INDEPENDENT

2003
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided with this paper
- All necessary working should be shown in every question
-

Total marks - 84

Attempt Questions 1 – 7

All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ... ...

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Question 1 (Start a new work booklet) **Marks**

a. i. Factorise completely $x^4+x^3-2x^2-2x$ **1**

ii. Hence sketch $y=x^4+x^3-2x^2-2x$ for $-2\le x\le 2$ **2**

b. Find $\int \sin^2 3x\,dx$ **3**

c. Calculate to the nearest minute the acute angle between the lines $x+y=4$ and $3x-4y=2$ **3**

d. Express $\dfrac{3^{501}+3^{500}}{4}$ as a power of 3 **1**

e. Use the table of standard integrals to show that $\int_0^a \dfrac{1}{\sqrt{x^2+a^2}}dx$ is independent of a **2**

Question 2 (Start a new work booklet)

a. Use the substitution $u=x-1$ to find $\int 5x\sqrt{x-1}\,dx$ **3**

b. The point $(a, 0)$ divides the interval AB joining the points A(9, 12) and B(1, b) in the ratio 2:1. Find a and b. **3**

c. i. Show that the Cartesian equation of the curve with parametric equations $x=6t$, $y=3t^2$ is $12y=x^2$ **1**

ii. Show that, if the tangents at the points $P(6p,3p^2)$ and $Q(6q,3q^2)$, intersect on the y-axis, then $p^2=q^2$ **2**

d. Solve $|x^2-5|=5x+9$ **3**

STUDENT NAME/NUMBER: ……………………

Question 3 (Start a new work booklet) — Marks

a. i. If $f(x)=e^{x+2}$, find the inverse function $f^{-1}(x)$ — **2**

ii. On the same axes, sketch the graphs of $y=f(x)$ and $y=f^{-1}(x)$ showing all important features. — **2**

b. The probability of throwing a 6 with an irregular die is $3/8$. In throwing the die 10 times, what is the probability of throwing a 6 exactly three times. — **2**

c. i. Expand $(1-x)^n$ using the Binomial Theorem — **1**

ii. Hence show that $\int_0^1 (1-x)^n dx=\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\ldots+\frac{(-1)^n}{n+1}\binom{n}{n}$ — **3**

iii. Show that $\frac{1}{n+1}=\sum_{r=0}^{n}(-1)^r.\frac{1}{r+1}\binom{n}{r}$ — **2**

STUDENT NAME/NUMBER: ……………………

Question 4 (Start a new work booklet) — Marks

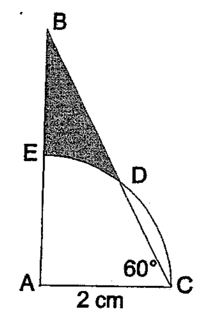
a. A golf ball is hit towards a tree 60 metres away and standing in the same horizontal plane as the ball. The tree is 20 metres high. The initial velocity of the ball is 30 metres per second. — **6**

Find the range of angles, in which the ball must be hit to clear the tree. Assume that $g=10$ ms^{-2}

b. A triangle ABC is right angled at A. ∠C is 60° and AC is 2 cm long. — **3**

A circular arc, centre A and radius 2 cm, cuts BC at D and AB at E, as shown.

Show that the area of the shaded portion BED is $\left(\sqrt{3}-\frac{\pi}{3}\right) cm^2$



c. i. Show that the function $f(x) = xe^x - 1$ has a zero between $x = 0$ and $x = 1$ — **1**

ii. Using $x=0.5$ as a first approximation, use Newton's Method once to obtain a second approximation to the zero. — **2**

Question 5 (Start a new work booklet)

Marks

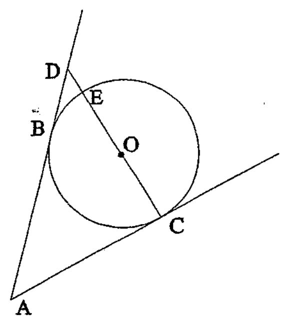
a. Use the derivative of $y=(x-2)e^{-x}$ to evaluate $\int_0^2 2e^{-x}(x-3)\,dx$ **3**

b. At time t minutes, the temperature T°C of a body in a room of constant temperature 20°C is decreasing according to the equation
$\frac{dT}{dt} = -k(T-20)$ for some constant $k > 0$.

i. Verify that $T = 20 + Ae^{-kt}$, where A is a constant, is a solution of the equation. **1**

ii. The initial temperature of the body is 90°C and it falls to 70°C after 10 minutes. Find the temperature after a further 5 minutes. **4**

c. The diagram shows two tangents, AB and AC, drawn from a common point, A, to a circle, centre O. **4**



The diameter CE produced cuts the tangent AB at the point D.

Copy the diagram into your workbook.

Show that $E\hat{D}A + 2 \times D\hat{E}B = 270°$

STUDENT NAME/NUMBER: ....

Question 6 (Start a new work booklet)

Marks

a. A particle, initially at $x = 1.5$, has velocity given by $v = \sqrt{9-4x^2}$.

Derive an expression for x in terms of time, t **3**

b. The sequence known as the Fibonacci Sequence is defined as follows:

$T_1 = 1,\ T_2 = 2$ and $T_n = T_{n-1} + T_{n-2}$ for $n > 2$

Use the Principle of Mathematical Induction to prove that

$T_1 + T_2 + T_3 + \ldots + T_n = T_{n+2} - 2$ **4**

c. At a particular dinner, each rectangular table has nine seats, five facing the stage and four with their backs to the stage.

In how many ways can 9 people be seated at the table if

i. John and Mary must sit on the same side? **3**
ii. John and Mary must sit on opposite sides? **2**

Question 7 (Start a new work booklet)

a. i. Find all solutions for $\frac{x}{1-x^2} \geq 0$ **3**

ii. Show that $\sec x \tan x = \frac{\sin x}{1-\sin^2 x}$ **1**

iii. Hence, or otherwise, find all solutions of $\sec x \tan x \geq 0$, $0 \leq x \leq 2\pi$ **3**

b. i. Find $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right]$ **3**

ii. Hence, or otherwise, show that $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1} x$ for $0 \leq x < 1$ **2**

NSW INDEPENDENT TRIAL EXAMS – 2003

MATHEMATICS – Extension 1 – HSC Trial Exam

SOLUTIONS

Q1 (a)

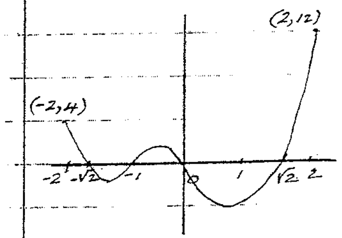
(i) $x^3(x+1) - 2x(x+1)$

$= x(x+1)(x^2-2)$

$= x(x+1)(x+\sqrt{2})(x-\sqrt{2})$

(ii) $f(2) > 0 \quad = 12$

$f(-2) = 4$



(b) $\int \sin^2 3x \, dx$

$= \frac{1}{2}\int (1 - \cos 6x)\, dx$

$= \frac{1}{2}(x - \frac{1}{6}\sin 6x) + c$

(c) $y = -x + 4 \quad m_1 = -1$

$y = \frac{3}{4}x - \frac{1}{2} \quad m_2 = 3/4$

$\tan\theta = \frac{3/4 - (-1)}{1 + 3/4 \cdot (-1)}$

$= 7$

$\theta = 81°52'$ to nearest minute

(d) $3^{500}\left(\frac{3+1}{4}\right) = 3^{500}$

(e) $I = \left[\ln(x + \sqrt{x^2+a^2})\right]_0^a$

$= \ln(a + \sqrt{a^2+a^2})$
$\quad - \ln(0 + \sqrt{a^2})$

$= \ln\left(\frac{a + \sqrt{2}a}{a}\right)$

$= \ln(1+\sqrt{2})$

Q2 (a)

$dx = du$

$u = x - 1 \Rightarrow x = u + 1$

$I = \int (u+1)\sqrt{u}\, du$

$= \int (u^{3/2} + u^{1/2})\, du$

$= \frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + c$

$= \frac{2}{5}(x-1)^2\sqrt{x-1}$
$\quad + \frac{2}{3}(x-1)\sqrt{x-1} + c$

$= \sqrt{x-1}\left(\frac{2}{5}(x-1)^2 + \frac{2}{3}(x-1)\right) + c$

(b) A(9,12) —2— (a,0) —1— B(1,b)

$a = \frac{2(1) + 1(9)}{1+2} = 11/3$

$0 = \frac{2(b) + 1(12)}{1+2}$

$2b + 12 = 0.$

$b = -6 \quad a = 11/3$

(c) (i) $x = 6t \quad y = 3t^2$

$t = \frac{x}{6}$

$y = 3\left(\frac{x}{6}\right)^2$

$= \frac{x^2}{12}$

$12y = x^2$

(ii) Tangents

at P: $y = px - 3p^2$

Q: $y = qx - 3q^2$

Solve → $x = 3(p+q)$

As tangents meet on y-axis $\therefore x = 0$

$\therefore p + q = 0$

$p = -q$

$p^2 = q^2$

OR $x = 0 \Rightarrow y = -3p^2 = -3q^2$

$\therefore p^2 = q^2$

(d) $|x^2 - 5| = 5x + 9$

$5x + 9 \geq 0$

$x \geq -9/5$

$x^2 - 5 = 5x + 9$

$x^2 - 5x - 14 = 0$

$(x-7)(x+2) = 0$

$x = 7, -2$

$x = 7$ as $-2 < -9/5$

$-x^2 + 5 = 5x + 9$

$x^2 + 5x + 4 = 0$

$(x+4)(x+1) = 0$

$x = -4, -1$

$x = -1$ as $-4 < -9/5$

$x = -1, 7$

Q3 (a)

(i) $f(x) = e^{x+2}$

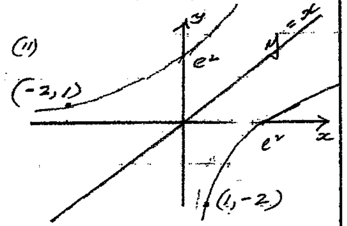
$y = e^{x+2}$

Inverse $\quad x = e^{y+2}$

$\ln x = y + 2$

$f^{-1}(x): y = \ln x - 2$

(ii)



(b) $(x+y)^{10} \quad P(6) = x = 3/8$
$P(\bar{6}) = y = 5/8$

$\therefore P(3 \times 6) = {}^{10}C_3 \times \left(\frac{3}{8}\right)^3 \times \left(\frac{5}{8}\right)^7$

Q3 (c)

(i) $(1-x)^n$

$= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(-x)$
$\quad + \binom{n}{2}1^{n-2}(-x)^2 + \ldots$
$\quad + \binom{n}{n}(-x)^n$

$= \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + (-1)^n\binom{n}{n}x^n$

(ii) Integrating term by term

$\int_0^1 (1-x)^n\, dx$

$= \left[\binom{n}{0}x - \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \ldots + \binom{n}{n}(-1)^n\frac{x^{n+1}}{n+1}\right]_0^1$

$= \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{(-1)^n}{n+1}\binom{n}{n}$

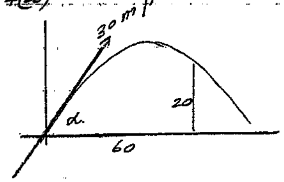
(iii) $LHS = \left[\frac{(1-x)^{n+1}}{-1(n+1)}\right]_0^1$

$= 0 - \frac{1^{n+1}}{-1(n+1)}$

$= \frac{1}{n+1}$

$RHS = \sum_{r=0}^{n} \frac{(-1)^r}{r+1}\binom{n}{r}$

Q4 (a)



$\ddot{x} = 0 \Rightarrow x = 30t\cos\alpha$

$60 = 30t\cos\alpha$

$t = 2/\cos\alpha$

$\ddot{y} = -10 \Rightarrow$ "

$y = -5t^2 + 30t\sin\alpha$

$\therefore -5t^2 + 30t\sin\alpha > 20$

$-5\left(\frac{4}{\cos^2\alpha}\right) + 30\frac{2}{\cos\alpha}\sin\alpha > 20$

$-20\sec^2\alpha + 60\tan\alpha > 20$

$(1 + \tan^2\alpha) = \sec^2\alpha$

$-20 - 20t^2 + 60t > 20$

$t^2 - 3t + 2 < 0$

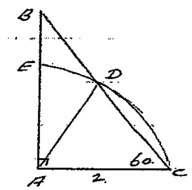
$(t-2)(t-1) < 0$

$1 < t < 2.$

$\therefore 45° < \tan\alpha < 63°26'$

(b) Shaded A =

$\Delta BAC - (\text{Sect } ADE + \Delta ADC)$



$A\hat{D}C = 60°$ (AD = AC)

$\therefore D\hat{A}C = 60° \quad DA = 2$

$D\hat{A}E = 30°$

$\tan 60° = \frac{BA}{2}$

$BA = 2\sqrt{3}.$

$\therefore \Delta BAC = \frac{1}{2} \cdot 2 \cdot 2\sqrt{3}$

$= 2\sqrt{3}\ cm^2$

Sect ADE $= \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\pi}{6}$

$= \frac{\pi}{3}\ cm^2$

$\Delta ADC = \frac{1}{2} \cdot 2 \cdot 2 \cdot \sin 60°$

$= \sqrt{3}\ cm^2$

$\therefore A = 2\sqrt{3} - \left(\frac{\pi}{3} + \sqrt{3}\right)$

$= \left(\sqrt{3} - \frac{\pi}{3}\right)\ cm^2$

(c) (i) $f(0) = 0 \cdot e^0 - 1 < 0$

$f(1) = 1 \cdot e^1 - 1 > 0$

$\therefore$ Root between 0, 1.

(ii) $f'(x) = e^x(x+1)$

$f(0.5) = 0.5e^{0.5} - 1$

$f'(0.5) = e^{0.5}(0.5 + 1)$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.5 - \left(\frac{0.5e^{0.5} - 1}{e^{0.5}(1.5)}\right)$

$= 0.57$

Q5

(a) $y = (x-2)e^{-x}$

$y' = e^{-x} + (x-2)(-e^{-x})$

$= e^{-x}(1 - x + 2)$

$= (-e^{-x})(x-3)$

$\therefore \int_0^2 2e^{-x}(x-3)\, dx.$

$= -2\int_0^2 (-e^{-x})(x-3)\, dx$

$= -2\left[(x-2)e^{-x}\right]_0^2$

$= -2\{0 - (-2)e^0\}$

$= -4$

(b) (i) $T = 20 + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$Ae^{-kt} = T - 20$

$\therefore \frac{dT}{dt} = -k(T - 20)$

(ii) $t = 0 \quad T = 90$

$90 = 20 + Ae^0$

$A = 70$

$t = 10 \quad T = 70$

$70 = 20 + 70e^{-10k}$

$e^{-10k} = 50/70$

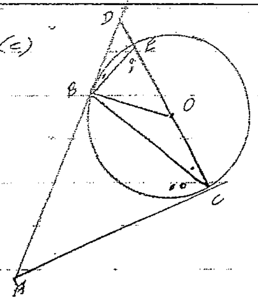
$-10k = \ln 5/7$

$k = 0.0336\ldots$

$t = 15, \ T = 20 + 70e^{-15k}$

$= 62.3°C$

Q5 (c)



(i) $E\hat{D}A + D\hat{E}B + D\hat{B}E = 180°$

($\angle$ sum $\triangle DEB$)

$D\hat{B}E = D\hat{C}B$

($\angle$ in alt seg th)

Sim $B\hat{C}A = B\hat{E}C$

$B\hat{C}A + D\hat{C}B = 90°$

(tang $AC \perp$ rad OC)

$\therefore B\hat{E}C + D\hat{B}E = 90°$

$B\hat{E}C = 180° - D\hat{E}B$

$\therefore D\hat{B}E = D\hat{E}B - 90°$

$\therefore$ (i) $\Rightarrow E\hat{D}A + 2D\hat{E}B - 90° = 180°$

$E\hat{D}A + 2D\hat{E}B = 270°$

Q6 (a) $\frac{dx}{dt} = \sqrt{9-4x^2}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{9/4 - x^2}}$

$t = \frac{1}{2}\sin^{-1}\frac{x}{3/2} + c$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

$0 = \frac{1}{2}\sin^{-1} 1 + c$

$c = -\frac{\pi}{4}$

$t = \frac{1}{2}\sin^{-1}\frac{2x}{3} - \frac{\pi}{4}$

$2t + \frac{\pi}{2} = \sin^{-1}\frac{2x}{3}$

$2x/3 = \sin(2t + \frac{\pi}{2})$

$x = \frac{3}{2}\sin(2t + \frac{\pi}{2})$

(b) for $n = 3$

$T_3 = 1 + 2 = 3$

$T_4 = 2 + 3 = 5$

$T_5 = 5 + 3 = 8$

$T_1 + T_2 + T_3 = T_5 - 2$

$1 + 2 + 3 = 8 - 2$ True for $n = 3$.

Assume

$T_1 + T_2 + \dots + T_k = T_{k+2} - 2$

$(T_1 + T_2 + \dots + T_k) + T_{k+1}$

$= (T_{k+2} - 2) + T_{k+1}$

$= (T_{k+2} + T_{k+1}) - 2$

$= T_{k+3} - 2$

etc

(c)(i) Facing John/Mary

$(4+3+2+1) \times 2$

Not Facing $(3+2+1) \times 2$

32 ways

$\therefore$ $32 \times 7!$ ways

(ii) John Facing Mary Not

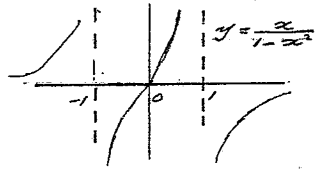
$= 5 \times 4 = 20$

$\therefore$ $40 \times 7!$ ways

Q7 (a)

(i) $\times (1-x^2)^2 \quad x(1-x^2) \geq 0$

$x(1-x)(1+x) \geq 0$



$\therefore 0 \leq x < 1$ OR $x < -1$

(undefined at $x = \pm 1$)

(ii) $\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \frac{\sin x}{\cos^2 x}$

$= \frac{\sin x}{1 - \sin^2 x}$

(iii) $\frac{S}{1-S^2} \geq 0$

By (i) $\sin x < -1$ (No Soln)

$0 \leq \sin x < 1$.

$\therefore 0 \leq x < \frac{\pi}{2}$

OR $\frac{\pi}{2} < x \leq \pi$

(b)(i) $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right]$

$= \frac{1}{1 + \left(\frac{x}{\sqrt{1-x^2}}\right)^2} \times \frac{\sqrt{1-x^2}\cdot 1 - \frac{x(-2x)\frac{1}{2}}{\sqrt{1-x^2}}}{1-x^2}$

$= \frac{1-x^2}{1} \times \frac{1-x^2+x^2}{(1-x^2)\sqrt{1-x^2}}$

$= \frac{1}{\sqrt{1-x^2}}$

(ii) $\therefore \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$= \int \frac{1}{\sqrt{1-x^2}}\,dx$

$= \sin^{-1}x + c$

$x = 0 \Rightarrow \tan^{-1}0 = \sin^{-1}0 + c$

$0 = 0 + c$

$c = 0$.

$\therefore \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$

$0 \leq x < 1$

as $\frac{x}{\sqrt{1-x^2}}$ has domain $0 \leq x < 1$