NSW INDEPENDENT SCHOOLS

2006
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used
- Write using black or blue pen
- A table of standard integrals is provided at the back of the paper
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

This paper MUST NOT be removed from the examination room

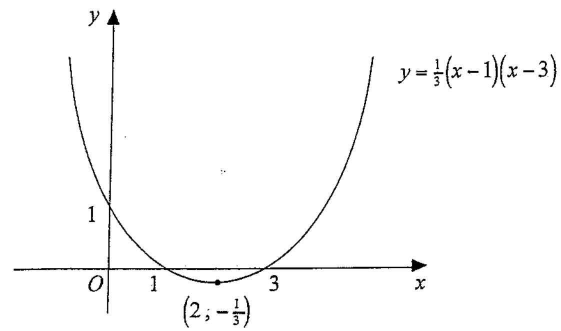
STUDENT NUMBER/NAME..........................

Student name / number..............................

Marks

Question 1 **Begin a new page**

(a) The diagram below shows the graph of the function $f(x)=\frac{1}{3}(x-1)(x-3)$.



On separate diagrams sketch the following graphs, showing clearly any intercepts on the axes, the coordinates of any turning points and the equations of any asymptotes:

(i) $y=|f(x)|$ **1**

(ii) $y=f(|x|)$ **1**

(iii) $y=\dfrac{1}{f(x)}$ **2**

(iv) $y=\ln\{f(x)\}$ **2**

(b) Consider the function $f(x)=x+\dfrac{1}{x-1}$.

(i) Find the coordinates and the nature of the stationary points on the curve $y=f(x)$. **3**

(ii) Show that the line $y=-3x$ is a tangent to the curve and find the coordinates of the point of contact. **2**

(c) Consider the curve given by the equation $x^2-y^2+xy+5=0$.

(i) Show that $\dfrac{dy}{dx}=\dfrac{2x+y}{2y-x}$. **2**

(ii) Hence find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y=x$. **2**

Student name / number..............................

Marks

Question 2 **Begin a new page**

(a) Find $\int \frac{x^2}{x^2+1}\,dx$. 2

(b) Use the substitution $u = e^x$ to find $\int \frac{e^x}{e^{2x}-1}\,dx$. 3

(c)(i) If $n = -1$ show that $\int_1^e x^n \ln x \; dx = \frac{1}{2}$. 2

(ii) If $n \neq -1$ show that $\int_1^e x^n \ln x \; dx = \frac{ne^{n+1}+1}{(n+1)^2}$. 3

(d)(i) Use the substitution $u = \tan\frac{x}{2}$ to show that $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,dx = 4-2\sqrt{3}$. 3

(ii) Hence find the exact value of $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{1+\sin x}\,dx$. 2

Student name / number..............................

Marks

Question 3 **Begin a new page**

(a) The equation $z^2-(a+bi)z-6i=0$, where a and b are real, has roots α and β such that $\alpha^2+\beta^2=5$.

(i) Show that $a^2-b^2=5$ and $ab=-6$. 2

(ii) Hence find any values of a and b. 2

(b) $z_1 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$ and $z_2 = 1+i$ are two complex numbers.

(i) Express z_2 and $z_1 z_2$ in modulus/argument form. 2

(ii) Express z_1 in the form $a+ib$ where a and b are real. 2

(c)(i) On an Argand diagram shade the region containing all the points representing complex numbers z such that both $|z-1| \le 1$ and $0 \le \arg z \le \frac{\pi}{6}$. 2

(ii) Find the area of the shaded region in simplest exact form. 2

(d) The complex number z is given in modulus/argument form by $z = r(\cos\theta + i\sin\theta)$. 3

Show that $\frac{z}{z^2+r^2}$ is real.

Student name / number..............................

Marks

Question 4 **Begin a new page**

(a) The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $(0<a<b)$ has eccentricity e. S is the focus of the hyperbola on the positive x axis. The line through S perpendicular to the x axis intersects the hyperbola at P and Q.

(i) Show that $PQ=\frac{2b^2}{a}$. **2**

(ii) If P and Q have coordinates $(9,24)$ and $(9,-24)$ respectively, write down two equations in a and b then solve these equations algebraically to show that $a=3$ and $b=6\sqrt{2}$. **3**

(iii) For these values of a and b, sketch the graph of the hyperbola showing clearly the intercepts on the x axis, the coordinates of the foci, and the equations of the directrices and asymptotes. **4**

(b) $T\left(ct,\frac{c}{t}\right)$ is a point on the rectangular hyperbola $xy=c^2$. The point M is the foot of the perpendicular drawn from the origin O to the tangent to the hyperbola at T.

(i) Show that the tangent to the hyperbola at T has equation $x+t^2y-2ct=0$. **2**

(ii) Find the coordinates of M. **2**

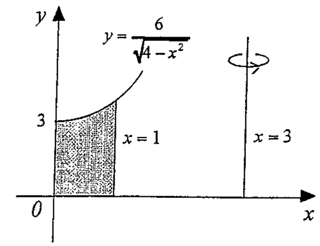
(iii) Find the equation of the locus of M as T moves on the rectangular hyperbola. **2**

Student name / number..............................

Marks

Question 5 **Begin a new page**

(a)



A mould for a section of concrete piping is made by rotating the region bounded by the curve $y=\frac{6}{\sqrt{4-x^2}}$ and the x axis between the lines $x=0$ and $x=1$ through one complete revolution about the line $x=3$. All measurements are in metres.

(i) By considering strips of width δx parallel to the axis of rotation, show that the volume V m^3 of the concrete used in the piping is given by **4**

$$V=12\pi\int_0^1 \frac{3-x}{\sqrt{4-x^2}}\,dx.$$

(ii) Hence find the volume of the concrete used in the piping, giving your answer correct to the nearest cubic metre. **4**

(b)(i) If α is a double root of the polynomial equation $P(x)=0$, show that α is a root of $P'(x)=0$. **2**

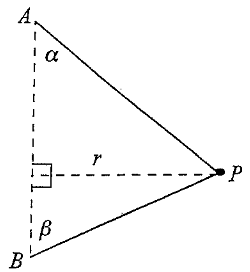
(ii) The equation $x^3+qx+r=0$, where q and r are non-zero real numbers, has a double root α. Show that $\alpha=-\frac{3r}{2q}$ and hence that $4q^3+27r^2=0$. Deduce that $q<0$. **5**

Student name / number..............................

Marks

Question 6 **Begin a new page**

(a)



A and B are two fixed points with B vertically below A. P is a particle of mass M kg. Two strings with ends fixed at A, B are fastened to P. Particle P moves in a horizontal circle of radius r metres with constant angular velocity ω radians per second so that both strings AP and BP remain taut, making angles α, β respectively with the vertical. The tensions in the strings AP and BP are T_1 Newtons and T_2 Newtons respectively. The acceleration due to gravity is g ms^{-2}.

(i) Draw a diagram showing the forces acting on the particle P. **1**

(ii) Explain why $T_1\cos\alpha - T_2\cos\beta = Mg$ and $T_1\sin\alpha + T_2\sin\beta = Mr\omega^2$. **3**

(iii) Hence show that $T_2 = \dfrac{M(r\omega^2\cos\alpha - g\sin\alpha)}{\sin(\alpha+\beta)}$. Find the corresponding expression for T_1. **2**

(iv) Find the smallest possible value of ω for the motion to continue as described. Explain what happens if ω drops below this value. **2**

(b)(i) If $z = \cos\theta + i\sin\theta$, use De Moivre's theorem to show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ for positive integers $n \geq 1$. **2**

(ii) By expanding $\left(z+\dfrac{1}{z}\right)^5$ show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$. **3**

(iii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5\theta \, d\theta$. **2**

Student name / number..............................

Marks

Question 7 **Begin a new page**

(a) a, b, c, d are positive real numbers, each strictly less than $\frac{\pi}{2}$.

(i) Show that $\cos a \cos b \leq \left\{\cos\frac{1}{2}(a+b)\right\}^2$. **4**

(ii) Hence show that $\cos a \cos b \cos c \cos d \leq \left\{\cos\frac{1}{4}(a+b+c+d)\right\}^4$. **2**

(iii) Deduce that $\cos a \cos b \cos c \leq \left\{\cos\frac{1}{3}(a+b+c)\right\}^3$. **2**

(b) In the Binomial expansion of $(1+ax)^n$, where n is a positive integer, the terms in x^2 and x^3 are $63x^2$ and $189x^3$.

(i) Show that $(n-2)a = 9$. **3**

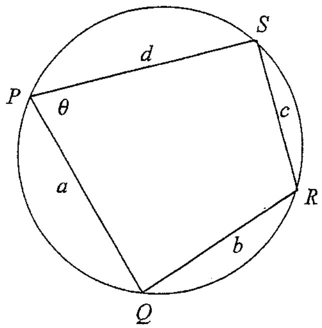
(ii) Hence show that $5n^2 - 47n + 56 = 0$ and find the values of n and a. **4**

Student name / number..............................

Marks

Question 8 **Begin a new page**

(a)



PQRS is a cyclic quadrilateral with sides of length a, b, c, d as shown in the diagram. The interior angle at P has size θ and the quadrilateral has area A.

(i) Show that $\sin\theta = \dfrac{2A}{ad+bc}$ and $\cos\theta = \dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)}$. **4**

(ii) Hence show that $A^2 = \frac{1}{16}\left\{4(ad+bc)^2 - \left(a^2+d^2-b^2-c^2\right)^2\right\}$. **2**

(iii) If $s = \frac{1}{2}(a+b+c+d)$ deduce that $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. **4**

(b) Use Mathematical Induction to show that $\tan(2n-1)\frac{\pi}{4} = (-1)^{n+1}$ for all positive integers $n \geq 1$. **5**

Question 1

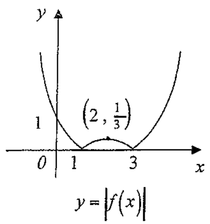
a. Outcomes assessed : E6

Marking Guidelines

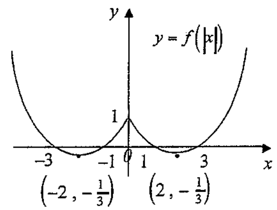
	Criteria	Marks
i	• shows curve with correct shape, intercepts and turning point	1
ii	• shows curve with correct shape, intercepts and turning points	1
iii	• shows both vertical asymptotes and x axis as horizontal asymptote	1
	• shows curve with correct shape and turning point	1
iv	• shows both vertical asymptotes and no curve for $1 < x < 3$	1
	• shows curve with correct shape and intercepts	1

Answer

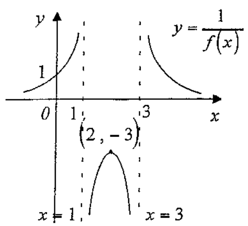
i.



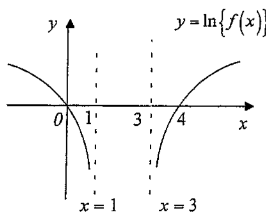
ii.



iii.



iv.



b. Outcomes assessed : P4, H6

Marking Guidelines

	Criteria	Marks
i	• solves $f'(x) = 0$ to find the x coordinates of the stationary points	1
	• identifies the maximum turning point giving both coordinates	1
	• identifies the minimum turning point giving both coordinates	1
ii	• shows equation for x coord. of any intersection point has a repeated root	1
	• deduces line is tangent to curve and gives both coordinates of point of contact	1

Answer

i. $f(x) = x + \dfrac{1}{x-1}$

$f'(x) = 1 - \dfrac{1}{(x-1)^2} \qquad \therefore f'(x) = 0 \Rightarrow x = 0, 2$

$f''(x) = \dfrac{2}{(x-1)^3} \qquad \therefore \begin{cases} f''(0) = -2 < 0 \\ f''(2) = \ \ 2 > 0 \end{cases}$

$\therefore (0, -1)$ is a maximum turning point

and $(2, 3)$ is a minimum turning point.

ii. Line $y = -3x$ meets the curve where

$$x + \frac{1}{x-1} = -3x$$
$$4x(x-1) + 1 = 0$$
$$4x^2 - 4x + 1 = 0$$
$$(2x-1)^2 = 0$$

Since this equation has a repeated root $\frac{1}{2}$, the line is tangent to the curve with point of contact $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

c. Outcomes assessed : H5

Marking Guidelines

	Criteria	Marks
i	• finds expression for derivative by implicit differentiation	1
	• rearranges to give required result	1
ii	• finds y in terms of x when derivative takes value 1.	1
	• substitutes in equation of curve to find coordinates of required points	1

Answer

i. $x^2 - y^2 + xy + 5 = 0$

$$2x - 2y\frac{dy}{dx} + 1.y + x\frac{dy}{dx} = 0$$
$$(2x+y) - \frac{dy}{dx}(2y - x) = 0$$
$$\frac{dy}{dx}(2y - x) = 2x + y$$
$$\frac{dy}{dx} = \frac{2x+y}{2y-x}$$

ii. Tangent parallel to line $y = x$ when $\dfrac{dy}{dx} = 1$.

$\left.\begin{array}{l} 2x + y = 2y - x \\ y = 3x \end{array}\right\}$ and $x^2 - y^2 + xy + 5 = 0$

$$x^2 - (3x)^2 + x(3x) + 5 = 0$$
$$-5x^2 + 5 = 0$$
$$x^2 = 1$$

Hence tangents to curve at $(1, 3)$ and $(-1, -3)$ are parallel to line $y = x$.

Question 2

a. Outcomes assessed : H8, HE4

Marking Guidelines

Criteria	Marks
• rearranges integrand into appropriate form	1
• writes primitive	1

Answer

$$\int \frac{x^2}{x^2+1}\,dx = \int\left(1 - \frac{1}{x^2+1}\right)dx = x - \tan^{-1}x + c$$

b. Outcomes assessed : HE6, E8

Marking Guidelines

Criteria	Marks
• converts to integral in u	1
• uses partial fractions to find primitive in u	1
• writes primitive as function of x	1

Answer

$u = e^x$

$du = e^x dx$

$$\begin{aligned}\int \frac{e^x}{e^{2x}-1}\,dx &= \int \frac{1}{e^{2x}-1}\; e^x dx \\ &= \int \frac{1}{u^2-1}\; du \\ &= \tfrac{1}{2}\int\left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \\ &= \tfrac{1}{2}\left\{\ln(u-1) - \ln(u+1)\right\} + c \\ &= \tfrac{1}{2}\ln\left(\frac{e^x-1}{e^x+1}\right) + c\end{aligned}$$

c. Outcomes assessed : HE6, E8

Marking Guidelines

Criteria	Marks
i • recognizes a pattern (or uses substitution) to find primitive	1
• substitutes and simplifies	1
ii • applies integration by parts	1
• substitutes in and simplifies first part	1
• finds new primitive and completes substitution and simplification	1

Answer

i. $\int_1^e x^{-1} \ln x \; dx = \tfrac{1}{2}\left[(\ln x)^2\right]_1^e = \tfrac{1}{2}(1^2 - 0^2) = \tfrac{1}{2}$

ii. Let $I = \int_1^e x^n \ln x \; dx$

$$\begin{aligned}I &= \tfrac{1}{n+1}\left[x^{n+1}\ln x\right]_1^e - \tfrac{1}{n+1}\int_1^e x^{n+1}\frac{1}{x}\; dx \\ &= \tfrac{1}{n+1}\left(e^{n+1}.1 - 0\right) - \tfrac{1}{n+1}\int_1^e x^n \; dx\end{aligned}$$

$$\begin{aligned}I &= \tfrac{1}{n+1}e^{n+1} - \tfrac{1}{(n+1)^2}\left[x^{n+1}\right]_1^e \\ &= \tfrac{1}{(n+1)^2}\left\{(n+1)e^{n+1} - \left(e^{n+1}-1\right)\right\} \\ &= \frac{ne^{n+1}+1}{(n+1)^2}\end{aligned}$$

d. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
i • converts dx to du and x limits to u limits	1
• uses expression for $\sin x$ in terms of u to convert integrand to function of u	1
• finds primitive and evaluates by substitution of limits	1
ii • uses an appropriate substitution to write new integral in terms of integral from i.	1
• evaluates new integral	1

Answer

i. $u = \tan\frac{x}{2}$

$du = \tfrac{1}{2}\sec^2\frac{x}{2}\, dx$

$2du = \left(1+u^2\right)dx$

$dx = \frac{2}{1+u^2}\, du$

$x = \frac{\pi}{3} \Rightarrow u = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$x = \frac{2\pi}{3} \Rightarrow u = \tan\frac{\pi}{3} = \sqrt{3}$

$$\begin{aligned}1+\sin x &= 1 + \frac{2u}{1+u^2} \\ &= \frac{1+u^2+2u}{1+u^2}\end{aligned}$$

$$\begin{aligned}I &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\, dx \\ &= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1+u^2}{(1+u)^2}\cdot\frac{2}{1+u^2}\, du \\ &= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} 2\,(1+u)^{-2}\, du \\ &= -2\left[\frac{1}{1+u}\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\end{aligned}$$

$$\begin{aligned}I &= -2\left\{\frac{1}{1+\sqrt{3}} - \frac{1}{1+\frac{1}{\sqrt{3}}}\right\} \\ &= -2\left\{\frac{1}{1+\sqrt{3}} - \frac{\sqrt{3}}{\sqrt{3}+1}\right\} \\ &= -2\frac{\left(1-\sqrt{3}\right)^2}{1-3} \\ &= 1 - 2\sqrt{3} + 3 \\ &= 4 - 2\sqrt{3}\end{aligned}$$

ii. Let $t = \pi - x$

$dt = -dx$

$x = \frac{\pi}{3} \Rightarrow t = \frac{2\pi}{3}$

$x = \frac{2\pi}{3} \Rightarrow t = \frac{\pi}{3}$

$$\begin{aligned}J &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{1+\sin x}\, dx \\ &= \int_{\frac{2\pi}{3}}^{\frac{\pi}{3}} \frac{\pi-u}{1+\sin(\pi-u)}\, . -du \\ &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi-u}{1+\sin u}\, du \\ &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi}{1+\sin u}\, du - \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{u}{1+\sin u}\, du\end{aligned}$$

$$\begin{aligned}\therefore J &= \pi\left(4-2\sqrt{3}\right) - J \\ 2J &= \pi\left(4-2\sqrt{3}\right) \\ J &= \pi\left(2-\sqrt{3}\right)\end{aligned}$$

Question 3

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • uses sum and product of roots to write expression relating a and b	1
• equates real and imaginary parts to obtain required result	1
ii • solves simultaneously to obtain quartic equation in either a or b	1
• finds values of a and b	1

Answer

i. $z^2 - (a+bi)z - 6i = 0$

$(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$(a+bi)^2 = 5 + 2(-6i)$

$\left(a^2-b^2\right) + 2abi = 5 - 12i$

Equating real and imaginary parts,

$a^2 - b^2 = 5$ and $ab = -6$

ii. $a^4 - a^2b^2 = 5a^2$

$a^4 - 5a^2 - 36 = 0$

$\left(a^2-9\right)\left(a^2+4\right) = 0$

$\therefore \left.\begin{matrix} a = 3 \\ b = -2 \end{matrix}\right\}$ or $\left.\begin{matrix} a = -3 \\ b = 2 \end{matrix}\right\}$

b. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes z_2 in modulus/argument form	1
• writes $z_1 z_2$ in modulus/argument form	1
ii • writes z_1 as quotient of two complex numbers each in form $a+ib$	1
• realizes denominator to write z_1 in form $a+ib$	1

Answer

i. $z_1 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$

$z_2 = 1+i$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\arg(z_1 z_2) = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$

$|z_1 z_2| = \sqrt{2}$

$\therefore z_1 z_2 = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ii. $z_1 z_2 = \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$$z_1 = \frac{\frac{\sqrt{2}}{2}\left(1+\sqrt{3}\,i\right)}{1+i}$$

$$= \frac{\frac{\sqrt{2}}{2}\left(1+\sqrt{3}\,i\right)(1-i)}{2}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4} + i\,\frac{\sqrt{6}-\sqrt{2}}{4}$$

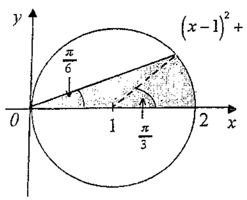
c. Outcomes assessed : H5, E3

Marking Guidelines

Criteria	Marks
i • selects region inside correct circle	1
• shades appropriate region inside this circle	1
ii • writes numerical expression for area	1
• simplifies, giving answer in exact form	1

Answer

i.



ii. Area is A sq.units where

$A = \frac{1}{2}\times 1\times 1\times \sin\frac{2\pi}{3} + \frac{1}{2}\times 1^2 \times \frac{\pi}{3}$

$= \frac{\sqrt{3}}{4} + \frac{\pi}{6}$

d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
• uses De Moivre's theorem to write expression for z^2	1
• uses double angle formulae to write expression for $z^2 + r^2$ in terms of $\cos\theta$ and $\sin\theta$	1
• rearranges to show required result	1

Answer

$z^2 = r^2\left(\cos 2\theta + i\sin 2\theta\right)$

$z^2 + r^2 = r^2\left(1 + \cos 2\theta + i\sin 2\theta\right)$

$= r^2\left(2\cos^2\theta + i\,.\,2\sin\theta\cos\theta\right)$

$\therefore z^2 + r^2 = 2r^2\cos\theta\left(\cos\theta + i\sin\theta\right)$

$= 2r^2\cos\theta\,.\,z$

$\therefore \frac{z}{z^2+r^2} = \frac{1}{2r^2\cos\theta}$ which is real.

Question 4

a. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • finds expression for y coordinates of P, Q in terms of b and e	1
• substitutes for e to find PQ in terms of a and b	1
ii • uses length of PQ to write one equation	1
• uses equation of the hyperbola to write a second equation	1
• solves simultaneously to find a and b	1
iii • shows curve with correct shape and x intercepts	1
• shows coordinates of foci	1
• shows equations of directrices	1
• shows equations of asymptotes	1

Answer

i. Vertical line through S has equation $x = ae$

At P, Q: $\frac{(ae)^2}{a^2} - \frac{y^2}{b^2} = 1$

$y^2 = b^2\left(e^2 - 1\right)$

$= \frac{b^4}{a^2}$

Hence P, Q have coordinates $\left(ae\,,\pm\frac{b^2}{a}\right)$

$\therefore PQ = \frac{2b^2}{a}$

ii. $PQ = 48 \Rightarrow b^2 = 24a$

Also $\frac{9^2}{a^2} - \frac{24^2}{b^2} = 1$

$\frac{9^2}{a^2} - \frac{24}{a} = 1$

$a^2 + 24a - 81 = 0$

$(a+27)(a-3) = 0$

$\therefore 0 < a < b \Rightarrow \begin{cases} a = 3 \\ b = 6\sqrt{2} \end{cases}$

iii.

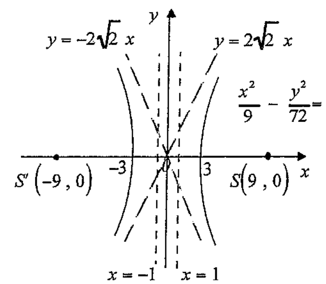
$b^2 = a^2\left(e^2 - 1\right)$

$72 = 9\left(e^2 - 1\right)$

$\therefore e = 3,\quad ae = 9,\quad \frac{a}{e} = 1$

Foci : $(\pm 9\,, 0)$

Directrices: $x = \pm 1$



b. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • finds the gradient of the tangent in terms of t	1
• finds the equation of the tangent	1
ii • writes the equation of OM	1
• solves simultaneously with equation of tangent to find coordinates of M in terms of t	1
iii • applies any appropriate method to eliminate t to obtain direct relation between x, y at M	1
• obtains equation of locus of M in simplified form	1

Answer

i. $y=\frac{c}{t}\Rightarrow\frac{dy}{dt}=-\frac{c}{t^2}$

$x=ct\Rightarrow\frac{dx}{dt}=c$

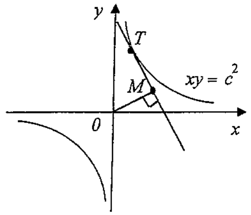
$\therefore\frac{dy}{dx}=-\frac{1}{t^2}$

Tangent at T has equation

$$y-\frac{c}{t}=-\frac{1}{t^2}(x-ct)$$

$$t^2y-ct=-x+ct$$

$$x+t^2y-2ct=0$$



ii. OM has equation $y=t^2x$.

At M: $x+t^4x-2ct=0$

M is point $\left(\frac{2ct}{1+t^4},\frac{2ct^3}{1+t^4}\right)$

iii. At M, $x\neq0$, $y\neq0$ and

$$xy=4c^2\frac{t^4}{(1+t^4)^2}$$

$$=4c^2\frac{x^4t^4}{(x^2+x^2t^4)^2}$$

$$=4c^2\frac{x^2y^2}{(x^2+y^2)^2}$$

$\therefore$ locus of M has equation

$$(x^2+y^2)^2=4c^2xy$$

Question 5

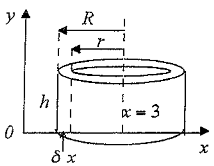
a. Outcomes assessed : H8, HE4, HE6, E1, E7

Marking Guidelines

Criteria	Marks
i • identifies inner and outer radii of a typical cylindrical shell in terms of x and δx	1
• expresses the volume of such a shell in terms of its height h, x and δx	1
• writes V as a limiting sum of the volumes of these cylindrical shells	1
• identifies h as a function of x and writes V as a definite integral	1
ii • splits the integral into a sum and obtains the inverse sine primitive for one component	1
• obtains the primitive function of the remaining component	1
• evaluates either component by substitution	1
• completes the calculation of V to specified accuracy.	1

Answer

i.



$h=\frac{6}{\sqrt{4-x^2}}$

$R=3-x$

$r=3-(x+\delta x)$

$\delta V=\pi h(R^2-r^2)$

$=\pi h(R+r)(R-r)$

$=\pi h\{2(3-x)-\delta x\}\delta x$

Ignoring terms in $(\delta x)^2$,

$$V=\lim_{\delta x\to0}\sum_{x=0}^{x=1}\pi\frac{6}{\sqrt{4-x^2}}2(3-x)\,\delta x$$

$$=12\pi\int_0^1\frac{3-x}{\sqrt{4-x^2}}\,dx$$

ii. $V=36\pi\int_0^1\frac{1}{\sqrt{4-x^2}}\,dx+6\pi\int_0^1-2x(4-x^2)^{-\frac{1}{2}}\,dx$

$$=36\pi\left[\sin^{-1}\tfrac{x}{2}\right]_0^1+12\pi\left[\sqrt{4-x^2}\right]_0^1$$

$$=36\pi\left(\tfrac{\pi}{6}-0\right)+12\pi\left(\sqrt{3}-2\right)$$

$$=6\pi\left(\pi+2\sqrt{3}-4\right)$$

Volume is 49 m^3 (to the nearest m^3)

b. Outcomes assessed : E2, E4

Marking Guidelines

Criteria	Marks
i • writes $P(x)$ with factor $(x-\alpha)^2$	1
• shows $P'(x)$ has a factor $(x-\alpha)$	1
ii • writes two equations for α in terms of q and r using $P(\alpha)=P'(\alpha)=0$	1
• solves simultaneously to obtain required expression for α	1
• substitutes this expression for α in original equation	1
• rearranges to obtain required relation between q and r	1
• uses the facts that q and r are real and non-zero to deduce $q<0$	1

Answer

i. If α is a double root, then for some polynomial $Q(x)$,

$$P(x)=(x-\alpha)^2Q(x)$$

$$P'(x)=2(x-\alpha)Q(x)+(x-\alpha)^2Q'(x)$$

$$=(x-\alpha)\{2Q(x)+(x-\alpha)Q'(x)\}$$

Hence $P'(\alpha)=0$.

ii. $P(x)=x^3+qx+r$

$P'(x)=3x^2+q$

$P(\alpha)=0\Rightarrow\ \alpha^3=-q\alpha-r$ (1)

$P'(\alpha)=0\Rightarrow3\alpha^2=-q$ (2)

$(1)\div(2)\Rightarrow\ \tfrac{1}{3}\alpha=\alpha+\frac{r}{q}$

$-\tfrac{2}{3}\alpha=\frac{r}{q}$

$\alpha=-\frac{3r}{2q}$

Then

$$\left(-\frac{3r}{2q}\right)^3+q\left(-\frac{3r}{2q}\right)+r=0$$

$$-27r^3-12q^3r+8q^3r=0$$

$$-r(4q^3+27r^2)=0$$

$\therefore r\neq0\ \Rightarrow\ 4q^3+27r^2=0$

But q and r are real and non-zero,

$\therefore27r^2>0$ and hence $4q^3<0$.

$\therefore q<0$.

Question 6

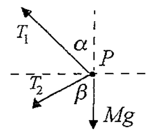
a. Outcomes assessed : E1, E5

Marking Guidelines

Criteria	Marks
i • shows the weight force and both tension forces at appropriate angles to the vertical	1
ii • invokes Newton's 2nd Law to deduce direction and magnitude of resultant force on P	1
• uses fact that vertical component of resultant is zero to justify first equation	1
• uses magnitude, direction of horizontal component of resultant to justify second equation	1
iii • eliminates T_1 from the simultaneous equations to obtain expression for T_2	1
• eliminates T_2 from the simultaneous equations (or substitutes) to obtain expression for T_1	1
iv • uses $T_2\geq0$ to find minimum value for ω	1
• states that lower string is slack and describes new position and size of circle of motion	1

Answer

i. Forces on P



ii. Since P is performing uniform circular motion in a horizontal circle, by Newton's second law, resultant force on P is directed horizontally toward the centre of the circle and has magnitude $Mr\omega^2$. Resolving components vertically and horizontally :

vertical component of resultant is zero $\Rightarrow T_1\cos\alpha - T_2\cos\beta = Mg$ (1)

horizontal component of resultant is $\overleftarrow{Mr\omega^2}$ $\Rightarrow T_1\sin\alpha + T_2\sin\beta = Mr\omega^2$ (2)

iii.

$$-\sin\alpha\times(1)+\cos\alpha\times(2)\Rightarrow T_2(\sin\alpha\cos\beta+\cos\alpha\sin\beta) = -\sin\alpha\,.\,Mg+\cos\alpha\,.\,Mr\omega^2$$

$$T_2\sin(\alpha+\beta) = M(r\omega^2\cos\alpha - g\sin\alpha)$$

$$T_2 = \frac{M(r\omega^2\cos\alpha - g\sin\alpha)}{\sin(\alpha+\beta)}$$

$$\sin\beta\times(1)+\cos\beta\times(2)\Rightarrow T_1(\sin\beta\cos\alpha+\cos\beta\sin\alpha) = \sin\beta\,.\,Mg+\cos\beta\,.\,Mr\omega^2$$

$$T_1\sin(\alpha+\beta) = M(r\omega^2\cos\beta + g\sin\beta)$$

$$T_1 = \frac{M(r\omega^2\cos\beta + g\sin\beta)}{\sin(\alpha+\beta)}$$

iv. While string AP is taut for all values of ω, string BP is taut provided $T_2 > 0$.

$T_2 > 0 \Rightarrow r\omega^2\cos\alpha > g\sin\alpha$

$\omega^2 > \dfrac{g\tan\alpha}{r}$ Hence least value of ω is $\sqrt{\dfrac{g\tan\alpha}{r}}$.

If ω falls below this value, string BP goes slack and particle performs circular motion in a horizontal circle of smaller radius with the plane of the circle at a greater distance below A. The angle at A will then be $\theta < \alpha$.

b. Outcomes assessed : H8, E3

Marking Guidelines

Criteria	Marks
i • uses De Moivre's theorem to write expression for z^n	1
• writes expression for reciprocal then obtains required result by addition	1
ii • writes Binomial expansion	1
• simplifies and rearranges, grouping terms to facilitate use of result from i.	1
• applies result from i.	1
iii • substitutes expression from ii. in integrand then finds primitive function	1
• evaluates integral after substitution of limits	1

Answer

i. $z = \cos\theta + i\sin\theta$

By De Moivre's theorem, $z^n = \cos n\theta + i\sin n\theta$

Then $\dfrac{1}{z^n} = \dfrac{\overline{z^n}}{|z^n|} = \cos n\theta - i\sin n\theta$ Hence $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

ii. $$\left(z+\frac{1}{z}\right)^5 = z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 + 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5$$

$$= \left(z^5+\frac{1}{z^5}\right) + 5\left(z^3+\frac{1}{z^3}\right) + 10\left(z+\frac{1}{z}\right)$$

$$\therefore (2\cos\theta)^5 = 2\cos5\theta + 10\cos3\theta + 20\cos\theta$$

$$\therefore \cos^5\theta = \tfrac{1}{16}(\cos5\theta + 5\cos3\theta + 10\cos\theta)$$

iii. $$\int_0^{\frac{\pi}{2}}\cos^5\theta\,d\theta = \tfrac{1}{16}\int_0^{\frac{\pi}{2}}(\cos5\theta+5\cos3\theta+10\cos\theta)\,d\theta$$

$$= \tfrac{1}{16}\left[\tfrac{1}{5}\sin5\theta + \tfrac{5}{3}\sin3\theta + 10\sin\theta\right]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{16}\left(\tfrac{1}{5} - \tfrac{5}{3} + 10\right)$$

$$= \tfrac{8}{15}$$

Question 7

a. Outcomes assessed : H5, PE3, E2

Marking Guidelines

Criteria	Marks
i • uses trig identities to write $\{\cos\frac{1}{2}(a+b)\}^2$ in terms of $\cos a\cos b$ and $\sin a\sin b$	1
• writes a similar expression for $\{\cos\frac{1}{2}(a-b)\}^2$	1
• eliminates terms in $\sin a\sin b$ by addition	1
• uses $\{\cos\frac{1}{2}(a-b)\}^2 \le 1$ to deduce required reesult	1
ii • applies result from i. to both pairs a, b and c, d, when considering $\cos a\cos b\cos c\cos d$	1
• applies result from i. to pair $\frac{1}{2}(a+b)$, $\frac{1}{2}(c+d)$	1
iii • applies result from ii. with $d = \frac{1}{3}(a+b+c)$	1
• rearranges to obtain required inequality	1

Answer

i. $$\{\cos\tfrac{1}{2}(a+b)\}^2 = \tfrac{1}{2}\{1+\cos(a+b)\} = \tfrac{1}{2}\{1+\cos a\cos b - \sin a\sin b\}$$

$$\{\cos\tfrac{1}{2}(a-b)\}^2 = \tfrac{1}{2}\{1+\cos(a-b)\} = \tfrac{1}{2}\{1+\cos a\cos b + \sin a\sin b\}$$

$$\therefore \{\cos\tfrac{1}{2}(a+b)\}^2 + \{\cos\tfrac{1}{2}(a-b)\}^2 = 1+\cos a\cos b$$

$$\therefore \{\cos\tfrac{1}{2}(a+b)\}^2 + 1 \ge 1+\cos a\cos b \qquad \text{since } \{\cos\tfrac{1}{2}(a-b)\}^2 \le 1$$

$$\therefore \cos a\cos b \le \{\cos\tfrac{1}{2}(a+b)\}^2$$

ii. $$\cos a\cos b\cos c\cos d \le \{\cos\tfrac{1}{2}(a+b)\}^2\{\cos\tfrac{1}{2}(c+d)\}^2$$

$$= \{\cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(c+d)\}^2 , \quad \text{where } 0<\tfrac{1}{2}(a+b)<\tfrac{\pi}{2},\ 0<\tfrac{1}{2}(c+d)<\tfrac{\pi}{2}$$

$$\le \left\{\left(\cos\tfrac{1}{2}\left[\tfrac{1}{2}(a+b)+\tfrac{1}{2}(c+d)\right]\right)^2\right\}^2$$

$$= \{\cos\tfrac{1}{4}(a+b+c+d)\}^4$$

iii $\cos a \cos b \cos c \cos\frac{1}{3}(a+b+c) \le \left\{\cos\frac{1}{4}\left[a+b+c+\frac{1}{3}(a+b+c)\right]\right\}^4$, since $0 < \frac{1}{3}(a+b+c) < \frac{\pi}{2}$

$$= \left\{\cos\tfrac{1}{3}(a+b+c)\right\}^4$$

$$\therefore \cos a \cos b \cos c \le \left\{\cos\tfrac{1}{3}(a+b+c)\right\}^3$$

b. Outcomes assessed : P4, HE3

Marking Guidelines

Criteria	Marks
i • equates coefficients of x^2 to obtain one equation in n and a	1
• equates coefficients of x^3 to obtain a second equation in n and a	1
• considers these equations simultaneously to obtain required result	1
ii • substitutes for a in either of the original equations	1
• rearranges to obtain given quadratic in n	1
• solves this quadratic for n, subject to the condition that n is a positive integer	1
• writes corresponding value of a	1

Answer

i. ${}^nC_2\, a^2 = 63 \Rightarrow \dfrac{n(n-1)}{2}a^2 = 63$

${}^nC_3\, a^3 = 189 \Rightarrow \dfrac{n(n-1)(n-2)}{3\times 2}a^3 = 189$

$$\therefore \frac{(n-2)a}{3} = \frac{189}{63}$$

$$(n-2)a = 9$$

ii. $(n-2)^2 63 = \dfrac{n(n-1)}{2}(n-2)^2 a^2$

$$= \frac{n(n-1)}{2}81$$

$$14(n^2-4n+4) = 9(n^2-n)$$

$$5n^2-47n+56=0$$

$$(5n-7)(n-8)=0$$

$\therefore n = 8$ and $a = \frac{3}{2}$ (since n is integral)

Question 8

a. Outcomes assessed : H5, P4, PE3

Marking Guidelines

Criteria	Marks
i • writes A as the sum of the areas of two triangles using supplementary opposite angles	1
• rearranges to obtain required result	1
• writes two expressions for QS^2 using the cosine rule	1
• equates these and rearranges to obtain required result	1
ii • uses sum of squares of expressions for $\sin\theta$, $\cos\theta$ is equal to1	1
• rearranges to obtain required expression for A^2	1
iii • uses difference of squares to factorise expression into two factors	1
• rearranges to rewrite each of these two factors as a difference of squares	1
• uses difference of squares again to write A^2 as a product of four factors	1
• introduces s and rearranges to obtain required result	1

Answer

i. Opposite $\angle s$ of a cyclic quadrilateral are supplementary.

$\therefore A = \frac{1}{2}ad\sin\theta + \frac{1}{2}bc\sin(\pi-\theta)$

$= \frac{1}{2}\sin\theta(ad+bc)$

$\therefore \sin\theta = \dfrac{2A}{ad+bc}$

Using the cosine rule in ΔQPS, ΔQRS

$$QS^2 = a^2+d^2-2ad\cos\theta$$

$$QS^2 = b^2+c^2-2bc\cos(\pi-\theta)$$

$$= b^2+c^2+2bc\cos\theta$$

$$\therefore\quad 0 = a^2+d^2-b^2-c^2-2\cos\theta(ad+bc)$$

$$\therefore \cos\theta = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}$$

ii. $1 = \sin^2\theta+\cos^2\theta \Rightarrow 1 = \left(\dfrac{2A}{ad+bc}\right)^2 + \left(\dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)}\right)^2$

$$4(ad+bc)^2 = 16A^2 + (a^2+d^2-b^2-c^2)^2$$

$$\therefore A^2 = \tfrac{1}{16}\left\{4(ad+bc)^2-(a^2+d^2-b^2-c^2)^2\right\}$$

iii. $A^2 = \frac{1}{16}\left\{2(ad+bc)-(a^2+d^2-b^2-c^2)\right\}\left\{2(ad+bc)+(a^2+d^2-b^2-c^2)\right\}$

$= \frac{1}{16}\left\{(b+c)^2-(a-d)^2\right\}\left\{(a+d)^2-(b-c)^2\right\}$

$= \frac{1}{16}\left\{(b+c)-(a-d)\right\}\left\{(b+c)+(a-d)\right\}\left\{(a+d)-(b-c)\right\}\left\{(a+d)+(b-c)\right\}$

$= \frac{1}{16}(2s-2a)(2s-2d)(2s-2b)(2s-2c)$

$= (s-a)(s-b)(s-c)(s-d)$

$\therefore A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$

b. Outcomes assessed : HE2, E9

Marking Guidelines

Criteria	Marks
• defines a sequence of statements $S(n)$, $n = 1, 2, 3, \ldots$ and shows the first is true	1
• writes the LHS of $S(k+1)$ in the form $\tan\left\{(2k-1)\frac{\pi}{4}+\frac{\pi}{2}\right\}$	1
• uses appropriate trig. identities to write this in terms of LHS of $S(k)$	1
• shows $S(k+1)$ is true if $S(k)$ is true	1
• completes process of Mathematical Induction	1

Answer

Let $S(n)$, $n = 1, 2, 3, \ldots$ be the sequence of statements $\tan(2n-1)\frac{\pi}{4} = (-1)^{n+1}$.

Consider $S(1)$: $LHS = \tan\frac{\pi}{4} = 1 = (-1)^2 = RHS$. Hence $S(1)$ is true.

If $S(k)$ is true : $\tan(2k-1)\frac{\pi}{4} = (-1)^{k+1}$ **

Consider $S(k+1)$: $\quad LHS = \tan\left\{2(k+1)-1\right\}\frac{\pi}{4}$

$$
\begin{aligned}
&= \tan\left\{(2k-1)\tfrac{\pi}{4}+\tfrac{\pi}{2}\right\} \\
&= -\cot(2k-1)\tfrac{\pi}{4} \\
&= -\left\{\tan(2k-1)\tfrac{\pi}{4}\right\}^{-1} \\
&= -\left\{(-1)^{k+1}\right\}^{-1} \qquad \text{if } S(k) \text{ is true, using } ** \\
&= -\left\{(-1)^{-1}\right\}^{k+1} \\
&= -(-1)^{k+1} \\
&= (-1)^{(k+1)+1} \\
&= RHS
\end{aligned}
$$

Hence if $S(k)$ is true then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true and then $S(3)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n \geq 1$.