NSW INDEPENDENT SCHOOLS

PRELIMINARY EXAMINATION

2001
MATHEMATICS

MATHEMATICS EXTENSION 1

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt **ALL** questions.
- **ALL** questions are of equal value.
- Write your Student Name / Number on every page of the question paper and your answer sheets .
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- The answers to the six questions are to be handed in separately, clearly marked Question 1, Question 2 etc.
- ***This question paper must not be removed from the examination room.***

STUDENT NUMBER / NAME: ....

NAME/STUDENT NUMBER: ....

Question 1 (Start a new work book)

Marks

a. The line, l, has an x-intercept of -6 and a y-intercept of 5. The line, k, has an x-intercept of 3 and a y-intercept of 4. Find the acute angle between the lines giving your answer to the nearest minute **3**

b. In how many ways can the letters of the word APPLE be arranged

i. in a row? **1**
ii. so that a vowel comes first and last? **2**

c. Find the ratio in which the point C(6, 2) divides the interval AB if A and B have coordinates (4, -2) and (9, 8) respectively. **3**

d. Find the gradient of the normal to the curve $y = \dfrac{1}{\sqrt{x^2 - 3}}$ at the point (2, 1) **3**

Question 2 (Start a new work book)

a. Find all values of x for which $\dfrac{1}{1 - x} \le 3$ **3**

b. Find the value(s) of k for which $kx^2 + 2kx + 3 = 0$ has roots which differ by 1 **3**

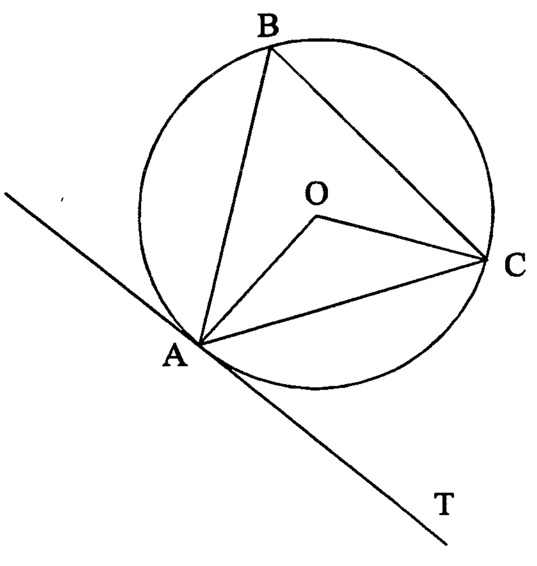
c. Use the t results to show that $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = t$ **3**

d. In the diagram, not drawn to scale, O is the centre of the circle. **3**

AT is a tangent to the circle. $\angle OAB$ is 36°, $\angle CAT$ is $3x$° and $\angle OCB$ is x°.

Copy the diagram into your workbook.

Find the value of x.



NAME/STUDENT NUMBER: ....

Question 3 (Start a new work book) **Marks**

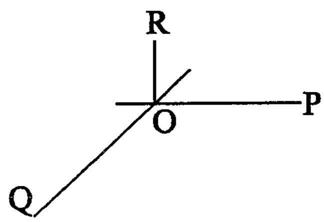
a. For the function, $y = f(x)$, you are given the following information: **4**

$f'(h) = 0;\ f(h) = -5;\ f(x) = f(-x);\ f(0) = 10$

Draw a neat sketch of a possible curve, clearly showing all of the information given.

b. In the diagram, which is not to scale, the points P, Q and O are in the same plane. R is a point vertically above O. P and Q are 750 metres apart and $\angle POQ$ is 120°. **4**

If $\angle QRO$ is 30° and $\angle PRO$ is 60°, find the height of R above O



c. If $x = 1$ is a double root of the equation $6x^4 - 7x^3 + cx^2 + 13x - 4 = 0$,

i. show that $c = -8$ **1**

ii. Hence find the other roots. **3**

Question 4 (Start a new work book)

a. i. If $\sin x - \cos x = A\sin(x - \alpha)$, where α is acute, find A and α **2**

ii. Hence or otherwise, solve $\sin x - \cos x = \sqrt{2}$ for $0° \le x \le 360°$ **2**

b. The chord PQ of the parabola $x^2 = 12y$ passes through the fixed point $(4, -3)$. Show that, if the tangents at P and Q intersect at the point T, then the locus of T is the line $2x - 3y + 9 = 0$ **3**

c. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two distinct points which lie on the parabola $y^2 = 4ax$. P and Q are both in the first quadrant such that $\angle PSQ$ is 90° where S is the focus $(a, 0)$ of the parabola.

i. Show that $(pq + 1)^2 = (p - q)^2$ **2**

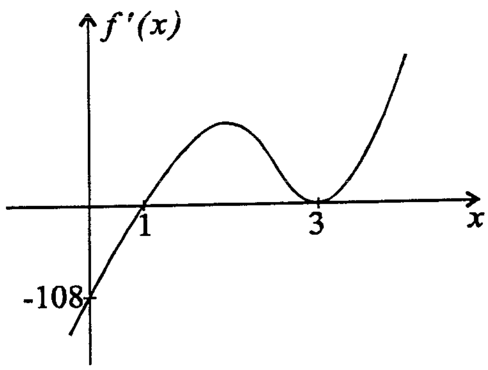
ii. Explain why $pq \ne -1$ **1**

iii. Show that the tangents at P and Q are not perpendicular **2**

NAME/STUDENT NUMBER: ....

Question 5 (Start a new work book) **Marks**

a. The graph shown, which is not drawn to scale, shows $f'(x)$, the derivative of the function $y = f(x)$. The derivative has a zero of degree 2 at $x = 3$ and cuts the x-axis at $x = 1$.



i. Explain why $f'(x) = k(x - 1)(x - 3)^2$, where k is a constant **1**

ii. Evaluate k **1**

iii. The curve, $y = f(x)$, has a turning point at (1, 0). Explain why this turning point is a minimum. **1**

iv. What is the nature of the turning point on $y = f(x)$ at the point where $x = 3$? Explain. **1**

v. Draw a possible sketch of $y = f(x)$ **2**

vi. Find an expression for $y = f(x)$ **2**

b. Solve $2\cos^2 2\theta - \cos 2\theta - 1 = 0$ for $0° \le \theta \le 360°$ **4**

NAME/STUDENT NUMBER: ....

Question 6 (Start a new work book)

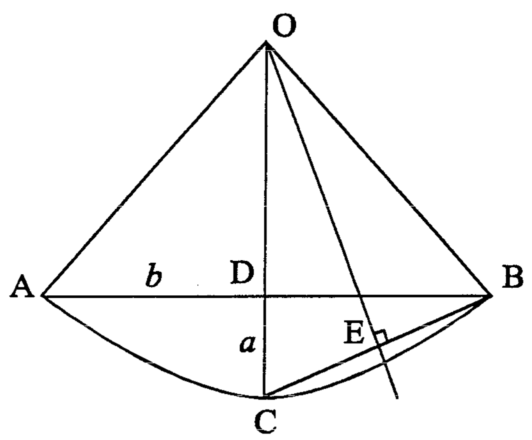
a. i. A rope is hanging vertically from the top of a tree with 3 metres of it lying on the ground. When the rope is stretched so that its end just touches the ground, it reaches a point 8 metres from the base of the tree. **2**

Find the length of the rope.

ii. If a metres of the rope was lying on the ground and the rope reached b metres from the base of the tree when stretched, show that the length, r, of the rope is given by **2**

$$r = \frac{1}{2}\left(a + \frac{b^2}{a}\right)$$

b. In the diagram below, O is the centre of a circle passing through A, B and C. OC is the perpendicular bisector of AB, with DC = a and AD = DB = b. OE is perpendicular to BC.



i. Prove that $\Delta OEB \;|||\; \Delta BDC$ **3**

ii. Hence find an expression for OB in terms of a and b **3**

iii. Explain the significance of your result with relation to part a.ii. above. **2**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

NSW INDEPENDENT TRIAL EXAMS -2001

PRELIMINARY EXAM: MATHEMATICS EXTENSION 1
SAMPLE SOLUTIONS

Question 1:

a. $m_l = \frac{5}{6}$; $m_k = -\frac{4}{3}$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$
$$= \left|\frac{\frac{5}{6} - -\frac{4}{3}}{1 + \frac{5}{6} \times -\frac{4}{3}}\right|$$
$$= \left|-19\tfrac{1}{2}\right| = 19\tfrac{1}{2}$$
$$\therefore \theta = 87°04'$$

b. i. $\frac{5!}{2!} = 60$ ii. $2 \times 3 \times 2 \times 1 \times 1 = 12$

c.
$$x = \frac{kx_2 + lx_1}{k + l}$$
$$6 = \frac{k \times 9 + l \times 4}{k + l}$$
$$6k + 6l = 9k + 4l$$
$$2l = 3k$$
$$k{:}l = 2{:}3$$

d.
$$y = (x^2 - 3)^{-\frac{1}{2}}$$
$$y' = -\tfrac{1}{2}(x^2 - 3)^{-3/2} \times 2x$$
$$= -x(x^2 - 3)^{-3/2}$$
$$\text{At } x = 2,\ m = -2 \times 1 = -2$$
$$\therefore m_{\text{normal}} = \tfrac{1}{2}$$

Question 2

a. There are critical values at $x = 1$ and at

$$\frac{1}{1 - x} = 3$$
$$1 = 3 - 3x$$
$$3x = 2$$
$$x = \frac{2}{3}$$

Testing in $x < \frac{2}{3}$: if $x = 0$: $\frac{1}{1 - 0} \leq 3$ ✓

Solution: $x \leq \frac{2}{3}$ and $x > 1$

b. If the roots differ by 1, $\alpha - \beta = 1$
Also, $\alpha + \beta = -2$ and $\alpha\beta = \frac{3}{k}$

Solving, $\alpha = -\frac{1}{2}$, $\beta = -\frac{3}{2}$

Now $\alpha \times \beta = -\frac{1}{2} \times -\frac{3}{2} = \frac{3}{4} = \frac{3}{k}$

So $k = 4$

c.
$$\text{LHS} = \frac{1 + \frac{2t}{1 + t^2} - \frac{1 - t^2}{1 + t^2}}{1 + \frac{2t}{1 + t^2} + \frac{1 - t^2}{1 + t^2}}$$
$$= \frac{1 + t^2 + 2t - 1 + t^2}{1 + t^2 + 2t + 1 + t^2}$$
$$= \frac{2t^2 + 2t}{2 + 2t}$$
$$= \frac{2t(t + 1)}{2(1 + t)}$$
$$= t = \text{RHS}$$

d. $\angle ABC = 3x$ (The angle between a tangent and a chord through the point of contact equals the angle in the alternate segment)

$\angle AOC = 6x$ (The angle at the centre is twice the angle at the circumference subtended by the same arc)

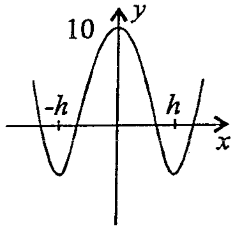
reflex $\angle AOC = 360 - 6x$ (Angles at a point add to 360°)

$(360 - 6x) + x + 3x + 36 = 360$
(Angle sum of a quadrilateral is 360°)

$\therefore x = 18°$

Question 3

a.



b. If h is the length OR:
In triangle OQR, $\tan R = OQ/h$
so $OQ = h \times \tan 30 = \frac{h}{\sqrt{3}}$
In triangle OPR, $\tan R = OP/h$
so $OP = h \times \tan 60 = h\sqrt{3}$

By the cosine rule in triangle QOP:

$$750^2 = \frac{h^2}{3} + 3h^2 - 2.\frac{h}{\sqrt{3}}.h\sqrt{3}.\cos 120$$

$$562500 = \frac{13h^2}{3}$$

$$h^2 = 129807.6923$$
$$h = 360.2883 = 360m \text{ (rounded)}$$

c. i. $6\times1^4 - 7\times1^3 + c\times1^2 + 13\times1 - 4 = 0$

Hence $c = -8$

ii. Dividing $6x^4 - 7x^3 - 8x^2 + 13x - 4$ by $(x - 1)^2 = x^2 - 2x + 1$ gives $6x^2 + 5x - 4$ whence $x = -\frac{4}{3}, \frac{1}{2}$

Question 4

a. i. $A\sin(x - \alpha) = A\sin x\cos\alpha - A\cos x\sin\alpha$
so $A\cos\alpha = 1$
and $A\sin\alpha = 1$
$\therefore A = \sqrt{2}$
and $\tan\alpha = 1 \Rightarrow \alpha = 45°$
so $\sin x - \cos x = \sqrt{2}\sin(x - 45)$

ii

$$\sqrt{2}\sin(x - 45) = \sqrt{2}$$
$$\sin(x - 45) = 1$$
$$x - 45 = 90$$
$$x = 135°$$

b. Let the coordinates of T be (x_0, y_0). Then, with $a = 3$, the equation of the chord of contact from T is given by $xx_0 = 6(y + y_0)$. But this chord passes through $(4, -3)$ so $4x_0 = 6(-3 + y_0)$. replacing x_0 and y_0 with x and y gives $2x - 3y + 9 = 0$

c. i. $m_{QS} = \frac{2aq - 0}{aq^2 - a} = \frac{2q}{q^2 - 1}$
Similarly, $m_{PS} = \frac{2ap - 0}{ap^2 - a} = \frac{2p}{p^2 - 1}$

But PS is perpendicular to QS so

$$\frac{2p}{p^2 - 1} \times \frac{2q}{q^2 - 1} = -1$$
$$4pq = -(p^2 - 1)(q^2 - 1)$$
$$4pq = -p^2q^2 + p^2 + q^2 - 1$$
$$p^2q^2 + 2pq + 1 = p^2 + q^2 - 2pq$$
$$(pq + 1)^2 = (p - q)^2$$

ii. If $pq = -1$, then $(p - q)^2 = 0$.
Hence $p = q$. But P and Q are distinct and $p \neq q$. So $pq \neq -1$

iii. $y = \sqrt{4ax}$
$y = 2\sqrt{a}\sqrt{x}$
$y' = \frac{\sqrt{a}}{\sqrt{x}}$
Gradient at P is $\frac{\sqrt{a}}{\sqrt{ap^2}} = \frac{1}{p}$

The gradient at Q is similarly $\frac{1}{q}$
If perpendicular, $\frac{1}{p} \times \frac{1}{q} = -1 \Rightarrow pq = -1$
which is not possible from part ii.

Question 5

a. i. The curve cuts at $x = 1$ so $x - 1$ is a factor. It also has a double root at $x = 3$ so $x - 3$ is a double factor. Allowing for differences in coefficients, the result follows.

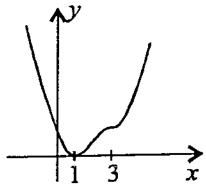
ii. When $x = 0$, $y' = -108$
$-108 = k(-1)(-3)^2 \Rightarrow k = 12$

iii. $y' < 0$ to the left of $x = 0$ so the curve is decreasing there. $y' > 0$ to the right so the curve is increasing there. Hence, $x = 1$ is a minimum.

Question 5 (continued)

iv. It is an increasing horizontal point of inflexion because it is a stationary point and the curve is increasing on both left and right.

v.



vi. $f'(x) = 12x^3 - 84x^2 + 180x - 108$

$f(x) = 3x^4 - 28x^3 + 90x^2 - 108x + C$

but $f(1) = 0 \Rightarrow C = 43$

so $f(x) = 3x^4 - 28x^3 + 90x^2 - 108x + 4$

b. $2\cos^2 2\theta - \cos 2\theta - 1 = 0$

$(2\cos 2\theta + 1)(\cos 2\theta - 1) = 0$

$\cos 2\theta = -\frac{1}{2} \qquad \cos 2\theta = 1$

If $0 \le \theta \le 360,\ 0 \le 2\theta \le 720$

For $\cos 2\theta = -\frac{1}{2}$

$\Rightarrow 2\theta = 120,\ 240,\ 480,\ 600$

so $\theta = 60,\ 120,\ 240,\ 300$

For $\cos 2\theta = 1$

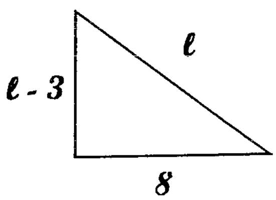
$\Rightarrow 2\theta = 0,\ 360,\ 720$

so $\theta = 0,\ 180,\ 360$

$\therefore \theta = 0,\ 60,\ 120,\ 180,\ 240,\ 300,\ 360$

Question 6

a.i. Let the length of the rope be l. Then



$l^2 = (l - 3)^2 + 8^2$

$l^2 = l^2 - 6l + 9 + 64$

$6l = 73$

$l = 12\frac{1}{6}$

ii. In the diagram, replace 3 with a and 8 with b. Then

$l^2 = (l - a)^2 + b^2$

$l^2 = l^2 - 6la + a^2 + b^2$

$2la = a^2 + b^2$

$l = \frac{1}{2}\left(a + \frac{b^2}{a}\right)$

b. i. OC ⊥ AB (given) so ∠BDC = 90°
OE ⊥ BC (given) so ∠OEB = 90°
So ∠BDC = ∠OEB

Now ∠AOC = ∠BOC (Perpendicular bisector of a chord bisects the angle subtended by the chord)

Also, ∠DBC = ½∠AOC (The angle at the centre is twice the angle at the circumference standing on the same arc)

and ∠EOB = ½∠COB (OE bisects ∠COB)

so ∠EOB = ∠DBC

Hence ΔOEB ||| ΔBDC (Two pairs of corresponding angles are equal)

ii. Hence $\frac{OB}{EB} = \frac{BC}{DC}$ so the radius, r (= OB), is

$r = OB = \frac{BC}{DC} \times EB$ but $EB = \frac{1}{2}BC$

so $r = \frac{1}{2}\frac{BC^2}{DC}$

But $BC^2 = DB^2 + DC^2 = b^2 + a^2$

and $DC = a$

so $r = \frac{1}{2}\frac{b^2 + a^2}{a} = \frac{1}{2}\left(\frac{b^2}{a} + a\right)$

iii. This is a geometric solution to the problem solved algebraically in part ii. OD is the height of the tree, OB represents the rope stretched to its fullest. DC is the "excess" when the rope hangs vertically i.e. the amount lying on the ground.