NSW INDEPENDENT SCHOOLS

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1998

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Write your student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the seven questions are to be handed in separately clearly marked Question 1, Question 2, etc..
- *The question paper must be handed to the supervisor at the end of the examination.*

STUDENT NUMBER / NAME..............................

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Question 1 (Start a new page) **Marks**

a. Given $\log_a(x^2y) = m$ and $\log_a\left(\dfrac{y}{x}\right) = n$, find $\log_a(xy)$ in terms of m and n **3**

b. Solve $2\tan^3\theta + 7\tan^2\theta + 2\tan\theta - 3 = 0$ for $0° \le \theta \le 360°$, giving your answers to the nearest minute. **3**

c. Solve $\dfrac{1}{x} \ge \sqrt{x}$ **3**

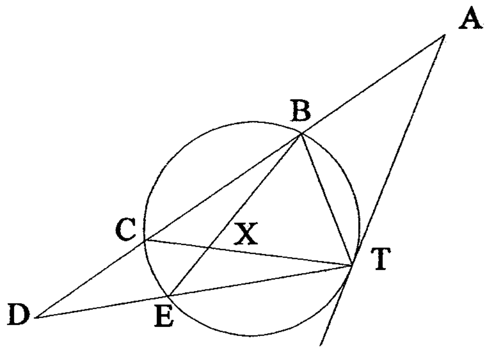
d. Given A(–1, 3) and B(2, –3), divide AB in the ratio 1:2 **1**

e. Show that $n! + (n-1)! + (n-2)! = n^2(n-2)!$ **2**

Question 2 (Start a new page)

a. Find the exact value of $\cos\left(2\tan^{-1}\dfrac{5}{12}\right)$ **3**

b. In the figure, **not drawn to scale**, AT is a tangent to the circle at T. ABCD and TED are straight lines. BE and CT intersect at X. **4**



Copy the diagram into your workbook.

i. Prove that
$\angle CXE - \angle ATB = \angle ATB - \angle CDE$

ii. If BC = DE = x, DC = 6 and ET = 15, find the value of x, giving reasons.

c. Find the exact value of $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4 - x^2}}$ **1**

d. Find $\displaystyle\int \cos^2\theta \, d\theta$ and hence find $\displaystyle\int \frac{dx}{(1 + x^2)^2}$ using the substitution $x = \tan\theta$ **4**

Question 3 (Start a new page) **Marks**

a. Ship Abel is headed due North at 10 km/h while ship Bessel, which is 10 km due West of ship Abel, is headed due East at 15 km/h. **3**

i. Find their distance apart after t hours.

ii. At what rate are the ships sailing away from each other after 2 hours?

b. Express $\cos x - 2\sin x$ in the form $A\cos(x + \alpha)$ and hence, or otherwise, solve $\cos x - 2\sin x = \sqrt{5}$ for $0° \le x \le 360°$ to the nearest minute. **3**

c. Use mathematical induction to prove the following result for positive integral values of n: **6**

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$$
$$\dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Question 4 (Start a new page)

a. i. Expand $(1 + 2x)^5(1 + 4x)^5$ in ascending powers of x as far as the term containing x^3 **5**

ii. Hence, noting that $(1.0608)^5 = (1.02)^5(1.04)^5$, evaluate $(1.0608)^5$ to four significant figures.

iii. A city's population grows for five years at a rate of 4% per year and then for another five years at a rate of 2% per year. If the initial population was 1 million people, what was the population, to the nearest thousand, at the end of the ten years?

b. i. On the same graph, sketch the curves $y = x^2$ and $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ **4**

ii. Use your graph to estimate the first positive solution of the equation $\sin x - x^2 = 0$

iii. Using Newton's Method once and your answer to part ii. as your first estimate, find a better approximation to the solution of the equation $\sin x - x^2 = 0$

(Question 4 is continued on the next page)

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Question 4 (continued) **Marks**

c. The line L_1 has equation $2x - y + 5 = 0$ and P is a point with coordinates $(-1, 2)$. L_2 goes through P and makes an angle, θ, with L_1 such that $\tan\theta = \frac{1}{3}$. 3

Find the equation(s) of L_2.

Question 5 (Start a new page)

a. Ron Aldo kicks a soccer ball off the ground from 25 metres out at an angle of 30° to the horizontal. The ball hits the top bar which is 2.4 metres above the ground. Neglecting air resistance and assuming the acceleration due to gravity is 10 m/s², find 6

i. the horizontal and vertical components of displacement using integration

ii. the Cartesian equation of motion for the path of the ball.

iii. the initial velocity of the ball.

iv. the height to which a goalkeeper standing under the path of the motion and 2 metres from the goal line would have to jump to touch the ball.

b. A sphere of ice cream sits on top of a cone. As the ice cream melts, maintaining its spherical shape, the radius reduces at a constant rate of 0.25 centimetres per minute. [The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$] 2

Find the rate of change of the volume of the ice cream with respect to time when the radius is 4 cm.

c. Using the expansion 4

$$1 + \binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots + \binom{2n}{2n}x^{2n} = (1 + x)^{2n}$$

show that

i. $1 - 2\binom{2n}{1} + 4\binom{2n}{2} - \ldots + \binom{2n}{2n}4^n = 1$

ii. $\binom{2n}{1} - 4\binom{2n}{2} + \ldots - n\binom{2n}{2n}4^n = -2n$

Question 6 (Start a new page) **Marks**

a. An object hanging from the end of a light spring is undergoing simple harmonic motion between the points P and Q, which are 6 cm apart. Initially, the object is at rest at P. After $\frac{\pi}{2}$ seconds, the particle is first at Q. **5**

i. Write down an expression for the velocity of the object as a function of its displacement from the centre of the motion.

ii. Find the exact value of the speed of the object when its displacement from the centre of the motion is 1.5 cm.

iii. What is the magnitude of the object's maximum acceleration and at what point does this occur?

b. The rate at which a body cools in air is given by the difference between the temperature, T°C, at any time, t minutes, and the temperature, A°C, of the surrounding air. This rate is given by the differential equation **5**

$$\frac{dT}{dt} = k(T - A), \text{ where } k \text{ is a constant.}$$

i. Show that $T = A + Pe^{kt}$, where P is a constant, is a solution of the differential equation.

ii. A hot cup of coffee cools from 90°C to 70°C in 8 minutes, the temperature of the air being 22°C. Find the time required for the cup to cool to a drinkable temperature of 60°C.

iii. Use the equation for T to describe the behaviour of T as t becomes large.

c. The acceleration of an object is given by $\ddot{x} = 3e^{3x}$. If its velocity is – 6 cm/s when the object is at the origin, find the velocity when it is 3 cm to the left of the origin. Give your answer to one decimal place. **2**

Question 7 (Start a new page) **Marks**

a. The velocity of a particle is given by $\frac{dx}{dt} = \sqrt{3x + 1}$ **3**

If the particle is initially 5 cm to the right of the origin,

i. find the equation for its displacement in terms of t.

ii. find the particle's displacement after 3 seconds.

b. i. Using the results for $\cos(A + B)$ and $\cos(A - B)$, express $\sin x . \sin 5x$ as the difference of two cosines. **9**

ii. Hence

1. evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x . \sin 5x \, dx$

2. find, in terms of sines, the sum of the first six terms of the series

$\sin x . \sin 5x + \sin 2x . \sin 8x + \sin 3x . \sin 13x + \sin 4x . \sin 20x + \ldots$

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$