NSW INDEPENDENT SCHOOLS

PRELIMINARY EXAMINATION

2000

MATHEMATICS

MATHEMATICS EXTENSION 1

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt **ALL** questions.
- **ALL** questions are of equal value.
- Write your Student Name / Number on every page of the question paper and your answer sheets .
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- The answers to the six questions are to be handed in separately, clearly marked Question 1, Question 2 etc.
- ***This question paper must not be removed from the examination room.***

STUDENT NUMBER / NAME: ....

Question 1 (Start a new page) **Marks**

a. Solve $\dfrac{2x+3}{x-2} < 1$ — 3

b. i. Find the roots of $x^3 - 6x^2 + 5x + 12 = 0$. — 2

ii. Hence sketch $y = x^3 - 6x^2 + 5x + 12$. — 1

c. When $P(x) = x^3 + x^2 - a$ is divided by $x - 2$, the remainder is 4. Find the remainder when $P(x)$ is divided by x — 2

d. i. Sketch the curve $y = \dfrac{1}{x-1}$, clearly showing the y-intercept, P. — 1

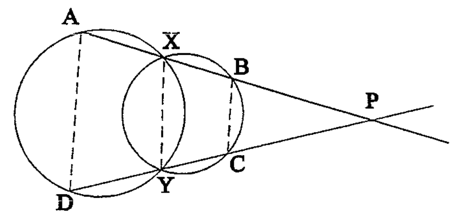
ii. Show that the equation of the tangent at P is given by — 2
$$x + y + 1 = 0$$

iii. Find the perpendicular distance from the point Q(2, 1) to the line
$$x + y + 1 = 0$$ — 1

Question 2 (Start a new page)

a. Find the acute angle between the lines $y = 2x - 7$ and $3x - 5y - 6 = 0$ — 2

b. Two circles intersect at the points X and Y. The straight lines AXB and DYC intersect at P, as shown. Let $\angle CBX = \alpha$



i. Prove that $AD \parallel BC$ — 2

ii. If PB = 5, BX = 3 and PC = 4, find YC — 2

c. Find the point P which divides the interval joining A(2, –4) and B(3, –3) externally in the ratio 2:3 — 2

d. In the "6 from 36" pools competition, punters select six numbers from thirty six possibilities numbered from 1 to 36.

i. How many different combinations are possible? — 1

ii. Jai always picks the numbers 16 and 31. How many different combinations can Jai select? — 1

iii. A System 8 entry allows a punter to choose eight numbers, any six of which can win. If a normal entry (ie choosing 6 numbers only) costs 50 cents, how much should a System 8 entry cost? — 2

Question 3 (Start a new page)

Marks

a. The roots of the quadratic equation $x^2 + 4x + 2 = 0$ are α and β

i. Find $(\alpha - 2)(\beta - 2)$ **1**

ii. Show that the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ is -5 **2**

iii. Hence or otherwise, find a quadratic equation which has roots **2**

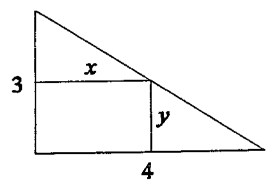
$$\frac{1}{\alpha^3} \text{ and } \frac{1}{\beta^3}$$

b. i. Show that $(a\sqrt{b} - c)^2 = a^2b + c^2 - 2ac\sqrt{b}$ **1**

ii. Hence or other wise find integers a, b and c such that **2**

$$a\sqrt{b} - c = \sqrt{24 - 16\sqrt{2}}$$

c. The diagram shows a right angled triangle with sides 3 cm and 4 cm. A rectangle is inscribed inside the triangle.



i. Show that the area of the triangle can be given by **1**

$$A = xy + \frac{(4 - x)y}{2} + \frac{(3 - y)x}{2}$$

ii. Hence show that $y = \frac{12 - 3x}{4}$ **1**

iii. Show that the maximum area of the rectangle is 3 cm² **2**

Question 4 (Start a new page)

Marks

a. $P(2ap,\ ap^2)$, $Q(2aq,\ aq^2)$ are points on the parabola $x^2 = 4ay$.

i. Show that the tangent at P has equation $y = px - ap^2$ **1**

ii. Find the point of intersection, M, of the tangents at P and Q. **2**

iii. If $q = 3p$, find the locus of M **2**

b. Differentiate $(2x - 3)^2(x + 1)^3$, giving your answer in simplest form. **3**

c. i. Express $\cos\theta - 2\sin\theta$ in the form $A\cos(\theta + \alpha)$, $A > 0$, $0 < \alpha < 90°$ **2**

ii. Hence solve the equation $\cos\theta - 2\sin\theta = 1$, $0 \le \theta \le 360°$ **2**

Question 5 (Start a new page)

a. John and Mary go to the cinema with three other couples. They sit together as a group in a single row.

i. In how many ways can they be arranged? **1**

ii. In how many ways can they sit so that each couple is together? **1**

iii. John and Mary had an argument going into the cinema and decide they do not want to sit together. How many arrangements are possible if the other couples are still sitting with their partners? **2**

b. Hozni is on a ship off the coast and notices two lighthouses on shore. The lighthouse on Broken Head bears 226° from the ship. The lighthouse on Skunk Head bears 197° from the ship and 182° from the Broken Head lighthouse. The lighthouses are 2.3 kilometres apart.

i. Find the distance of Hozni's ship from the Broken Head lighthouse. **2**

ii. If the angle of elevation of the Broken Bay lighthouse from Hozni's ship is 6°, calculate the height of the lighthouse above sea level. **2**

[Question 5 is continued on the next page]

Question 5 (continued) **Marks**

c. i. Show that $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \tan^2 x$ **2**

ii. Hence show that the exact value of $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$ **2**

Question 6 (Start a new page)

a. A is the point (x, y) and B is the point $(4, 0)$. O is the origin.

i. Find expressions for the gradient of OA and AB. **1**

ii. If OA is perpendicular to AB, show that $x^2 + y^2 = 4x$. **1**

iii. What kind of curve does this equation describe? **2**

Give a geometrical reason why this should be the case.

b. i. On the same diagram, sketch and clearly label the graphs of **2**

$$y = |x - 2| \text{ and } y = |x + 5|$$

Show clearly any intercepts on the axes and any intersection points.

ii. Sketch on a separate diagram the graph of **2**

$$y = |x - 2| - |x + 5|$$

Show clearly any intercepts on the axes and any intersection points.

iii. Hence, or otherwise, solve $|x - 2| < |x + 5|$ **2**

iv. For what value(s) of a do the equations **2**

$$y = |x - 2| - a \text{ and } y = |x + 5|$$

have no simultaneous solution?

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Q1 (a) $\dfrac{2x+3}{x-2} < 1$

Critical value at $x=2$ and

$\dfrac{2x+3}{x-2} = 1$

$2x+3 = x-2$

$x = -5$

$x<-5$; $-5<x<2$; $x>2$

Test $x=0$: $\frac{3}{-2} < 1$ true

$\therefore -5 < x < 2$

(b)(i) Test $x = \pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

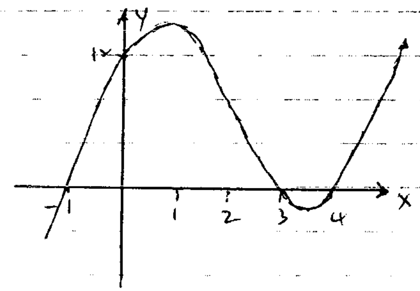
$P(-1) = 0$

$P(3) = 0$

$P(4) = 0$

$\therefore$ Roots are $x=-1, x=3, x=4$

(ii)



(c) $P(2) = 2^3 + 2^2 - a = 4$

$-a = -8$

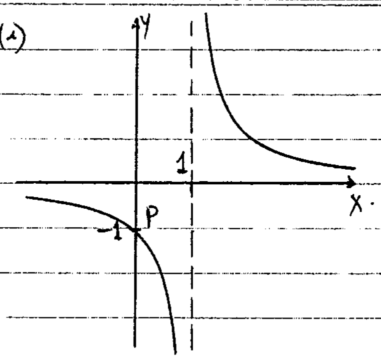
$a = 8$

$\therefore P(x) = x^3 + x^2 - 8$

$P(0) = -8$

$\therefore$ Remainder is -8

(d)(i)



(ii) $y' = \dfrac{-1}{(x-1)^2}$

At $x=0$, $y' = -1$ and $y = -1$

$\therefore y - y_1 = m(x - x_1)$

$y - {}^-1 = -1(x-0)$

$y+1 = -x$

$x+y+1 = 0$

(iii) $d = \left|\dfrac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}\right|$

$= \left|\dfrac{2+1+1}{\sqrt{1^2+1^2}}\right|$

$= \dfrac{4}{\sqrt{2}}$ or $2\sqrt{2}$

Q2 (a) $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_2 m_1}\right|$

$= \left|\dfrac{2 - 3/5}{1 + 2\times 3/5}\right|$

$= \dfrac{7}{11}$

$\therefore \theta = 32°28'$

(b)(i) $\hat{XYC} = 180 - \alpha$ (opp. angles of cyclic quad.)

and $\hat{DAX} = 180 - \alpha$ (exterior angle of cyclic quad.)

so $\hat{DAX} + \hat{XBC} = 180$

and $AD \parallel DC$ (cointerior angles add to 180°)

(ii) Now $PX.PB = PY.PC$

$8 \times 5 = 4 \times PY$

$PY = 10$

and $YC = 6$

(c) $P\left(\dfrac{kx_2 + lx_1}{k+l}, \dfrac{ky_2 + ly_1}{k+l}\right)$

$= \left(\dfrac{-2\times3 + 3\times2}{-2+3}, \dfrac{-2\times{}^-3 + 3\times{}^-4}{-2+3}\right)$

$= (0, -6)$

(d) (i) ${}^{36}C_6 = 1947792$

(ii) ${}^{36}C_4 = 58905$

(iii) ${}^{8}C_6 = 28$

Q3 (a) $\alpha+\beta = -4$

$\alpha\beta = 2$

(i) $(\alpha-2)(\beta-2) = \alpha\beta - 2(\alpha+\beta) + 4$

$= 2 - 2\times -4 + 4$

$= 14$

(ii) $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} = \dfrac{\alpha^3+\beta^3}{\alpha^3\beta^3}$

$= \dfrac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{(\alpha\beta)^3}$

But $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 16 - 2\times 2 = 12$

$\therefore \dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} = \dfrac{-4\times(12-2)}{(2)^3}$

$= -5$

(iii) Quadratic will be of the form

$x^2 - \left(\dfrac{1}{\alpha^3}+\dfrac{1}{\beta^3}\right)x + \dfrac{1}{\alpha^3\beta^3} = 0$

$x^2 + 5x + \dfrac{1}{8} = 0$

$8x^2 + 40x + 1 = 0$

(b)(i) $(a\sqrt{b} - c)^2 = a^2b - 2ac\sqrt{b} + c^2$

$= a^2b + c^2 - 2ac\sqrt{b}$

(ii) Squaring both sides

$a^2b + c^2 - 2ac\sqrt{b} = 24 - 16\sqrt{2}$

Clearly, $b = 2$

$\therefore 2a^2 + c^2 = 24$

$ac = 8$

$\Rightarrow a = +2, c = 4$ or $a = -2, c = -4$

$\therefore a = \pm 2, b = 2; c = \pm 4$

(c) (i) Area of rectangle $= xy$

Area of top triangle $= \frac{1}{2}x(3-y)$

Area of other triangle $= \frac{1}{2}y(4-x)$

$\therefore A = xy + \dfrac{(4-x)y}{2} + \dfrac{(3-y)x}{2}$

(ii) But $A = 6$ $\left(\frac{1}{2}\times 3\times 4\right)$

$\therefore xy + \dfrac{(4-x)y}{2} + \dfrac{(3-y)x}{2} = 6$

$2xy + 4y - xy + 3x - xy = 12$

$4y + 3x = 12$

$y = \dfrac{12-3x}{4}$

(iii) Area of rectangle $= xy$

$A = x.\dfrac{(12-3x)}{4}$

$= 3x - \frac{3}{4}x^2$

This gives an inverted parabola with vertex (maximum point) at $x = -b/2a$

$= -\dfrac{3}{2\times -3/4} = 2$

$\therefore$ Maximum area when $x = 2$

so $A = 3\times 2 - \frac{3}{4}\times 2^2$

$= 3\text{ cm}^2$

Q4 (a) (i) $y = x^2/4a$

$dy/dx = 2x/4a = x/2a$

At P, $x = 2ap$ so $m = p$

$\therefore y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$= px - 2ap^2$

$\therefore y = px - ap^2$

(ii) Tangents at

P: $y = px - ap^2$

Q: $y = qx - aq^2$

Solving: $(p-q)x = a(p^2 - q^2)$

$x = a(p+q)$

$\Rightarrow y = apq$

$\therefore M(a(p+q), apq)$

(iii) $X = a(p+q)$, $Y = apq$

and $q = 3p$

$\therefore X = a(p+3p) = 4ap$

+ $p = \dfrac{X}{4a}$

so $Y = a.\dfrac{X}{4a}.3.\dfrac{X}{4a}$

$16aY = 3X^2$

i.e. $X^2 = \dfrac{16aY}{3}$

(b) $\dfrac{dy}{dx} = u.\dfrac{dv}{dx} + v.\dfrac{du}{dx}$

$= (2x-3)^2.3(x+1)^2 + (x+1)^3 4(2x-3)^1$

$= (x+1)^2(2x-3)^1[3(2x-3) + 4(x+1)]$

$= (x+1)^2(2x-3)(10x-5)$

$= 5(x+1)^2(2x-3)(2x-1)$

(c)(i) $A\cos(\theta+\alpha) = A\cos\theta\cos\alpha - A\sin\theta\sin\alpha$

$\therefore A\cos\alpha = 1$

$A\sin\alpha = 2$

$\Rightarrow A = \sqrt{5}$; $\tan\alpha = 2$

so $\alpha = 63°26'$

$\therefore \cos\theta - 2\sin\theta = \sqrt{5}\cos(\theta + 63°26')$

(ii) $\sqrt{5}\cos(\theta + 63°26') = 1$

$\cos(\theta + 63°26') = 1/\sqrt{5}$

$\therefore \theta + 63°26' = 63°26', 296°34', 423°26'$

$\therefore \theta = 0°, 233°08', 360°$

Q5 a. (i) $8! = 40320$

(ii) There are 4 pairs so the pairs can be arranged in $4!$ ways. But each pair can sit in 2 ways

$\therefore$ Total $= 4! \times 2^4 = 384$

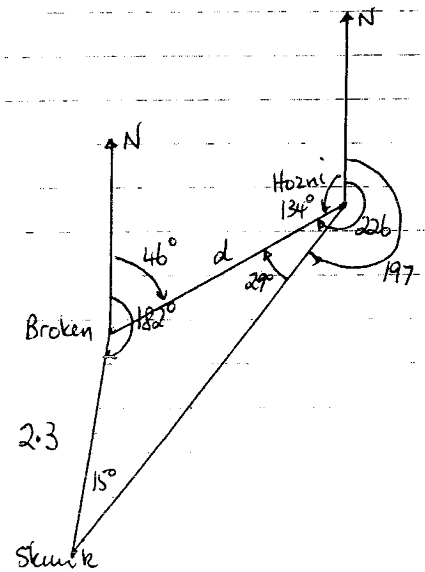
(iii) There are 3 pairs plus John and Mary i.e 5 "units" with each pair sitting in 2 ways

$\therefore$ Total $= 5! \times 2^3 = 960$.

Excluding those situations where John + Mary are together gives

$960 - 384 = 576$

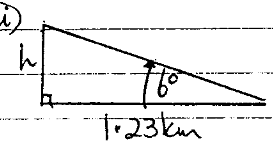
(b)



(i) $\dfrac{d}{\sin 15} = \dfrac{2.3}{\sin 29}$

$\therefore d = 1.23$ kms

(ii)



$\tan 6° = \dfrac{h}{1.23}$

$h = 1.23 \times \tan 6°$

$= 0.13$ km

$= 130$ metres

(c) (i) $LHS = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

$= \dfrac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)}$

$= \dfrac{2\sin^2 x}{2\cos^2 x}$

$= \tan^2 x = RHS$

(ii) Let $x = 22\frac{1}{2}$

$\tan^2 22\frac{1}{2} = \dfrac{1 - \cos 45}{1 + \cos 45}$

$= \dfrac{1 - 1/\sqrt{2}}{1 + 1/\sqrt{2}}$

$= \dfrac{\sqrt{2} - 1}{\sqrt{2} + 1} \times \dfrac{\sqrt{2} - 1}{\sqrt{2} - 1}$

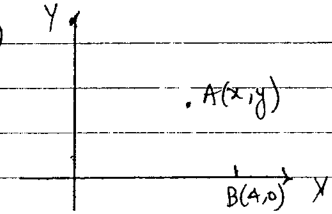
$\tan^2 22\frac{1}{2} = (\sqrt{2} - 1)^2$

$\therefore \tan 22\frac{1}{2} = \pm(\sqrt{2} - 1)$

But $\tan 22\frac{1}{2} > 0$

$\therefore \tan 22\frac{1}{2} = \sqrt{2} - 1$

Q6 (a)



(i) $M_{OA} = y/x$; $M_{AB} = \dfrac{y}{x - 4}$

(ii) $M_{OA} \times M_{AB} = -1$

$\dfrac{y}{x} \times \dfrac{y}{x - 4} = -1$

$y^2 = -x^2 + 4x$

So $x^2 + y^2 = 4x$

(iii) $x^2 - 4x + y^2 = 0$

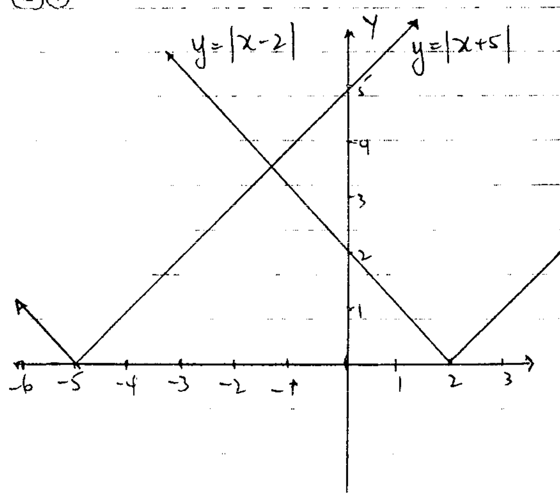
$x^2 - 4x + 4 + y^2 = 4$

$(x - 2)^2 + y^2 = 2^2$

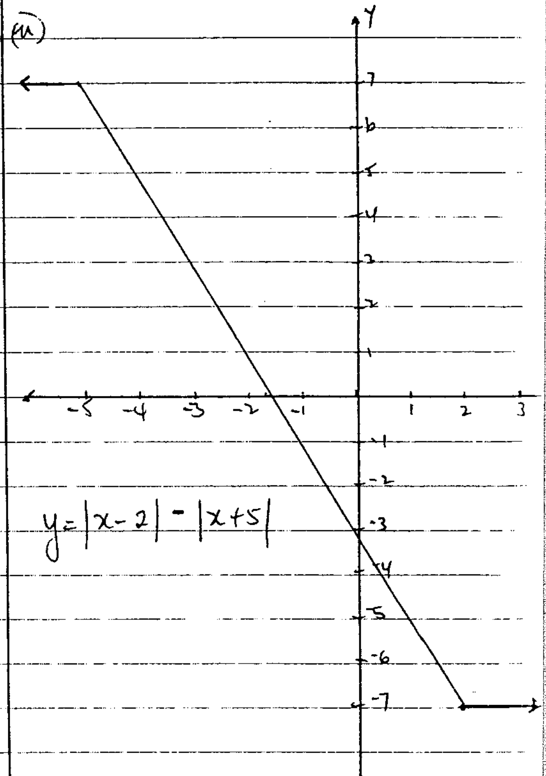
$\therefore$ Circle, centre $(2, 0)$ radius 2

$\angle OAB$ is the angle in a semi-circle (which equals $90°$)

(b) (i)



(ii)



(iii) $|x - 2| < |x + 5|$

$|x - 2| - |x + 5| < 0$

From graph, $x > -1.5$

(iv) From graph, $a > 7$ or $a < -7$

[since, solving simultaneously

$|x - 2| - a = |x + 5|$

$\therefore a = |x - 2| - |x + 5|$

From graph (ii) the RANGE of $y = |x - 2| - |x + 5|$ is between -7 and 7. Therefore, there will be no solution if $y > 7$ or $y < -7$]