in a river was measured at regular intervals starting from midnight. The height h metres, by which the water level exceeded normal levels was recorded.

The rate at which h increased at time t, is given by $R(t) = 0.5(1 + e^{0.5t}\sin t)$.



(i) Deduce from the graph, the approximate time after midnight at which the water level peaked. **1**

(ii) Consider the table below and use Simpson's Rule with five function values to estimate the area under the curve shown between $t = 0$ and $t = 3\frac{1}{3}$ correct to two decimal places. **2**

t	0	$\frac{5}{6}$	$\frac{5}{3}$	$\frac{5}{2}$	$\frac{10}{3}$
$R(t)$	0.5	1.061	1.645	1.544	0

(iii) The town needs to be evacuated if the water rises to 4 m above the normal levels. If the water level was 0.25 m higher than the normal levels at midnight, was the town evacuated? **1**

(iv) Now, if the water level returned to normal at time p, find the value of $\int_0^p R(t)\,dt$. **1**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Mathematics Trial HSC Examination 2012 – Solutions

Section 1

1. D
2. A
3. C
4. B
5. A
6. B
7. C
8. C
9. D
10. C

Section 2

Question 11

(a) 1.320 (4 s.f.)

(b) $(2p+1)(4p^2-2p+1)$

(c) 10% of shots = 25
Average $= 3\times 25+10$
$= 85$ shots

(d) (i) $AB=\sqrt{4^2+2^2}=\sqrt{20}=2\sqrt{5}$

(ii) $m_{AB}=\frac{2}{4}=\frac{1}{2}$

(iii) Gradient of line $m=-2$

$-2\times\frac{1}{2}=-1$

Gradients multiply to -1.

$\therefore l\perp m$.

(iv) Midpoint of AB is $M=(-2,1)$

Sub $(-2,1)$ into eqn of m.

LHS $=2\times -2+1+3=0=$ RHS.

$\therefore$M lies on the lines l and m and is the point of intersection of the two lines.

(v) m is the perpendicular bisector of AB. $\therefore$ all points on m are equidistant from A and B.

$\therefore PA=PB$.

(vi) $PQ=\sqrt{5^2+(-10)^2}=\sqrt{125}=5\sqrt{5}$.

(vii) Diagonal PQ bisects the diagonal AB.

(viii) Area $=\frac{1}{2}\times 2\sqrt{5}\times 5\sqrt{5}=25\text{ u}^2$

Question 12

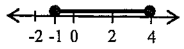
(i) $|3-2x|\le 5$

$-5\le 3-2x\le 5$

$-8\le -2x\le 2$

$4\ge x\ge -1$

$-1\le x\le 4$



(ii) $2\log(x-5)=\log(2x-7)$

$\log\left((x-5)^2\right)=\log(2x-7)$

$(x-5)^2=2x-7$

$x^2-10x+25=2x-7$

$x^2-12x+32=0$

$(x-4)(x-8)=0$

$x=4, 8$

$x=8\left(x\ne 4 \because \log(-1)\notin\mathbb{R}\right)$

(iii) $\sum_{k=1}^{10}(10-3k)=7+4+1+\ldots+(-20)$

$=\frac{10}{2}(7-20)$

$=-65$

(iv) $4-x=y-4 \quad \therefore x+y=8$

$\frac{3}{x}=\frac{y}{3} \quad \therefore xy=9$

$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=\frac{8}{9}$

(v) $\mathrm{P(G)}=\frac{1}{11}$; $\mathrm{P(B)}=\frac{4}{11}$; $\mathrm{P(R)}=\frac{6}{11}$

(i) $\mathrm{P(BB)}=\frac{4}{11}\times\frac{4}{11}=\frac{16}{121}$

(ii) P(at least 1 G or R) $=1-$ P(BB) $=1-\frac{16}{121}=\frac{105}{121}$

(iii) There is $\frac{1}{11}$ chance that a green marble is removed and $\frac{10}{11}$ chance that the marble removed is not green.

$\therefore \mathrm{P(G)}=\frac{1}{11}\times\frac{1}{11}+\frac{10}{11}\times\frac{2}{11}=\frac{21}{121}$

Question 13

(a) (i) $\frac{1}{1-x^2}\times -2x=-\frac{2x}{1-x^2}$

(ii) $3x^2\times e^{3x}+x^3\times 3e^{3x}=3x^2e^{3x}(1+x)$

(b) (i) $\int\frac{dx}{\sqrt{3x-2}}=\int(3x-2)^{-\frac{1}{2}}dx$

$=\frac{(3x-2)^{\frac{1}{2}}}{\frac{1}{2}\times 3}=\frac{2}{3}\sqrt{3x-2}+C$

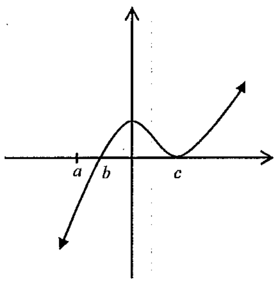
(ii) $\int\frac{x^2-3x}{x^3}dx=\int\left(\frac{1}{x}-\frac{3}{x^2}\right)dx$

$=\log_e x+\frac{3}{x}+C$

(c) $\int_{-1}^{1}x(x-3)\,dx=\int_{-1}^{1}\left(x^2-3x\right)dx$

$=\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_{-1}^{1}=\left(\frac{1}{3}-\frac{3}{2}\right)-\left(\frac{-1}{3}-\frac{3}{2}\right)=\frac{2}{3}$

(d)



(e) (i) $f'(x)=2\cos x(-\sin x)-\cos x$

$=-\cos x(2\sin x+1)$

(ii) For stationary points, $f'(x)=0$

$-\cos x(2\sin x+1)=0$

$\cos x=0$ or $\sin x=-\frac{1}{2}$

$x=\frac{3\pi}{2}$ or $x=\frac{7\pi}{6}$

Test nature

x	3	$\frac{7\pi}{6}$	4	$\frac{3\pi}{2}$	5
$f'(x)$	1.26	0	–0.34	0	0.26
	/	—	\	—	/

$\therefore f(x)$ has a local maximum at $x=\frac{7\pi}{6}$

and a local minimum at $x=\frac{3\pi}{2}$.

Question 14

(a) Focal length $a=4$

Equation is $(x+1)^2=16(y-3)$

(b) (i) $\frac{dy}{dx}=\frac{1}{x^2}\times 2x=\frac{2}{x}$. At $x=e$, $\frac{dy}{dx}=\frac{2}{e}$.

$\therefore$ gradient of tangent at P is $\frac{2}{e}$.

Consider gradient of $OP=\frac{2-0}{e-0}=\frac{2}{e}$.

$\therefore OP$ is the tangent at P or the tangent at P passes through the origin.

(ii) Gradient of normal $=-\frac{e}{2}$.

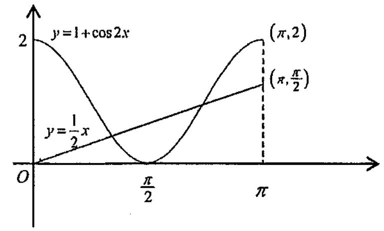
Equation of normal is $y-2=-\frac{e}{2}(x-e)$

For y-intercept, sub $x=0$.

$y=2+\frac{e^2}{2}=\frac{4+e^2}{2}$

(iii) Area $\Delta OPQ=\frac{1}{2}\times e\times\frac{4+e^2}{2}=\frac{4e+e^2}{2}$ units^2

(c) (i) and (ii)



(iii) For x-coordinates of the intersection points, solve the two equations simultaneously.

$1+\cos 2x=\frac{x}{2}$

$2+2\cos 2x=x$

$x-2\cos 2x=2$

(iv) From the graph, there are two solutions to the equation $x-2\cos 2x=2$ for $0\le x\le\pi$.

(d) (i) $4000=5000e^{2k}$

$e^{2k}=\frac{4}{5}$ or $k=\frac{1}{2}\ln\frac{4}{5}$ or $0.5\ln 0.8$

$N=5000e^{7k}$ where $k=0.5\ln 0.8$

$N=2289.73$ (2dp)

(ii) $250 = 5000e^{kt}$ where $k = 0.5\ln 0.8$

$t = \frac{1}{k}\ln\frac{250}{5000} = 26.85$

$\therefore$ it takes 27 days for the beetle population to fall below 250.

Question 15

(a) $y = x^2 - 1 \Rightarrow x^2 = y + 1$

$V = \pi\int_0^1 x^2 dy = \pi\int_0^1 (y+1)\,dy$

$= \pi\left[\frac{y^2}{2} + y\right]_0^1 = \pi\left[\left(\tfrac{1}{2}+1\right) - (0)\right]$

$= \frac{3\pi}{2}$ units^3

(b) (i) y- coordinate is $p^2 + 1$.

(ii) $d = \frac{\left|2p + 5\left(p^2+1\right) - 4\right|}{\sqrt{2^2+5^2}} = \frac{\left|5p^2+2p+1\right|}{\sqrt{29}}$

(iii) For intersection points, solve $d = 0$.

$5p^2 + 2p + 1 = 0$

$\Delta = b^2 - 4ac = 4 - 4\times 5 = -16$

$\Delta < 0$; $\therefore$ no real solutions.

$\therefore$ the line does not intersect the parabola.

(c) (i) For $x = c^-$, $f'(x) > 0$, $x = c$, $f'(x) = 0$ and for $x = c^+$, $f'(x) < 0$.

$\therefore f(x)$ has a max. stationary point at $x = c$.

(ii) Increasing $\Rightarrow f'(x) > 0$

Concave up $\Rightarrow f''(x) > 0$ or gradient of $f'(x)$ is +ve.

$a < x < b$ or $x > e$.

(iii) Stationary point of inflexion at $x = a$.

(d) (i) $\ddot{x} = \frac{2}{(t+1)^2}$; $\dot{x} = \frac{-2}{t+1} + C$

At $t = 0, \dot{x} = -1$

$-1 = -2 + C \Rightarrow C = 1$

$\therefore \dot{x} = \frac{-2}{t+1} + 1$

(ii) Sub $t = 1$, $\dot{x} = \frac{-2}{1+1} + 1 = 0$

$\therefore$ the particle is at rest at $t = 1$.

(iii) Initially the velocity is negative. It is zero at $t = 1$ and then it is positive.

$\therefore$ distance travelled is the sum of two areas.

$d = -\int_0^1\left(\frac{-2}{t+1}+1\right)dt + \int_1^2\left(\frac{-2}{t+1}+1\right)dt$

$= -\left[-2\ln(t+1)+t\right]_0^1 + \left[-2\ln(t+1)+t\right]_1^2$

$= -[(2\ln 2 - 1) - (0)] + [(-2\ln 3 + 2) - (-2\ln 2 + 1)]$

$= \ln\left(\frac{16}{9}\right)$m or 0.575 m (3 s.f.)

Question 16

(a) (i) $Q = (m, m^2 - 1)$

$PQ = \sqrt{(m-0)^2 + \left(m^2-2\right)^2}$

$= \sqrt{m^2 + m^4 - 4m^2 + 4} = \sqrt{m^4 - 3m^2 + 4}$

(ii) Let $D = PQ = \sqrt{m^4 - 3m^2 + 4}$

$\frac{dD}{dm} = \frac{1}{2\sqrt{m^4 - 3m^2 + 4}} \times \left(4m^3 - 6m\right)$

$\frac{dD}{dm} = 0 \Rightarrow 4m^3 - 6m = 0$

$2m\left(2m^2 - 3\right) = 0$

$m = 0, \pm\sqrt{\frac{3}{2}}$ $\quad m > 0 \Rightarrow m = \sqrt{\frac{3}{2}}$

Test nature:

m	1	$\sqrt{\frac{3}{2}}$	2
D'	−0.7	0	3.5
	\	—	/

$\therefore$ D is minimum at $m = \sqrt{\frac{3}{2}}$ and $Q = \left(\sqrt{\frac{3}{2}}, \frac{1}{2}\right)$

(b) (i) Let $\angle APB = x$

$\angle APB + \angle ABP + \angle A = 180°$ (angle sum of ΔAPB)

$\angle ABP = 90 - x$ ($\angle A = 90°$ $\because$ $ABCD$ is a square)

$\angle QBC = 90 - \angle ABP$ ($\angle B = 90°$)

$= 90 - (90 - x) = x$

$\therefore \angle ABP = \angle QBC = x$ (as required)

(ii) In ΔAPB,

$PB = \sqrt{AP^2 + AB^2}$ (Pythagoras Theorem)

$= \sqrt{2^2 + 1^2}$ (Given $AB = 2$ and $AP = \frac{1}{2}AB$)

$= \sqrt{5}$

$\sin x = \frac{AB}{PB} = \frac{2}{\sqrt{5}}$ $\quad$ (1)

$\cos x = \frac{AP}{PB} = \frac{1}{\sqrt{5}}$ $\quad$ (2)

In ΔQBC, $\sin x = \frac{QC}{BC} = \frac{2}{\sqrt{5}}$ from (1)

$\frac{QC}{2} = \frac{2}{\sqrt{5}}$

$QC = \frac{4}{\sqrt{5}}$ $\quad$ (3)

(iii) $\angle QCB = 90 - x$ (angle sum of ΔQBC)

$\therefore \angle QCD = x$ ($\angle C$=90°)

In ΔQCD, using the Cosine Rule,

$QD^2 = QC^2 + CD^2 - 2.QC.CD.\cos x$

$= \frac{16}{5} + 4 - 2\times\frac{4}{\sqrt{5}}\times 2\times\frac{1}{\sqrt{5}}$ from (2) and (3)

$= 4$

$QD = 2$

$\therefore QD = CD = 2$

(c) (i) At 3.20 am (from the graph) as this is the point when $R(t)$ becomes zero and the water level is no longer increasing.

(ii)

t	0	$\frac{5}{6}$	$\frac{5}{3}$	$\frac{5}{2}$	$\frac{10}{3}$
$R(t)$	0.5	1.061	1.645	1.544	0
weight w	1	4	2	4	1
$w\times R(t)$	0.5	4.244	3.29	6.176	0

Area $\approx \frac{\frac{5}{6}}{3}\left[0.5 + 4.244 + 3.29 + 6.176 + 0\right]$

≈ 3.95 (2dp)

(iii) The area under the curve is the total increase in the height of the water from midnight.

The water level peaks at 3.20 am

Peak water level = 0.25+3.95 = 4.2 m.

Yes, the town will need to be evacuated.

(iv) The area under the curve $R(t)$ represents the net change in the water level from $t = 0$ or midnight.

At midnight, the water level was 0.25 m higher than normal.

When the water level returns to the normal, the net change in the water level from midnight is −0.25.

$\therefore \int_0^p R(t)dt = -0.25$