NSW INDEPENDENT SCHOOLS

PRELIMINARY HSC EXAMINATION

1999

MATHEMATICS
3 UNIT ADDITIONAL

Time Allowed - One and a half hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Start each question on a new page.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- Clearly mark each question (Question 1, Question 2, ... etc) and each question part.
- Write your Student Name/Number on every page.
- *This question paper must not be removed from the examination room.*

STUDENT NUMBER/NAME: ....

Question 1 (Start a new page) — **Marks**

a. Find the exact value of $\tan 15°$. — 2

b. The equation $x^2 - (1 - 2k)x + k + 3 = 0$ has consecutive roots. Find the value of k. — 3

c. Solve the inequality $\dfrac{x}{2x - 1} \leq 5$. — 3

d. i. In how many ways can 8 committee members be selected from 12 people? — 4

ii. Two people say that they will only serve as committee members if both are selected. Otherwise, neither will serve. In how many ways can this be done to satisfy BOTH these conditions?

Question 2 (Start a new page)

a. Find the equation of the line through the point of intersection of the lines $2x + 3y - 7 = 0$ and $x - 2y + 1 = 0$ and perpendicular to the line $y = 1 - 3x$. — 3

b. i. On the same axes, sketch the curves $y = x^2$ and $y = |x|$ — 3

ii. Hence, or otherwise, solve $x^2 < |x|$

c. The diagram shows the line interval AB ***trisected*** at P and Q. The coordinates of A are $(-6, -2)$ and of Q are $(2, 2)$. Find the coordinates of B. — 2

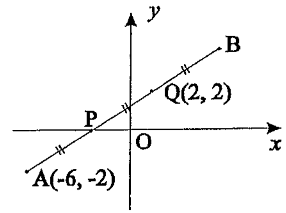


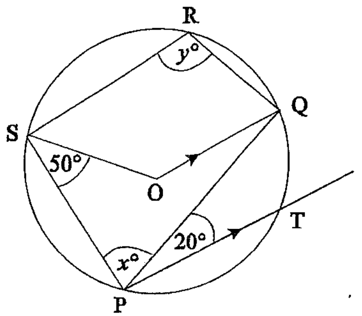
Diagram not to scale

Question 2 is continued on the next page

Question 2 (continued) — Marks

d. In the diagram, O is the centre of the circle and $PT \parallel OQ$. Find the values of x and y, giving reasons. — 4

Diagram not to scale



Question 3 (Start a new page)

a. The lines $3x - y + 2 = 0$ and $mx - y - 1 = 0$ intersect at 45°. Find the possible value(s) of m. — 3

b. For the parabola $y = x^2 - 4x - 1$, write down the coordinates of the focus and the equation of the directrix. — 2

c. Find the equation of the tangent to the curve $y = 2x\sqrt{x+1}$ at the point where $x = 3$. — 4

d. Make n the subject of the formula $S = \frac{n}{2}[2a + (n - 1)d]$. — 3

Question 4 (Start a new page)

a. A man, M, is standing in a horizontal plane which also contains a tower, AB, which is 10 metres high, and a building, XY, which is 95 metres high. The man is 210 metres from the tower and his bearing from it is 42°. His bearing from the building is 110°. The bearing of the tower from the building is 150°. — 4

Find the angle of elevation from the top of the tower to the top of the building.

Question 4 is continued on the next page.

Question 4 (continued) — Marks

b. The polynomial, $P(x)$, is given by $P(x) = x^4 + 6x^3 + 5x^2 - 12x$. — 5

i. Find $P(-4)$.

ii. Hence or otherwise find the factors of $P(x)$.

iii. Draw a neat sketch of $y = P(x)$ in the domain $-5 \le x \le 2$.

iv. Hence, or otherwise, solve $x^4 + 6x^3 + 5x^2 - 12x \le 0$

c. Sketch the inequality $y \le \sqrt{16 - x^2}$ — 3

Question 5 (Start a new page)

a. i. In how many ways can the letters of the word MATHS be arranged? — 3

ii. If it does not matter whether the "word" formed makes sense,

1. how many three letter "words" can be formed from the word MATHS if no repetitions are allowed?

2. how many three letter "words" can be formed if repetitions are allowed?

b. i. Show that $\frac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta = \tan\theta$. — 3

ii. Hence, or otherwise, solve $\frac{\sin^3\theta}{\cos\theta} = 1 - \sin\theta\cos\theta$ for $0° \le \theta \le 360°$

c. i. Find the Cartesian equation of the parabola $x = 4t$, $y = 2t^2$. — 6

ii. Show that the equation of the chord of contact from the point T(3, −2) is $3x - 4y + 8 = 0$.

iii. Find the coordinates of the points P and Q where the tangents from T touch the parabola.

iv. Find the gradients of the tangents at P and Q.

v. What conclusion can you draw about the chord of contact?

Question 6 (Start a new page) **Marks**

a. A is the point $(5, 0)$ and O is the origin. Given that the point $B(x, y)$ lies on the line $y = 1 - 3x$ and that OB is perpendicular to AB, find the coordinates of B. **4**

b. In the diagram, $\angle A > \angle B > 0$ and $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{4}{5}$, **8**

i. Show that $\tan A \cdot \tan B = \dfrac{1}{9}$

ii. If $PR = QR$, show that

$9(\tan A + \tan B) = 8$

iii. Hence find A and B.

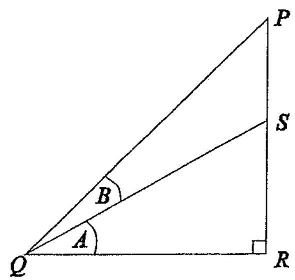


Diagram not to scale

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

NSW INDEPENDENT TRIAL EXAMS 1999 SOLUTIONS

MATHEMATICS : 3 UNIT PRELIMINARY EXAM SOLUTIONS 1999

QUESTION 1

a. $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$

$\tan(60-45) = \dfrac{\tan 60 - \tan 45}{1+\tan 60 . \tan 45}$

$= \dfrac{\sqrt{3}-1}{1+\sqrt{3}}$

b. Let the roots be α and $\alpha+1$

Then $2\alpha + 1 = 1-2k$ ... ①

$\alpha^2+\alpha = k+3$ ... ②

From ①, $\alpha = -k$

Substitute into ②:

$k^2 - k = k+3$

$k^2-2k-3=0$

$(k-3)(k+1)=0$

$\therefore k=3$ or $k=-1$

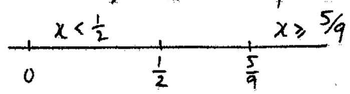
c. $\dfrac{x}{2x-1} \le 5$

Critical points at $x=\frac{1}{2}$

and $\dfrac{x}{2x-1} = 5$

$x = 10x-5$

$\Rightarrow x = 5/9$



Testing $x=0$: $0 \le 5$ ✓

$\therefore x < \frac{1}{2}$ or $x \ge 5/9$

d. (i) $^{12}C_8 = 495$

(ii) Either both are in, leaving 6 positions to be filled from 10

i.e. $^{10}C_6 = 210$

or both are out, leaving 8 positions to be filled from 10

i.e $^{10}C_8 = 45$

$\therefore$ 255 ways.

QUESTION 2

a. $2x+3y-7+\lambda(x-2y+1)=0$

$x(2+\lambda)+y(3-2\lambda)+\lambda-7=0$

$\therefore m = -\dfrac{(2+\lambda)}{3-2\lambda} = +\dfrac{1}{3}$

$6+3\lambda = -3+2\lambda$

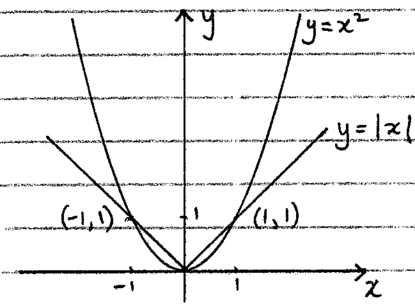
$9 = -\lambda$

$\lambda = -9$

$\therefore 2x+3y-7-9(x-2y+1)=0$

$\Rightarrow 7x-21y-16=0$

b. (i)



(ii) $-1 < x < 1$ (from graph)

c. $AQ : QB = 2:1$

Let $B(x,y)$

then $2 = \dfrac{2\times x + 1\times -6}{3}$

$\Rightarrow x = 6$

and $2 = \dfrac{2\times y + 1\times -2}{3}$

$\Rightarrow y = 4$

$\therefore B(6,4)$

d. $S\hat{O}Q = 2x^\circ$ (angle at centre) OBTUSE

$O\hat{Q}P = 20^\circ$ (alternate angles)

Reflex $\angle S\hat{O}Q = 360-2x^\circ$

$\therefore 50^\circ + 360 - 2x^\circ + 20^\circ + x^\circ = 360$ (angle sum of quadrilateral)

$\Rightarrow x = 70^\circ$

Hence $y = 110^\circ$ (opposite angle of cyclic quadr.)

QUESTION 3

a. $m_1 = 3$; $m_2 = m$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan 45 = \left|\frac{3 - m}{1 + 3m}\right|$$

$$\frac{3-m}{1+3m} = 1 \quad \text{or} \quad \frac{-(3-m)}{1+3m} = 1$$

$$3 - m = 1 + 3m \qquad -3 + m = 1 + 3m$$

$$2 = 4m \qquad -4 = 2m$$

$$m = \tfrac{1}{2} \qquad m = -2$$

$\therefore m = -2$ or $\frac{1}{2}$

b.
$$y = x^2 - 4x - 1$$
$$y + 1 + 4 = x^2 - 4x + 4$$
$$y + 5 = (x-2)^2$$

$\therefore$ Vertex is $(2, -5)$

focal length is $\frac{1}{4}$

So focus $(2, -4\frac{3}{4})$

& directrix $y = -5\frac{1}{4}$

c. $y = 2x(x+1)^{1/2}$

At $x = 3$, $y = 2 \times 3 \times 4^{1/2} = 12$

$$y' = 2(x+1)^{1/2} + 2x \cdot \tfrac{1}{2}(x+1)^{-\frac{1}{2}}$$

At $x = 3$, $y' = 2 \times 4^{1/2} + 2 \times 3 \times \frac{1}{2} \times 4^{-\frac{1}{2}}$
$= 5\frac{1}{2}$

$$y - y_1 = m(x - x_1)$$
$$y - 12 = \tfrac{11}{2}(x - 3)$$

$$2y - 24 = 11x - 33$$
$$\Rightarrow 11x - 2y - 9 = 0$$

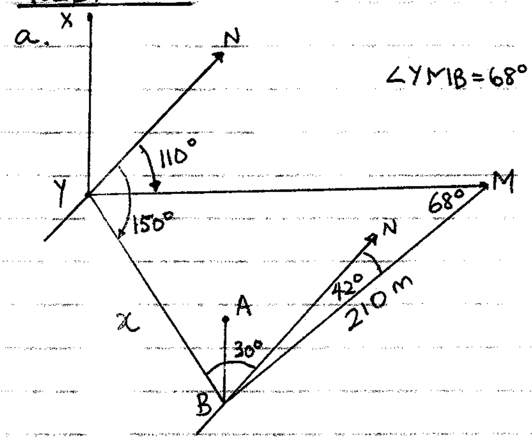
d. $S = \frac{n}{2}\left[2a + (n-1)d\right]$

$$2S = 2an + n^2 d - nd$$
$$\therefore n^2 d + n(2a - d) - 2S = 0$$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(2a-d) \pm \sqrt{(2a-d)^2 - 4 \cdot d \cdot {-2S}}}{2d}$$
$$= \frac{-(2a-d) \pm \sqrt{4a^2 - 4ad + d^2 + 8dS}}{2d}$$

QUESTION 4

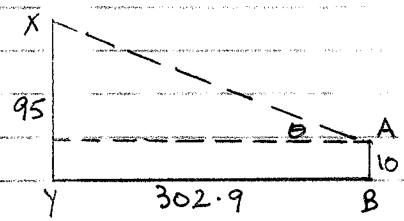
a.



$\angle YMB = 68°$

$$\frac{210}{\sin 40} = \frac{x}{\sin 68}$$

$$x = \frac{210 \sin 68}{\sin 40} = 302.9$$



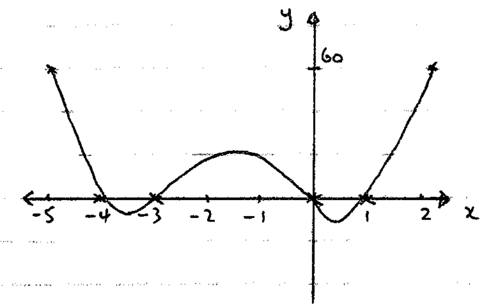
$$\tan\theta = \frac{85}{302.9}$$

$$\theta = 15°41'$$

b.(i) $P(-4) = (-4)^4 + 6(-4)^3 + 5(-4)^2 - 12(-4)$
$= 0$

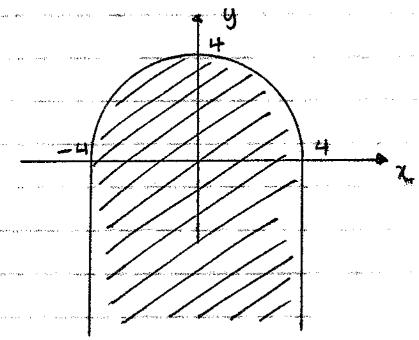
(ii) $P(x) = x(x+4)(x-1)(x+3)$
[either by division or by trial and error]

(iii) $P(-5) = 60$
$P(2) = 60$



(iv) $-4 \le x \le -3$ and $0 \le x \le 1$

c.



QUESTION 5

a.(i) $5! = 120$

(ii) 1. Order significant: ${}^5P_3 = 60$

2. $5 \times 5 \times 5 = 125$

b.(i) $LHS = \dfrac{\sin^3\theta + \sin\theta\cos^2\theta}{\cos\theta}$

$= \dfrac{\sin\theta(\sin^2\theta + \cos^2\theta)}{\cos\theta}$

$= \tan\theta$

$= RHS$

(ii) $\dfrac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta = 1$

$\therefore \tan\theta = 1$

$\therefore \theta = 45^\circ, 225^\circ$

c.(i) $t = x/4$

$\therefore y = 2 \times \dfrac{x^2}{16}$

$8y = x^2$

(ii) $a = 2$ & $T(3,-2)$

$x x_0 = 2a(y + y_0)$

$x \times 3 = 4(y - 2)$

$3x = 4y - 8$

$3x - 4y + 8 = 0$

(iii) $y = x^2/8$

$\therefore 3x - 4 \cdot \dfrac{x^2}{8} + 8 = 0$

$6x - x^2 + 16 = 0$

$x^2 - 6x - 16 = 0$

$(x+2)(x-8) = 0$

$\therefore x = -2, \quad y = \frac{1}{2}$

$x = 8, y = 8$

let $P(8, 8)$

$Q(-2, \frac{1}{2})$

(iv) Parameter at P: $4t = 8$ $\therefore t = 2$

& gradient is 2

at Q, $4t = -2$, $\therefore t = -\frac{1}{2}$

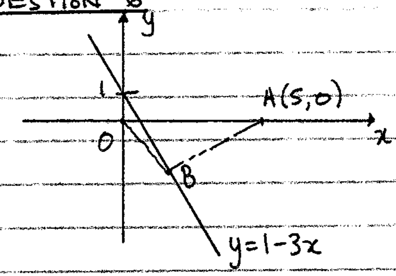
$\therefore$ gradient is $-\frac{1}{2}$

(v) Since $M_P \times M_Q = 2 \times -\frac{1}{2} = -1$,

tangents are perpendicular. Therefore, the chord of contact is a focal chord.

QUESTION 6

a.



Since B lies on $y = 1-3x$, the coordinates of B are $(x, 1-3x)$

$m_{OB} = \dfrac{1-3x}{x}$

$m_{AB} = \dfrac{1-3x}{x-5}$

$\therefore \dfrac{1-3x}{x} \times \dfrac{1-3x}{x-5} = -1$

$1 - 6x + 9x^2 = -x^2 + 5x$

$\therefore 10x^2 - 11x + 1 = 0$

$(10x - 1)(x - 1) = 0$

$x = 1/10 \quad y = 7/10$

and $x = 1 \quad y = -2$

$\therefore$ B is $(1, -2)$ OR $(\frac{1}{10}, \frac{7}{10})$

b.(i) $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{4}{5}$

$\dfrac{\cos A\cos B - \sin A\sin B}{\cos A\cos B + \sin A\sin B} = \dfrac{4}{5}$

$5\cos A\cos B - 5\sin A\sin B = 4\cos A\cos B + 4\sin A\sin B$

$\cos A\cos B = 9\sin A\sin B$

$\therefore \tan A\tan B = \frac{1}{9}$

(ii) Since ΔPQR is isosceles,

$A + B = 45^\circ$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$1 = \dfrac{\tan A + \tan B}{1 - 1/9}$

$\therefore \tan A + \tan B = 8/9$

and $9(\tan A + \tan B) = 8$

(iii) Let $\alpha = \tan A$, $\beta = \tan B$.

Then $\alpha\beta = \frac{1}{9}$

$\alpha + \beta = \frac{8}{9} \Rightarrow \beta = \frac{8}{9} - \alpha$

so $\alpha\left(\frac{8}{9} - \alpha\right) = \frac{1}{9}$

$\dfrac{8\alpha}{9} - \alpha^2 = \dfrac{1}{9}$

$9\alpha^2 - 8\alpha + 1 = 0$

$\alpha = \dfrac{8 \pm \sqrt{64 - 36}}{18}$

$\tan A = \dfrac{8 + \sqrt{28}}{18}$ or $\tan A = \dfrac{8 - \sqrt{28}}{18}$

$\Rightarrow A = 36^\circ 27'$ or $8^\circ 33'$

$\therefore A = 36^\circ 27'$

$B = 8^\circ 33'$