NSW INDEPENDENT SCHOOLS

MATHEMATICS

3 UNIT ADDITIONAL

PRELIMINARY EXAMINATION

1997

Time allowed - One and a half hours
(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- Each question attempted is to be handed in separately clearly marked Question 1, Question 2,.... etc
- ***The question paper must be handed to the supervisor at the end of the examination.***

STUDENT NUMBER / NAME.......................................

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Question 1 (Start a new page) **Marks**

a. Solve the inequality $\dfrac{2x}{x+1} > 3$ **3**

b. A school offers 4 subjects from KLA Group 1 and 6 subjects from KLA Group 2 to its Year 12 students. Calculate **3**

i. the number of ways a student can select any four subjects without restriction.

ii. the number of ways a student can select four subjects if at least one subject must come from KLA Group 1 and at least one from KLA Group 2.

c. Tangents are drawn to the curve $y = x^2$ at the points (1, 1) and (4, 16). Calculate the size of the acute angle between these two tangents at the point where they intersect. **3**

d. Sketch the following on separate number planes: **3**

i. $|x| + |y| = 2$ ii. $|x + y| = 2$ iii. $y = |x + 2|$

Question 2 (Start a new page)

a. The point (6, 6) divides the interval AB internally in the ratio 4:1. If A is the point (–2, 2), find the coordinates of B. **3**

b. Express $\dfrac{1 - x^{-1}}{x^{-1} - x^{-2}}$ in its simplest form. **3**

c. Show that $\tan 75° = 2 + \sqrt{3}$ **3**

d. The points (2, 11), (1, 6) and (0,5) lie on the parabola $y = ax^2 + bx + c$. Find the values of a, b and c. **3**

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Marks

Question 3 (Start a new page)

a. If α and β are the roots of the eqution $x - 7 + \frac{4}{x} = 0$, find the value of **6**

i. $\alpha + \beta$ and $\alpha\beta$

ii. $\alpha^2 + \beta^2$

iii. $\alpha^3 + \beta^3$

iv. $\alpha - \beta$

b. i. Show that the parabola $x^2 = 4ay$ can be represented by the parametric equations $x = 2at,\ y = at^2$ **6**

ii. Find the midpoint of the line joining the points $P(2ap,\ ap^2)$ and $Q(2aq,\ aq^2)$

iii. Show that, if PQ is a focal chord, $pq = -1$.

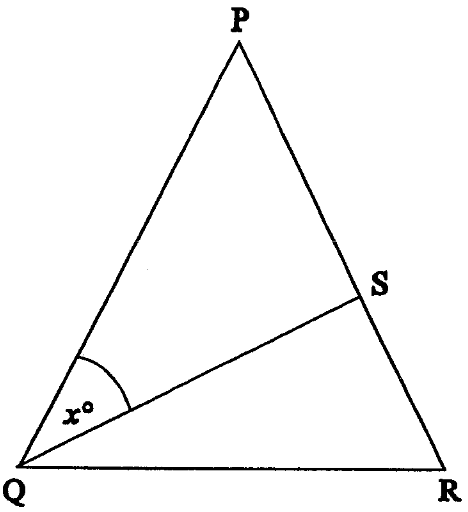
iv. Hence find the Cartesian equation of the locus of the midpoint of the focal chord PQ.

Question 4 (Start a new page) **Marks**

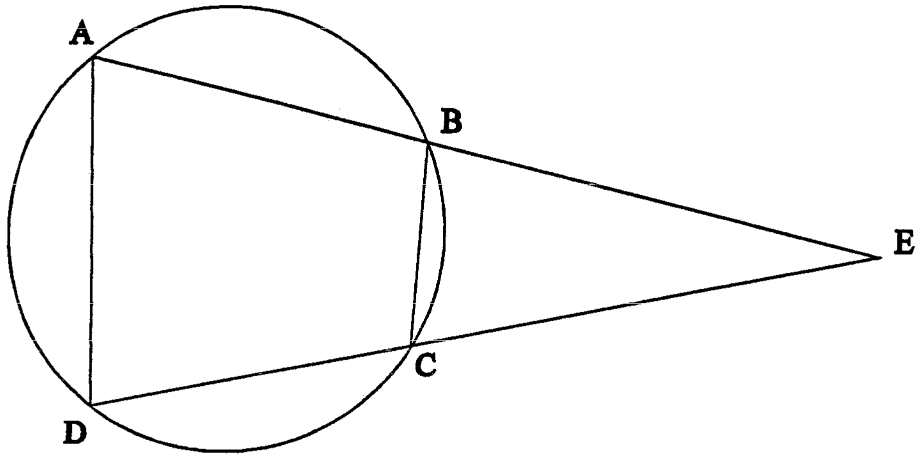
a. In the diagram, not drawn to scale, ΔPQR is isosceles and PS = QS = QR. 4

Copy the diagram onto your answer sheet.

i. Find the value of x, giving reasons.



b. 3



Copy the diagram onto your answer sheet.

i. Prove that $\Delta AED \; ||| \; \Delta CEB$

ii. Hence prove that AE × EB = DE × EC

c. i. Find the factors of the polynomial $P(x) = x^3 - 6x^2 + 11x - 6$ 5

ii. Sketch the curve, showing all intercepts on the axes.

iii. For which values of x is $x^3 - 6x^2 + 11x - 6 \geq 0$?

Question 5 (Start a new page) **Marks**

a. The tangent at P(a, b) on the curve $xy = 3$ cuts the y-axis at T(0, 6). Find the coordinates of P. **4**

b. A flagpole stands on a level parade ground. From a point due south of the pole, the angle of elevation to the top of the pole is 35°. From a point which bears 112° from the pole, the angle of elevation to the top of the pole is 28°. If the two points are 40 metres apart, find the height of the pole. **4**

c. i. By considering the identity $\cos 2x = \cos(x + x)$, show that $\cos 2x = 1 - 2\sin^2 x$. **4**

ii. Hence or otherwise, solve $\sin x = \cos 2x$ for $0° \le x \le 360°$

Question 6 (Start a new page)

a. Show that the derivative of $y = \sqrt{x + \sqrt{x + 1}}$ is given by **5**

$$\frac{dy}{dx} = \frac{2\sqrt{x + 1} + 1}{4\sqrt{x + 1}\sqrt{x + \sqrt{x + 1}}}$$ and hence find the equation of the

tangent to the curve at the point where $x = 0$.

b. A(–4, 9) and B(2, 4½) are two fixed points. A variable point P(x, y) moves so that its distance from A is twice its distance from B. **7**

i. Show that P moves on the circle $x^2 + y^2 - 8x - 6y = 0$

ii. Find the centre and radius of the circle.

iii. Sketch the graph of the circle showing clearly the coordinates of any points of intersection with the x-axis and the y-axis.

iv. Find the exact area of that part of the circle which lies in the first quadrant.

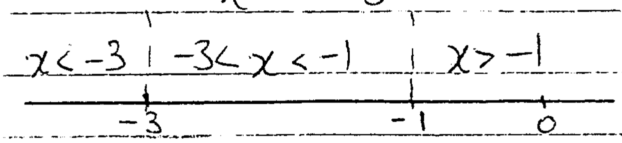
) $\dfrac{2x}{x+1} > 3$

Critical values at $x = -1$

and $\dfrac{2x}{x+1} = 3$

$2x = 3x + 3$

$x = -3$



Test $x = 0$: $0 > 3$ (not true)

$\therefore -3 < x < 1$

) (i) ${}^{10}C_4 = 210$

ii) Possibilities:

1 Group 1; 3 Group 2: ${}^4C_1 \times {}^6C_3 = 80$

2 Group 1; 2 Group 2: ${}^4C_2 \times {}^6C_2 = 90$

3 Group 1; 1 group 2: ${}^4C_3 \times {}^6C_1 = 24$

Total $= 194$

) $dy/dx = 2x$

At $x = 1$, $m = 2$

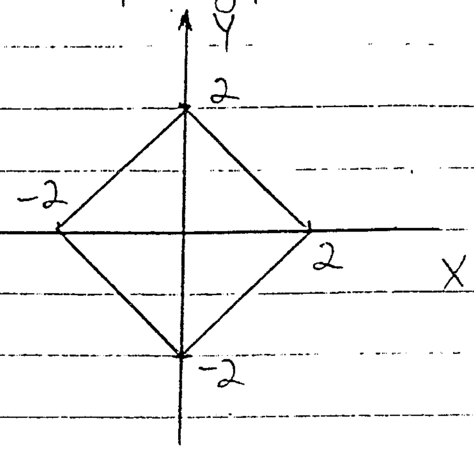
$x = 4$, $m = 8$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

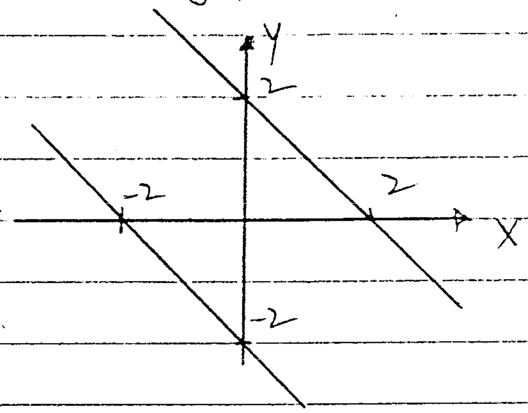
$= \left|\dfrac{8 - 2}{1 + 8 \times 2}\right| = \dfrac{6}{17}$

$\theta = 19°26$

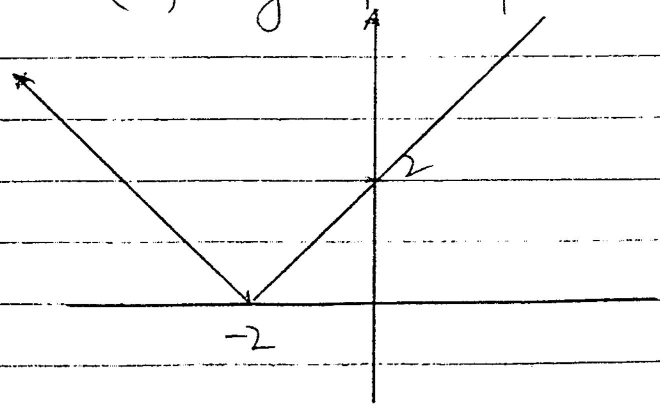
(d) (i) $|x| + |y| = 2$



(ii) $|x + y| = 2$



(iii) $y = |x + 2|$



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1 (a) $A(-2,2)$; $P(6,6)$; $B(x,y)$

$$\frac{4\times x+1\times 2}{4+1}=6 \Rightarrow x=8$$

$$\frac{4\times y+1\times 2}{4+1}=6 \Rightarrow x=7$$

$\therefore B(8,7)$

(b) $$\frac{1-x^{-1}}{x^{-1}-x^{-2}}=\frac{1-x^{-1}}{x^{-1}-x^{-2}}\times\frac{x^2}{x^2}$$
$$=\frac{x^2-x}{x-1}$$
$$=\frac{x(x-1)}{x-1}$$
$$=x \quad (x\neq 1, x\neq 0)$$

(c) $$\tan 75=\tan(45+30)$$
$$=\frac{\tan 45+\tan 30}{1-\tan 45\tan 30}$$
$$=\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$
$$=\frac{2\sqrt{3}+4}{2}$$
$$=\sqrt{3}+2$$

(d) $(2,11)$: $4a+2b+c=11$ ---(1)
$(1,6)$: $a+b+c=6$ ---(2)
$(0,5)$: $c=5$ ---(3)

From (2) + (3) $a+b=1$ --- (4)
From (1) + (3) $4a+2b=6$
or $2a+b=3$ --- (5)

Solve (4) + (5) gives
$a=2$
$b=-1$

$a=2$, $b=-1$, $c=5$

or $y=2x^2-x+5$

3. (a) $x - 7 + \frac{4}{x} = 0$

$x^2 - 7x + 4 = 0$

(i) $\alpha + \beta = 7$

$\alpha\beta = 4$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 7^2 - 2\times 4$

$= 41$

(iii) $\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$

$= 7(41-4)$

$= 259$

(iv) $(\alpha-\beta)^2 = \alpha^2+\beta^2-2\alpha\beta$

$= 41 - 2\times 4$

$= 33$

$\alpha - \beta = \pm 33$

) (i) If $x = 2at$, $x^2 = 4a^2t^2$

If $y = at^2$, $4ay = 4a\times at^2$

$= 4a^2t^2$

$\therefore$ $x = 2at$, $y = at^2$ satisfies $x^2 = 4ay$

(ii) $M\left(a[p+q], \frac{a[p^2+q^2]}{2}\right)$

(iii) PQ: $\frac{y - aq^2}{x - 2aq} = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{p+q}{2}$

Now PQ passes through $(0, a)$

At $S(0, a)$

$\frac{a - aq^2}{-2aq} = \frac{p+q}{2}$

$2 - 2q^2 = -2pq - 2q^2$

$\therefore pq = -1$

(iv) Let $X = a(p+q)$, $Y = \frac{a(p^2+q^2)}{2}$

$\frac{X^2}{a^2} = p^2 + q^2 + 2pq$

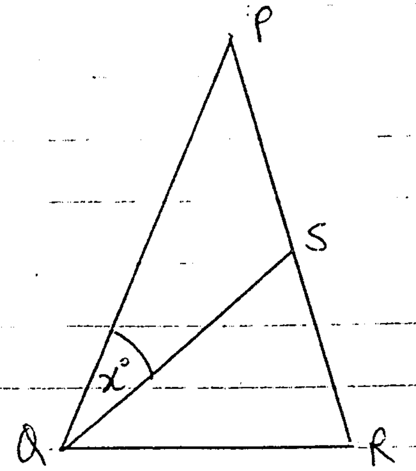
and $\frac{2Y}{a} = p^2 + q^2$

$\therefore \frac{X^2}{a^2} = \frac{2Y}{a} + 2\times -1 \quad (pq = -1)$

$X^2 = 2aY - 2a^2$

$X^2 = 2a(Y - a)$

(a)



$PS = SQ$ (given)

$\therefore \Delta SPQ$ is isosceles and

$\angle QPS = x$

$\therefore \angle QSR = x + x$ (exterior angle)

$= 2x$

Also ΔQPR is isosceles (given)

$\therefore \angle SRQ = \dfrac{180 - x}{2}$ (angle sum of triangle)

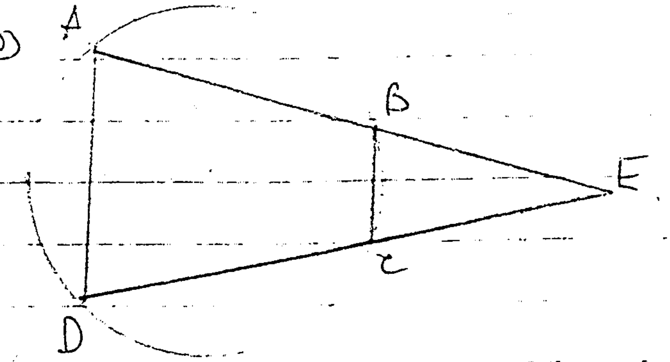
But $QS = QR$ (given)

$\therefore \Delta QSR$ is isosceles

and $2x = \dfrac{180 - x}{2}$ (base angles)

$\Rightarrow x = 36^\circ$

(b)



i) $\angle E$ is common

$\angle BCE = \angle DAB$ (exterior $\angle$, cyclic quad)

$\therefore \Delta AED \;|||\; \Delta CEB$ (two pairs equal angles)

(ii) $\dfrac{AE}{CE} = \dfrac{ED}{EB} = \dfrac{AD}{CB}$ (corr. sides in same ratio)

$\therefore AE.EB = DE.EC$

(c)(i) $P(x) = x^3 - 6x^2 + 11x - 6$

$P(1) = 1 - 6 + 11 - 6 = 0$

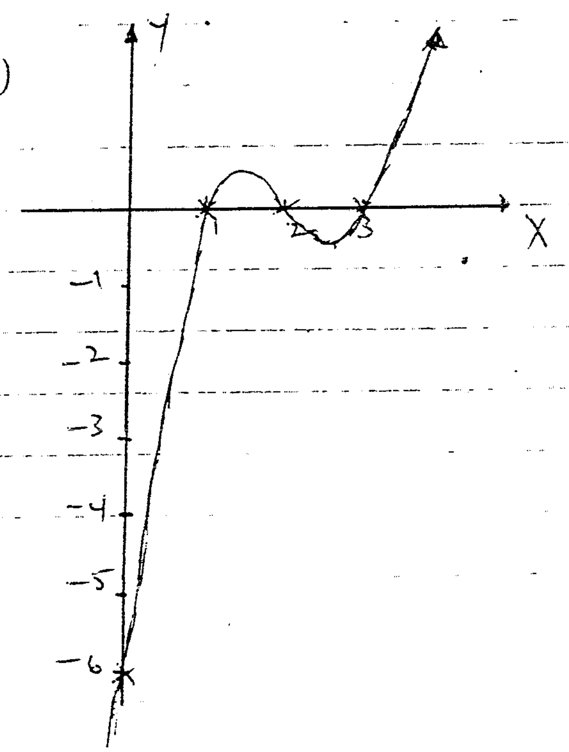
$\therefore (x-1)$ is a factor

$P(2) = 8 - 24 + 22 - 6 = 0$

$\therefore (x-2)$ is a factor

$\therefore P(x) = (x-1)(x-2)(x-3)$

(ii)



(iii) $x^3 - 6x^2 + 11x - 6 \geq 0$

when $1 \leq x \leq 2$ and $x \geq 3$

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(a) $y = 3x^{-1}$

$y' = -3x^{-2}$

At $x = a$, $m = -3/a^2$ & $b = \frac{3}{a}$

$\therefore y - \frac{3}{a} = \frac{-3}{a^2}(x - a)$

At $T(0,6)$

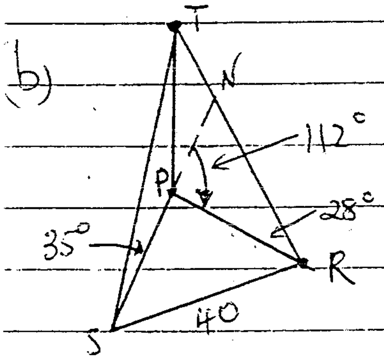
$6 - \frac{3}{a} = -\frac{3}{a^2}\,x - a$

$6 = \frac{6}{a}$

$a = 1, \quad b = 3$

$P(1,3)$

b)



Let $PS = x$ & $PR = y$, $TP = h$

then $\tan 35 = \frac{h}{x} \Rightarrow x = \frac{h}{\tan 35}$

& $x = h \tan 55$

Similarly $y = h \tan 62$

By cosine rule:

$40^2 = x^2 + y^2 - 2xy \cos 68$

$1600 = h^2 \tan^2 55 + h^2 \tan^2 62 - 2 \times h \tan 55 \times h \tan 62 \times \cos 68$

$\therefore h^2 = \frac{1600}{\tan^2 55 + \tan^2 62 - 2\tan 55 \tan 62 \cos 68}$

$h = 21.1869$ metres

$\doteqdot 21$ metres

(c)(i) $\cos(x + x) = \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x$

$= 1 - \sin^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

(ii) $\sin x = 1 - 2\sin^2 x$

$\therefore 2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}, \quad \sin x = -1$

$x = 30°, 150°$ & $270°$

5

(a) $y = \left(x + (x+1)^{1/2}\right)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}\left(x + (x+1)^{1/2}\right)^{-1/2} \cdot \left(1 + \frac{1}{2}(x+1)^{-\frac{1}{2}}\right)$

$= \frac{1}{2}\left(\frac{1}{\sqrt{x+\sqrt{x+1}}}\right) \cdot \left(1 + \frac{1}{2\sqrt{x+1}}\right)$

$= \frac{1}{2}\left(\frac{1}{\sqrt{x+\sqrt{x+1}}}\right)\left(\frac{2\sqrt{x+1}+1}{2\sqrt{x+1}}\right)$

$= \frac{2\sqrt{x+1}+1}{4\sqrt{x+1}\sqrt{x+\sqrt{x+1}}}$

At $x=0$, $y=1$, $\frac{dy}{dx} = \frac{3}{4}$

$\therefore y - 1 = \frac{3}{4}(x-0)$

$y = \frac{3}{4}x + 1$

b)(i) $2 \times PB = PA$

$4 \times PB^2 = PA^2$

$[(x-2)^2 + (y-4.5)^2] = (x+4)^2 + (y-9)^2$

$4x^2 - 16x + 16 + 4y^2 - 36y + 81$
$= x^2 + 8x + 16 + y^2 - 18y + 81$

$3x^2 + 3y^2 - 24x - 18y = 0$

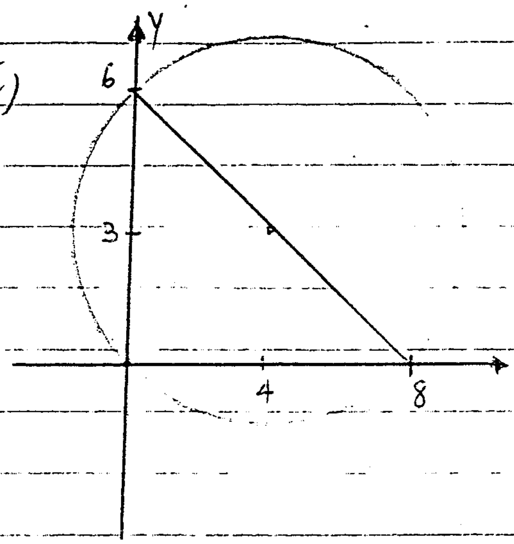
$x^2 + y^2 - 8x - 6y = 0$

(b)(ii) $x^2 - 8x + 16 + y^2 - 6y + 9 = 16 + 9$

$(x-4)^2 + (y-3)^2 = 25$

$\therefore$ Centre $(4,3)$ radius 5

(iii)



(iv) Join $(0,6)$ + $(8,0)$

Area of $\Delta = \frac{1}{2} \times 8 \times 6 = 24$

Area of semicircle $= \frac{1}{2}\pi \times 5^2$

$= \frac{25\pi}{2}$

$\therefore$ Area required $= 24 + \frac{25}{2}\pi$