NSW INDEPENDENT SCHOOLS

2001
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided on the last page
- All necessary working should be shown in every question

Total marks (84)

Attempt Questions 1 – 7

All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ... ...

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Question 1 (Start a new work book) — **Marks**

a. Determine the ratio in which the point C(6, 9) divides the interval AB, where A is the point $(-1, -5)$ and B the point $(3, 3)$. **3**

b. Solve the inequality $x - 1 \le \frac{1}{x - 1}$. **3**

c. For the polynomial $P(x) = x^3 - 2x^2 - x + 2$

i. show that $x - 1$ is a factor. **1**

ii. Hence, or otherwise, find all the factors of $P(x)$. **1**

d. i If $t = \tan\frac{\theta}{2}$, show that $\sin\theta = \frac{2t}{1 + t^2}$ and $\cos\theta = \frac{1 - t^2}{1 + t^2}$. **2**

ii Using these results, show that $\frac{1 - \cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$. **1**

iii Hence find the exact value of $\tan 15°$. **1**

Question 2 (Start a new work book)

a. For the parabola defined by the parametric equations $x = 4t,\ y = 2t^2$

i. by differentiation, show that the gradient of the tangent at the point, P, where $t = 3$, is 3. **1**

ii. find the gradient of the focal chord through P. **1**

iii. calculate the acute angle between the tangent at P and the focal chord through P. **2**

b. Use one iteration of Newton's method to find an approximation to the root of the equation $x\log_e x - 2x = 0$ near $x = 7$. Give your answer to 1 decimal place. **3**

c. Six people attend a dinner party.

i. In how many different ways can they be arranged around a round table? **1**

ii. In how many different ways can they be arranged if a particular couple must sit together? **1**

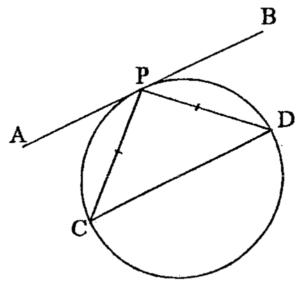
iii. What is the probability that, if the people are seated at random, the couple are sitting apart from each other? **1**

Marks

Question 2 (continued)

d. PC and PD are equal chords of a circle. A tangent, AB, is drawn at P. **2**

Prove that AB is parallel to CD



Question 3 (Start a new work book)

a. Jane, a netball goal shooter, has a 70% probability of scoring a goal at any attempt. In her next 10 attempts at scoring, what is the probability that she scores at least 8 times? Give your answer as a decimal to 2 significant figures. **3**

b. Show that the equation of the circle whose diameter is the join of the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ **2**

c. Use the Principle of Mathematical Induction to prove that $2^{3n} - 3^n$ is divisible by 5 for all positive integers n. **4**

d. The arc of the curve $y = \cos 2x$ between $x = 0$ and $x = \frac{\pi}{6}$ is rotated through 360° about the x-axis. **3**

Find the exact volume of the solid formed.

Question 4 (Start a new work book)

a. If $\binom{n}{r} = \binom{n}{r+1}$, where n and r are positive integers, show that n is odd. **3**

b. i. Express $x^2 + 6x + 13$ in the form $(x + a)^2 + b^2$ **1**

ii. Hence, using the substitution $u = x + 3$, find $\int \frac{dx}{x^2 + 6x + 13}$ **2**

Marks

Question 4 (continued)

c. Show that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$ **3**

d. If $y = ½(e^{x} - e^{-x})$, show that $x = \log_e\left(y + \sqrt{y^2 + 1}\right)$ **3**

Question 5 (Start a new work book)

a. A particle's motion is defined by the equation $v^2 = 12 + 4x - x^2$, where x is its displacement from the origin in metres and v its velocity in ms^{-1}. Initially, the particle is 6 metres to the right of the origin.

i. Show that the particle is moving in Simple Harmonic Motion **1**

ii. Find the centre, the period and the amplitude of the motion **3**

iii. The displacement of the particle at any time t is given by the equation $x = a\sin(nt + \theta) + b$.

Find the values of θ and b, given $0 \leq \theta \leq 2\pi$ **2**

b. Newton's Law of Cooling states that the rate of change in the temperature, $T°$, of a body is proportional to the difference between the temperature of the body and the surrounding temperature, $P°$.

i. If A and k are constants, show that the equation $T = P + Ae^{kt}$ satisfies Newton's Law of Cooling. **2**

ii. A cup of tea with a temperature of 100°C is too hot to drink. Two minutes later, the temperature has dropped to 93°C. If the surrounding temperature is 23°C, calculate A and k. **2**

iii. The tea will be drinkable when the temperature has dropped to 80°C. How long, to the nearest minute, will this take? **2**

Question 6 (Start a new work book)

Marks

a. A particle is projected horizontally with velocity, V ms^{-1}, from a point h metres above the ground. Take g ms^{-2} as the acceleration due to gravity.

i. Taking the origin at the point on ground immediately below the projection point, find expressions for x and y, the horizontal and vertical displacements respectively of the particle at time t seconds. **2**

ii. Hence show that the equation of the path of the particle is given by the equation $y = \dfrac{2hV^2 - gx^2}{2V^2}$. **2**

iii. Find how far the particle travels horizontally from its point of projection before it hits the ground. **2**

b. A particle moves in a straight line so that its velocity after t seconds is v ms^{-1} and its displacement is x.

i. Given that $\dfrac{d^2x}{dt^2} = 10x - 2x^3$ and that $v = 0$ when $x = -1$, find v in terms of x **3**

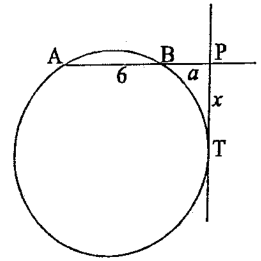
ii. Explain why the motion cannot exist between $x = -1$ and $x = 1$. **2**

iii. Describe briefly what would have happened if the motion had commenced at $x = 0$ with $v = 0$. **1**

Question 7 (Start a new work book)

Marks

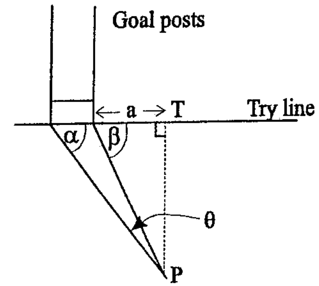
a. In the circle, the chord AB is 6 metres long. The chord is produced to the point P and BP is a metres.



A tangent to the circle cuts the chord at P. PT is x metres

Show that $x = \sqrt{a(a + 6)}$. **2**

b. In a rugby game, teams score points by placing the ball over the try line at the end of the field. A kicker may then take the ball back at right angles from the try line and attempt to kick the ball between the goal posts.



In the diagram, a try has been scored a metres to the right of the goal posts. The kicker has brought the ball back to the point P to attempt his kick. The kicker wants to maximise θ, his angle of view of the goalposts.

Let PT be x metres and assume that the goal posts are 6 metres wide.

i. Show that $\tan\theta = \dfrac{6x}{a^2 + 6a + x^2}$. **3**

ii. Letting $T = \tan\theta$, find the value of x for which T is a maximum. **2**

iii. Hence show that the maximum angle, θ, is given by $\theta = \tan^{-1}\left(\dfrac{3}{\sqrt{a^2 + 6a}}\right)$ **2**

iv. If a try is scored 10 metres to the right of the goal posts, find the maximum value of θ (to the nearest minute) and the corresponding value of x (to the nearest centimetre). **2**

v. Explain why the goal kicker, to maximise his angle of view of the goal posts, should imagine himself at the point of contact of a tangent to the circle passing through the goal posts. **1**

2001 INDEPENDENT TRIALS: MATHEMATICS EXTENSION 1
SAMPLE SOLUTIONS

Question 1:

a. $$x = \frac{kx_2 + lx_1}{k + l}$$
$$6 = \frac{k \times 3 + l \times -1}{k + l}$$
$$6k + 6l = 3k - l$$
$$3k = -7l$$
$$k:l = -7:3$$

i.e C divides AB externally in the ratio 7:3

b. Critical points: $x = 1$ and $x - 1 = \frac{1}{x - 1}$

Solving: $(x - 1)^2 = 1$
$x - 1 = \pm 1$
$\therefore x = 0,\ 2$

Testing regions $x < 0$, $0 < x < 1$, $1 < x < 2$ and $x > 2$ gives solutions

$$x < 0 \text{ and } 1 < x \le 2$$

c. i. $P(1) = 1^3 - 2 \times 1^2 - 1 + 2 = 0$. Hence $x - 1$ is a factor

ii. $P(x) = x^2(x - 2) - (x - 2) = (x - 2)(x^2 - 1) = (x - 2)(x - 1)(x + 1)$

d. i. Book work

ii. $$\frac{1 - \frac{1 - t^2}{1 + t^2}}{\frac{2t}{1 + t^2}} = \frac{1 + t^2 - 1 + t^2}{2t}$$
$$= t$$
$$= \tan\frac{\theta}{2}$$

iii. $$\tan 15° = \frac{1 - \cos 30°}{\sin 30°} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}$$

Question 2:

a. i. $\frac{dy}{dx} = \frac{dy/dx}{dx/dt} = \frac{4t}{4} = t$; therefore, $m = 3$

ii Focus (0, 2) and point (12, 18); therefore $m = \frac{4}{3}$

Question 2 (continued)

iii. $$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3 - \frac{4}{3}}{1 + 3 \times \frac{4}{3}}\right| = \frac{1}{3}$$
$\therefore \theta = 18°26'$

b. $$x' = x - \frac{f(x)}{f'(x)} = 7 - \frac{7\ln 7 - 2 \times 7}{\ln 7 - 1} = 6.5997199.. \text{ so } x = 6.6$$

c. i. $(n - 1)! = 5! = 120$
ii. Counting the couple as one, $4! \times 2! = 48$
iii. There are 48 ways they can sit together so there are $120 - 48 = 72$ ways to sit apart
$P(\text{sit apart}) = 72/120 = 3/5$

d. ∠APC = ∠PDC (angles between tangent and chord equals angle in the alt. segment)
∠PDC = ∠PCD (base angles in isosceles triangle are equal)
∴ ∠APC = ∠PCD and AB ∥ CD (if alternate angles are equal, lines are parallel)

Question 3

a. Let p = probability of scoring a goal = .7
Let q = probability of missing = 3
Let n = 10 and r = number of goals scored

Then $P(X = r) = \binom{n}{r} p^r q^{n-r}$ and
$$P(X \ge 8) = P(X = 8 \text{ or } X = 9 \text{ or } X = 10)$$
$$= \binom{10}{8} 0.7^8 \times 0.3^2 + \binom{10}{9} 0.7^9 \times 0.3 + \binom{10}{10} 0.7^{10}$$
$$= 0.382827864 = 0.38$$

b. Let P(x, y) be a point on the circle. Then ∠APB = 90° (angle in a semicircle is a rt angle)
Hence AP ⊥ PB and $m_{AP} m_{PB} = -1$
$$\therefore \frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$
whence $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

c. Let $f(n) = 2^{3n} - 3^n$
Then $f(1) = 2^3 - 3 = 5$ which is divisible by 5

Assume that $f(k) = 2^{3k} - 3^k$ is divisible by 5 for k a positive integer, and show that $f(k + 1)$ is therefore also divisible by 5

Question 3 (continued)

Then $f(k+1) = 2^{3(k+1)} - 3^{k+1}$
$= 2^{3k} \times 2^3 - 3^k \times 3$
$= 8 \times 2^{3k} - 3 \times 3^k$
$= 5 \times 2^{3k} + 3 \times 2^{3k} - 3 \times 3^k$
$= 5 \times 2^{3k} + 3 \times (2^{3k} - 3^k)$

The first term is clearly divisible by 5 and $2^{3k} - 3^k$ is also divisible by 5 by our assumption above.
Therefore $f(k+1)$ is divisible by 5 if $f(k)$ is divisible by 5

But $f(1)$ is divisible by 5, so $f(2)$ is divisible by 5 and so on for all positive integers n.

d $V = \pi\int_0^{\frac{\pi}{6}} \cos^2 2x\,dx$

$= \pi \times \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{2} \times \left[\left(\frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3}\right) - (0 - 0)\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right]$

Question 4

a $\binom{n}{r} = \binom{n}{r+1}$

$\therefore \frac{n!}{r!(n-r)!} = \frac{n!}{(r+1)!(n-r-1)!}$

$\frac{(n-r-1)!}{(n-r)!} = \frac{r!}{(r+1)!}$

$\frac{1}{n-r} = \frac{1}{r+1}$

$\therefore r + 1 = n - r$

$n = 2r + 1$

and since r is a positive integer, n is odd

b. i. $x^2 + 6x + 13 = x^2 + 6x + 9 + 4 = (x+3)^2 + 4$

ii $u = x + 3 \Rightarrow du = dx$ so $\int \frac{dx}{x^2 + 6x + 13} = \int \frac{dx}{(x+3)^2 + 4}$

$= \int \frac{du}{u^2 + 4}$

$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C$

$= \frac{1}{2}\tan^{-1}\frac{(x+3)}{2} + C$

Question 4 (continued)

c. Let $\alpha = \cos^{-1}\left(\frac{4}{5}\right)$ and $\beta = \cos^{-1}\left(\frac{3}{5}\right)$; then $\cos\alpha = \frac{4}{5}$ and $\cos\beta = \frac{3}{5}$

Therefore, $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{4}{5}$

Consider $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{4}{5} \times \frac{3}{5} - \frac{3}{5} \times \frac{4}{5}$

$= 0$

$\therefore \cos(\alpha + \beta) = 0$

$\alpha + \beta = \frac{\pi}{2}$

i.e. $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

d. $y = \frac{1}{2}(e^x - e^{-x})$

$2y = e^x - \frac{1}{e^x}$

$2ye^x = e^{2x} - 1$

$0 = e^{2x} - 2ye^x - 1$

$\therefore e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$

$= \frac{2y \pm 2\sqrt{y^2 + 1}}{2}$

$= y \pm \sqrt{y^2 + 1}$

but $e^x > 0$ and $\sqrt{y^2 + 1} > y$

$\therefore e^x = y + \sqrt{y^2 + 1}$

so $x = \ln(y + \sqrt{y^2 + 1})$

Question 5

a. i. Now $\ddot{x} = \frac{d}{dx}\left[\frac{1}{2}v^2\right]$ and $\frac{1}{2}v^2 = 6 + 2x - \frac{1}{2}x^2$

Therefore $\ddot{x} = 2 - x = -1(x - 2)$ so the motion is Simple Harmonic

ii. Centre of motion is 2 (where $\ddot{x} = 0$) and $n = 1$ so period $T = \frac{2\pi}{n} = 2\pi$

Extremes of motion occur when $v = 0$ i.e. when $6 + 2x - \frac{1}{2}x^2 = 0 \Rightarrow x = -2, 6$ so the amplitude is 4.

iii. Now $a = 4$, $n = 1$ and the centre of motion, $b = 2$ so $x = 4\sin(t + \theta) + 2$

Further when $t = 0$, $x = 6$ so $6 = 4\sin\theta + 2 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore x = 4\sin(t + \frac{\pi}{2}) + 2$

Question 5 (continued)

b i. Newton's Law is $\frac{dT}{dt} = k(T - P)$

If $T = P + Ae^{kt}$ then $\frac{dT}{dt} = k \times Ae^{kt} = k(T - P)$

ii. $100 = 23 + Ae^0 \Rightarrow A = 77$

and $93 = 23 + 77e^{k \times 2} \Rightarrow e^{2k} = \frac{70}{77}$

$\therefore k = \frac{1}{2}\ln\frac{70}{77} = -0.0476550899 = -0.0477$

iii. $80 = 23 + 77 \times e^{-0.0477 \times t} \Rightarrow t = \dfrac{\ln\frac{57}{77}}{-0.0477} = 6.31106047 \approx 6$ minutes

Question 6

a. i In the x direction: $\ddot{x} = 0 \Rightarrow \dot{x} = \int 0\,dt = C_1$

When $t = 0$, $\dot{x} = V \Rightarrow C_1 = V$

$\therefore \dot{x} = V$

$x = \int V dt = Vt + C_2$

When $t = 0$, $x = 0 \Rightarrow C_2 = 0$

$\therefore x = Vt$

In the y direction: $\ddot{y} = -g \Rightarrow \dot{y} = \int -g dt = -gt + C_3$

When $t = 0$, $\dot{y} = 0 \Rightarrow C_3 = 0$

$\therefore \dot{y} = -gt$

$y = \int -gt dt = -\frac{1}{2}gt^2 + C_4$

When $t = 0$, $y = h \Rightarrow C_4 = h$

$\therefore y = -\frac{1}{2}gt^2 + h$

ii. $x = Vt \Rightarrow t = \frac{x}{V}$. Substitute into $y = -\frac{1}{2}gt^2 + h$

$$y = -\frac{1}{2}g \times \left(\frac{x}{V}\right)^2 + h$$

$$= \frac{-gx^2}{2V^2} + h$$

$$= \frac{-gx^2 + 2V^2h}{2V^2}$$

iii. We require $y = 0$ thus $\frac{-gx^2 + 2V^2h}{2V^2} = 0 \Rightarrow x^2 = \frac{2V^2h}{g} \Rightarrow x = \pm\sqrt{\frac{2V^2h}{g}}$

But the particle is moving in a positive direction so $x = V\sqrt{\frac{2h}{g}}$

Question 6 (continued)

b. i. $\frac{d}{dx}[\frac{1}{2}v^2] = 10x - 2x^3$

$\therefore \frac{1}{2}v^2 = \int 10x - 2x^3 dx = 5x^2 - \frac{x^4}{2} + C$

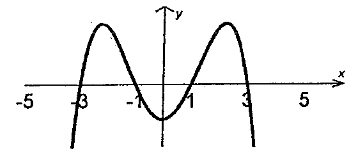
$v^2 = 10x^2 - x^4 + K$ and when $v = 0$, $x = -1 \Rightarrow K = -9$

$v^2 = 10x^2 - x^4 - 9$

$v = \pm\sqrt{10x^2 - x^4 - 9}$

ii $v^2 = -(x^4 - 10x^2 + 9) = -(x^2 - 1)(x^2 - 9) = -(x - 1)(x + 1)(x - 3)(x + 3)$

Hence $v = 0$ when $x = -3, -1, 1, 3$
From graph, between $x = -1$ and $x = 1$, $v^2 < 0$ so the motion cannot exist between $x = -1$ and $x = 1$



iii. If $x = 0$, then acceleration is zero. Since $v = 0$, the particle would remain stationary.

Question 7

a. Now $PT^2 = AP \times BP$ (On a circle, the square of the length of the tangent from an external point equals the product of the intercepts of the secant through the point)

Therefore $x^2 = a \times (a + 6) \Rightarrow x = \sqrt{a(a + 6)}$

b i. Now $\tan\alpha = \frac{x}{a + 6}$, $\tan\beta = \frac{x}{a}$, $\theta = \beta - \alpha$

$\therefore \tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$

$$= \frac{\frac{x}{a} - \frac{x}{a + 6}}{1 + \frac{x}{a} \times \frac{x}{a + 6}} = \frac{(a + 6)x - ax}{a(a + 6) + x^2} = \frac{6x}{a^2 + 6a + x^2}$$

ii. $\frac{dT}{dx} = \frac{(a^2 + 6a + x^2) \times 6 - 6x(2x)}{(a^2 + 6a + x^2)^2} = \frac{6a^2 + 36a - 6x^2}{(a^2 + 6a + x^2)^2} = 0$ when $x = \sqrt{a(a + 6)}$

When $x < \sqrt{a(a + 6)}$, $\frac{dT}{dx} > 0$; when $x > \sqrt{a(a + 6)}$, $\frac{dT}{dx} < 0$; therefore this is a max.

iii. $T = \tan\theta = \frac{6\sqrt{a(a + 6)}}{a^2 + 6a + (a^2 + 6a)} = \frac{3}{\sqrt{a^2 + 6a}} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{\sqrt{a^2 + 6a}}\right)$

iv. $x = 12.64911064 \approx 12.65$ m and $\theta = 13°20'33'' = 13°21'$

v. The maximum value of θ occurs when $x = \sqrt{a(a + 6)}$. Using the result from part a., we see that, because the square of the tangent equals the product of the intercepts of the secant, the goal posts and the point P from which the kick is taken lie on a circle, with PT a tangent