# NSW INDEPENDENT SCHOOLS

**2006**
**Higher School Certificate**
**Trial Examination**

# Mathematics
## Extension 1

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- Write using black or blue pen
- Write your student number and/or name at the top of every page
- All necessary working should be shown in every question
- A table of standard integrals is provided separately

**Total marks – 84**

Attempt Questions 1 – 7

All questions are of equal value

**This paper MUST NOT be removed from the examination room**

STUDENT NUMBER/NAME....................................

Student name / number ..................................

Marks

**Question 1** **Begin a new page**

(a) When the polynomial $P(x) = x^3 + ax + 1$ is divided by $(x+2)$ the remainder is 3. Find the value of $a$. **2**

(b) The acute angle between the lines $y = (m+2)x$ and $y = mx$ is $45°$.

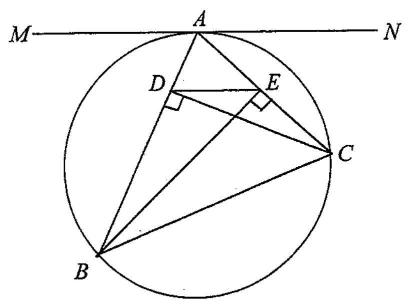
(i) Show that $\left|\dfrac{2}{m^2+2m+1}\right| = 1$. **1**

(ii) Hence find any values of $m$. **2**

(c)(i) Show that $\cot\theta - \cot 2\theta = \operatorname{cosec} 2\theta$. **2**

(ii) Hence find the exact value of $\cot 15°$. **1**

(d)




$ABC$ is a triangle inscribed in a circle. $MAN$ is the tangent at $A$ to the circle $ABC$. $CD$ and $BE$ are altitudes of the triangle.

(i) Copy the diagram.

(ii) Give a reason why $BCED$ is a cyclic quadrilateral. **1**

(iii) Hence show that $DE$ is parallel to $MAN$. **3**



Student name / number ..................................

Marks

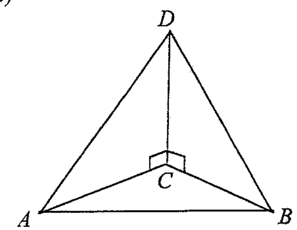
**Question 2** **Begin a new page**

(a) $A(-3, 4)$ and $B(1, 2)$ are two points. Find the coordinates of the point $P(x, y)$ which divides the interval $AB$ externally in the ratio 3 : 1. **2**

(b)(i) Solve the inequality $\dfrac{1}{1-x} < 1$. **2**

(ii) Hence find the set of values of $x$ for which the limiting sum $S$ of the geometric series $1 + x + x^2 + x^3 + \ldots$ is such that $S < 1$. **1**

(c)

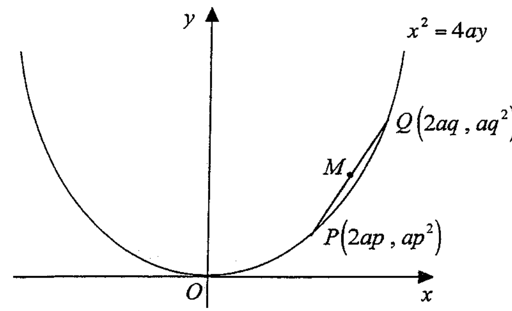



Three points $A$, $B$ and $C$ lie on horizontal ground. Points $A$ and $B$ are 30 metres apart and $\angle ACB = 120°$. A vertical flagpole $CD$ of height $h$ metres stands at $C$. From each of $A$ and $B$ the angle of elevation of the top $D$ of the flagpole is $30°$.

(i) Show that $AC = BC = h\sqrt{3}$. **1**

(ii) Hence find the value of $h$. **2**

(d)




(i) Find the coordinates of the point $T$ on the parabola $x^2 = 4ay$ such that the tangent to the parabola at $T$ is parallel to the line $y = x$. **1**

(ii) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points that move on the parabola $x^2 = 4ay$ such that the chord $PQ$ is always parallel to the line $y = x$. $M$ is the midpoint of $PQ$. Find the equation of the locus of $M$ and state any restrictions on this locus. **3**

Student name / number ..................................

Marks

**Question 3** **Begin a new page**

(a) Consider the function $f(x)=\dfrac{x-2}{x-1}$.

(i) Show that the function is increasing for all values of $x$ in its domain. **2**

(ii) Sketch the graph of the function showing clearly any intercepts on the coordinate axes and the equations of any asymptotes. **2**

(iii) Find the equation of the inverse function $f^{-1}(x)$. Deduce that the graph of the function $f(x)$ is symmetrical about the line $y=x$. **2**

(b) Consider the function $y=\frac{1}{2}\cos^{-1}(x-1)$.

(i) Find the domain and range of the function. **2**

(ii) Sketch the graph of the function showing clearly the coordinates of the endpoints. **1**

(iii) The region in the first quadrant bounded by the curve $y=\frac{1}{2}\cos^{-1}(x-1)$ and the coordinate axes is rotated through 360° about the $y$ axis. Find the volume of the solid of revolution, giving your answer in simplest exact form. **3**



Student name / number ..................................

Marks

**Question 4** **Begin a new page**

(a)(i) Show that the equation $x^3+2x-7=0$ has a root $\alpha$ such that $1<\alpha<2$. **2**

(ii) If an initial approximation of $1\cdot 5$ is taken for $\alpha$, use one application of Newton's method to find the next approximation, rounding your answer to one decimal place. **2**

(b) Use the substitution $x=u^2$, $u\geq 0$, to find the value of $\displaystyle\int_1^3 \frac{1}{(x+1)\sqrt{x}}\,dx$. **4**

Give your answer in simplest exact form.

(c) Five different fair dice are thrown together. Find the probability that

(i) the five scores are all different **2**

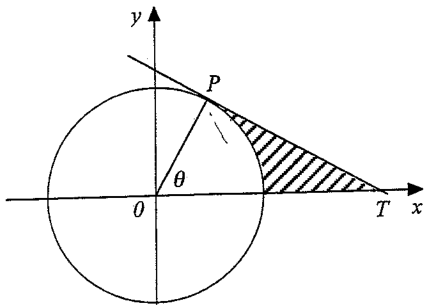
(ii) the five scores include at most one 6 **2**

Student name / number ..................................

Marks

**Question 5** **Begin a new page**

(a)




$P$ is a point on the circle $x^2 + y^2 = 1$ such that the radius $OP$ makes an angle $\theta$ with the positive $x$ axis, where $0 < \theta < \frac{\pi}{2}$. The tangent to the circle at $P$ cuts the $x$ axis at $T$.

(i) Show that the area A of the shaded region is given by $A = \frac{1}{2}(\tan\theta - \theta)$. **2**

(ii) If $\theta$ is increasing at a constant rate of $0\cdot1$ radians per second find the rate at which A is increasing when $\theta = 1$, giving your answer correct to 2 decimal places. **2**

(b) The number $N$ of individuals in a population at time $t$ years is given by $N = 100 - 60e^{+0\cdot1t}$.

(i) Sketch the graph of $N$ as a function of $t$ showing clearly the initial population size and the limiting population size. **2**

(ii) Find the exact time taken for the population to double its initial size and find the rate at which the population is increasing then. **2**

(c) Use Mathematical Induction to show that, for all positive integers $n \geq 1$, **4**

$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.



Student name / number ..................................

Marks

**Question 6** **Begin a new page**

(a) A particle is moving in a straight line with Simple Harmonic Motion. At time $t$ seconds it has displacement $x$ metres from a fixed point $O$ on the line, given by $x = 1 + 3\cos\frac{t}{2}$, velocity $v$ ms$^{-1}$ and acceleration $a$ ms$^{-2}$.

(i) Show that $a = -\frac{1}{4}(x-1)$. **1**

(ii) Find the distance travelled and the time taken by the particle over one complete oscillation of its motion. **2**

(iii) Find the time taken by the particle to travel the first 100 metres of its motion, giving your answer in seconds correct to two decimal places. **3**

(b) A golfer hits a golf ball from a point $O$ with speed 40 ms$^{-1}$ at an angle $\theta$ above the horizontal. The ball travels in a vertical plane where the acceleration due to gravity is 10 ms$^{-2}$.

(i) Write down expressions for the horizontal displacement $x$ metres, and the vertical displacement $y$ metres, of the golf ball from $O$ after time $t$ seconds. **1**

(ii) Hence show that the horizontal range $R$ metres of the golf ball until it returns to ground level is given by $R = 160\sin 2\theta$. **2**

(iii) The golfer is aiming over horizontal ground at a circular pond of radius 10 metres with centre 110 metres from $O$. Find the set of possible values of $\theta$ for the golf ball to land directly in the pond, giving your answers correct to the nearest degree. **3**

Student name / number ..................................

**Marks**

**Question 7** **Begin a new page**

(a) A particle is moving in a straight line. At time $t$ seconds it has displacement $x$ metres from a fixed point $O$ on the line, velocity $v$ ms$^{-1}$, given by $v = (k - x)^2$ for some constant $k > 0$, and acceleration $a$ ms$^{-2}$. Initially the particle is at $O$.

(i) Show that $x = \dfrac{k^2 t}{kt + 1}$. Hence show that $x < k$ for all values of $t$. **4**

(ii) Express $a$ in terms of $x$. Deduce that the particle is always moving to the right and always slowing down. **2**

(iii) Find the distance travelled and the time taken by the particle for its speed to drop to 1% of its initial value. **2**

(b)(i) Show that ${}^{n+1}C_r - {}^{n}C_r = {}^{n}C_{r-1}$ , $r = 1, 2, 3, \ldots, n$. **2**

(ii) Hence find the value of $\sum_{n=3}^{100} {}^{n}C_2$. **2**

## Question 1

**a. Outcomes assessed : PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • uses the remainder theorem to write an equation for $a$ | 1 |
| • finds the value of $a$ | 1 |

**Answer**

$P(x) = x^3 + ax + 1 \qquad \therefore P(-2) = 3 \Rightarrow -8 - 2a + 1 = 3 \qquad \therefore a = -5$

**b. Outcomes assessed : P4, H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • uses the formula for the angle between two lines to obtain the required equation for $m$ | 1 |
| ii • reduces this equation to a quadratic in $m$ | 1 |
| • solves this quadratic to find two values of $m$ | 1 |

**Answer**

i. $\left|\frac{(m+2)-m}{1+m(m+2)}\right| = \tan 45°$

$\therefore \left|\frac{2}{m^2+2m+1}\right| = 1$

ii. $\left|\frac{2}{(m+1)^2}\right| = 1$

$\therefore (m+1)^2 = 2$

$\therefore m + 1 = \pm\sqrt{2}$

$m = -1 \pm \sqrt{2}$

**c. Outcomes assessed : H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • expresses $\cot 2\theta$ in terms of $t$ where $t = \tan\theta$ | 1 |
| • simplifies LHS in terms of $t$ and recognizes expression for $\text{cosec}\, 2\theta$ | 1 |
| ii • applies identity to evaluate $\cot 15°$ | 1 |

**Answer**

i Let $t = \tan\theta$. Then

$$\cot\theta - \cot 2\theta = \frac{1}{t} - \frac{1-t^2}{2t} = \frac{2-(1-t^2)}{2t} = \frac{1+t^2}{2t} = \text{cosec}\, 2\theta$$

ii. $\cot 15° - \cot 30° = \text{cosec}\, 30°$

$\cot 15° - \sqrt{3} = 2$

$\cot 15° = 2 + \sqrt{3}$

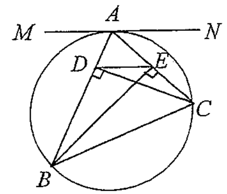
**d. Outcomes assessed : PE2, PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| ii • quotes appropriate test for a cyclic quadrilateral | 1 |
| iii • uses equality of exterior and interior opposite angles of $BCED$ with explanation | 1 |
| • applies alternate segment theorem with explanation | 1 |
| • quotes an appropriate test for parallel lines | 1 |

**Answer**

i




ii. Interval $BC$ subtends equal angles $BDC$ and $BEC$ at points $D$, $E$ on the same side of $BC$. Hence $BCED$ is a cyclic quadrilateral.

iii. $\angle ADE = \angle ACB$ (*exterior $\angle$ of cyclic quad. $BCED$ is equal to interior opp. $\angle$*)

But $\angle MAB = \angle ACB$ (*$\angle$ between tangent and chord $AB$ is equal to $\angle$ subtended by $AB$ in the alternate segment*)

Hence $\angle MAB = \angle ADE$

$\therefore DE \parallel MAN$ (*equal alternate $\angle$s on transversal $AB$*)

## Question 2

**a. Outcomes assessed : P4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • finds $x$ coordinate | 1 |
| • finds $y$ coordinate | 1 |

**Answer**

$A(-3, 4) \qquad B(1, 2)$

$3 \; : \; -1$

$P\left(\frac{3\times1+(-1)\times(-3)}{3+(-1)}, \frac{3\times2+(-1)\times4}{3+(-1)}\right)$

$\therefore P(3, 1)$ is the required point of external division.

**b. Outcomes assessed : H5, PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • applies a method which deals appropriately with the inequality and variable denominator | 1 |
| • finds possible values of $x$ | 1 |
| ii • uses condition for existence of limiting sum along with result from i. to find $x$ values. | 1 |

**Answer**

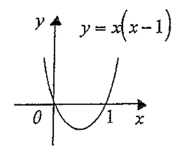
i. $\frac{1}{1-x} < 1$

$1 - x < (1-x)^2 \quad , \quad x \neq 1$

$0 < (1-x)^2 - (1-x)$

$0 < (1-x)\{(1-x) - 1\}$

$0 < x(x-1)$




$\therefore x < 0$ or $x > 1$

ii. $1 + x + x^2 + \ldots$ has limiting sum

$S = \frac{1}{1-x}$ provided $|x| < 1$.

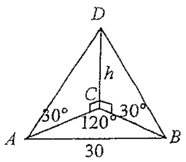
Hence $S < 1$ for $-1 < x < 0$.

c. **Outcomes assessed : H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • uses right triangle trigonometry to show result | 1 |
| ii • uses cosine rule to write equation for $h$ | 1 |
| • finds value of $h$ | 1 |

**Answer**




i.
$AC = BC = h\cot 30° = h\sqrt{3}$

ii. Using the cosine rule in $\Delta ABC$,

$30^2 = 3h^2 + 3h^2 - 6h^2\cos 120°$

$900 = 6h^2\left(1+\frac{1}{2}\right)$

$h^2 = 100$

$h = 10$

**d. Outcomes assessed : H5, PE3, PE4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • uses derivative to write equation for $x$ at $T$ then writes coordinates of $T$. | 1 |
| ii • uses gradient of $PQ$ to show sum of $p$ and $q$ is 2 | 1 |
| • writes coordinates of $M$ in terms of $p, q$ then deduces equation of locus | 1 |
| • states restriction $y > a$ | 1 |

**Answer**

i. $x^2 = 4ay$

$\frac{dy}{dx} = \frac{x}{2a}$

$\frac{dy}{dx} = 1 \Rightarrow x = 2a$

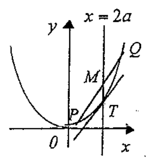
$\therefore T(2a\,,\,a)$

ii. $PQ$ has gradient $\frac{a\left(p^2-q^2\right)}{2a(p-q)} = \frac{p+q}{2}$

Hence $p+q=2$

$M\left(a(p+q)\,,\,\frac{a}{2}\left(p^2+q^2\right)\right)$

Hence locus of $M$ has equation $x = 2a$.




$M$ lies above $T$ $\therefore y > a$.

**Question 3**

**a. Outcomes assessed : P5, H6, HE4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • differentiates the function | 1 |
| • notes that the derivative is positive throughout the domain | 1 |
| ii • sketches hyperbola with correct intercepts on the coordinate axes | 1 |
| • shows equations of both asymptotes | 1 |
| iii • finds the equation of the inverse function | 1 |
| • uses reflection property of graphs of inverse functions to justify required deduction | 1 |

**Answer**

i.

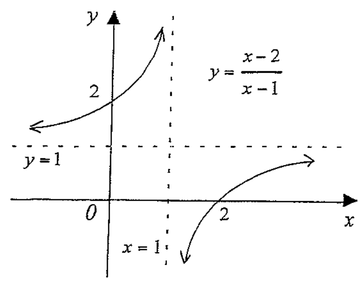
$f'(x) = \frac{1.(x-1)-(x-2).1}{(x-1)^2}$

$= \frac{1}{(x-1)^2}$

$f(x) = \frac{x-2}{x-1}$ has domain $x \neq 1$.

$\therefore f'(x) > 0$ and function is increasing throughout its domain.

ii.




iii.

$y = \frac{x-2}{x-1}$

$(x-1)y = x-2$

$xy - y = x - 2$

$x(y-1) = y-2$

$x = \frac{y-2}{y-1}$

Interchanging $x \leftrightarrow y$, inverse function is

$y = \frac{x-2}{x-1}$

$\therefore f^{-1}(x) = \frac{x-2}{x-1}$

The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ in the line $y = x$. But the two graphs are identical. Hence the graph of $f(x)$ must be symmetrical in the line $y = x$.

**b. Outcomes assessed : H8, HE4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • states domain of function | 1 |
| • states range of function | 1 |
| ii • sketches graph of correct shape showing coordinates of endpoints | 1 |
| iii • writes volume as integral in terms of $y$ | 1 |
| • finds primitive after using appropriate trig. identity | 1 |
| • evaluates by substitution of limits | 1 |

**Answer**

i. $y = \frac{1}{2}\cos^{-1}(x-1)$

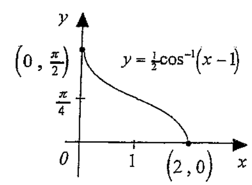
$-1 \le x-1 \le 1$

*Domain* $\{x : 0 \le x \le 2\}$

$0 \le \cos^{-1}(x-1) \le \pi$

*Range* $\left\{y : 0 \le y \le \frac{\pi}{2}\right\}$

ii.




iii. $x = 1 + \cos 2y$

$V = \pi\int_0^{\frac{\pi}{2}}(1+\cos 2y)^2\,dy$

$= \pi\int_0^{\frac{\pi}{2}}\left(1+2\cos 2y+\cos^2 2y\right)dy$

$= \pi\int_0^{\frac{\pi}{2}}\left(1+2\cos 2y+\frac{1}{2}(1+\cos 4y)\right)dy$

$= \pi\left[\frac{3}{2}y+\sin 2y+\frac{1}{8}\sin 4y\right]_0^{\frac{\pi}{2}}$

$= \pi\left\{\frac{3}{2}\left(\frac{\pi}{2}-0\right)+(\sin\pi-\sin 0)+\frac{1}{8}(\sin 2\pi-\sin 0)\right\}$

$= \frac{3}{4}\pi^2$

Question 4

**a. Outcomes assessed : PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • shows that values of $f(x)=x^3+2x-7$ at $x=1, x=2$ have opposite signs | 1 |
| • uses continuity of $f$ to deduce the existence of a root of the equation between 1 and 2 | 1 |
| ii • applies Newton's rule for next approximation | 1 |
| • calculates this approximation | 1 |

**Answer**

i. Let $f(x)=x^3+2x-7$

Then $f$ is continuous with

$f(1)=-4<0$ and $f(2)=5>0$

Hence $f(\alpha)=0$ for some $1<\alpha<2$.

ii. $f'(x)=3x^2+2$

$$\alpha_1 = 1\cdot 5 - \frac{f(1\cdot 5)}{f'(1\cdot 5)} = 1\cdot 5 - \frac{3\cdot 375+3-7}{6\cdot 75+2}$$

$\therefore \alpha_1 \approx 1\cdot 6$

**b. Outcomes assessed : HE6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • converts $dx$ into $du$ and simplifies new integrand | 1 |
| • finds $u$ limits | 1 |
| • finds primitive function in terms of $u$ | 1 |
| • substitutes limits and evaluates in simplest exact form | 1 |

**Answer**

$x=u^2\ ,\ u\ge 0$

$dx = 2u\ du$

$x=1 \Rightarrow u=1$

$x=3 \Rightarrow u=\sqrt{3}$

$$I=\int_1^3 \frac{1}{(x+1)\sqrt{x}}\,dx$$

$$=\int_1^{\sqrt{3}} \frac{1}{(u^2+1)u}\,2u\ du$$

$$=2\int_1^{\sqrt{3}} \frac{1}{(u^2+1)}\ du$$

$$I=2\left[\tan^{-1}u\right]_1^{\sqrt{3}}$$

$$=2\left(\tan^{-1}\sqrt{3}-\tan^{-1}1\right)$$

$$=2\left(\tfrac{\pi}{3}-\tfrac{\pi}{4}\right)$$

$$=\tfrac{\pi}{6}$$

**c. Outcomes assessed : HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • counts the number of arrangements of five different scores | 1 |
| • selects appropriate denominator and simplifies | 1 |
| ii • writes numerical expression for the sum of probabilities of *no 6* and *exactly one 6* | 1 |
| • calculates this probability | 1 |

**Answer**

i. $P(\textit{all different})=\dfrac{{}^6C_5\times 5!}{6^5}=\dfrac{5}{54}$

ii. $P(\textit{at most one } 6)={}^5C_0\left(\tfrac{1}{6}\right)^0\left(\tfrac{5}{6}\right)^5+{}^5C_1\left(\tfrac{1}{6}\right)^1\left(\tfrac{5}{6}\right)^4=2\left(\tfrac{5}{6}\right)^5\approx 0\cdot 804$

$\dfrac{6\times5\times4\times3\times2\times1}{6^5}$

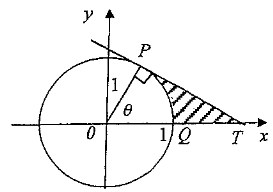
Question 5

**a. Outcomes assessed : H4, H5, HE5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • shows $PT=\tan\theta$ | 1 |
| • uses difference between area triangle and area of sector to obtain expression for $A$ | 1 |
| ii • expresses $\frac{dA}{dt}$ in terms of $\frac{d\theta}{dt}$ | 1 |
| • evaluates to find rate of increase of $A$ when $\theta=1$ | 1 |

**Answer**




i. $\angle OPT = 90°$

*(tangent ⊥ radius drawn to point of contact)*

$\therefore PT=\tan\theta$

$A=\tfrac{1}{2}OP\,.\,PT-\tfrac{1}{2}.1^2.\theta$

$=\tfrac{1}{2}(\tan\theta-\theta)$

ii. $\dfrac{dA}{dt}=\tfrac{1}{2}\left(\sec^2\theta\dfrac{d\theta}{dt}-\dfrac{d\theta}{dt}\right)$

$=\tfrac{1}{2}\left(\sec^2\theta-1\right)\dfrac{d\theta}{dt}$

$=\tfrac{1}{2}\tan^2\theta\times 0\cdot 1$

Hence when $\theta=1$, $A$ is increasing at a rate $0\cdot 12$ sq. units per second.

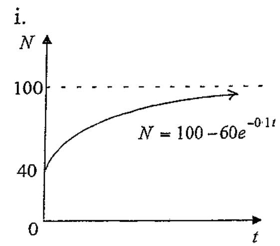
**b. Outcomes assessed : HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • sketches graph of correct shape showing initial population size | 1 |
| • shows limiting size as horizontal asymptote | 1 |
| ii • solves exponential equation for $t$ to find exact time for initial size to double | 1 |
| • differentiates then finds rate of increase. | 1 |

**Answer**

i.




ii. $t=0 \Rightarrow N=100-60e^0=40$

$N=80 \Rightarrow 60e^{-0\cdot 1t}=20$

$e^{-0\cdot 1t}=\tfrac{1}{3}$

$-0\cdot 1t=\ln\tfrac{1}{3}$

$t=\ln\tfrac{1}{3}\div(-0\cdot 1)$

Population doubles initial size in $10\ln 3$ years

$\dfrac{dN}{dt}=0\cdot 1\times 60e^{-0\cdot 1t}=\tfrac{1}{10}(100-N)$

Population is then increasing at a rate of 2 individuals per year.

**c. Outcomes assessed : HE2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • defines a sequence of statements and shows the first is true | 1 |
| • expresses the LHS of $S(k+1)$ in terms of the RHS of $S(k)$ (if true) | 1 |
| • rearranges algebraically to give RHS of $S(k+1)$ | 1 |
| • writes final explanation to complete process of induction | 1 |

Answer

Let $S(n)$ be the sequence of statements $1^2+3^2+5^2+\dots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1),\quad n=1,2,3,\dots$

*Consider* $S(1)$ : $LHS=1^2=1$ and $RHS=\frac{1}{3}.1.(2-1)(2+1)=1$. Hence $S(1)$ is true

*If* $S(k)$ *is true* : $1^2+3^2+5^2+\dots+(2k-1)^2=\frac{1}{3}k(2k-1)(2k+1)$ **

*Consider* $S(k+1)$ : $LHS=\{1^2+3^2+5^2+\dots+(2k-1)^2\}+(2k+1)^2$

$$=\tfrac{1}{3}k(2k-1)(2k+1)+(2k+1)^2 \qquad \textit{if } S(k) \textit{ is true, using } **$$
$$=\tfrac{1}{3}(2k+1)\{k(2k-1)+3(2k+1)\}$$
$$=\tfrac{1}{3}(2k+1)\{2k^2+5k+3\}$$
$$=\tfrac{1}{3}(2k+1)(k+1)(2k+3)$$
$$=\tfrac{1}{3}(k+1)\{2(k+1)-1\}\{2(k+1)+1\}$$
$$=RHS$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Therefore $S(n)$ is true for all positive integers $n \ge 1$.

**Question 6**

**a. Outcomes assessed : HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • finds expression for $a$ by differentiation (or by noting centre of oscillation and value of $n$) | 1 |
| ii • uses amplitude to find distance | 1 |
| • uses period to find time | 1 |
| iii • finds number of complete oscillations and corresponding time | 1 |
| • recognizes time taken for extra 4 m is time to first reach $O$ and writes equation for $t$ | 1 |
| • adds time for the extra 4m to time for 8 complete oscillations and calculates total time | 1 |

**Answer**

i. $x=1+3\cos\frac{t}{2}$

$v=-\frac{3}{2}\sin\frac{t}{2}$

$a=-\frac{3}{4}\cos\frac{t}{2}$

$\therefore a=-\frac{1}{4}(x-1)$

ii. Amplitude is 3m and period is $4\pi$ s.
Hence distance travelled is 12m and time taken is $4\pi$ s.

iii. Initially particle is at its far right extreme where $x=4$.
Also $100=8\times 12+4$. Hence time taken for 100m is time for 8 complete oscillations plus the time taken to travel directly from $x=4$ to $x=0$.

$x=0 \Rightarrow \cos\frac{t}{2}=-\frac{1}{3}$. First such $t$ is $2(\pi-\cos^{-1}\frac{1}{3})$ seconds.

Hence time taken to travel 100m is $8\times 4\pi+2(\pi-\cos^{-1}\frac{1}{3})\approx 104\cdot 35$ s.

**b. Outcomes assessed : H4, HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • writes expressions for $x$ and $y$ | 1 |
| ii • finds $t>0$ for which $y=0$ | 1 |
| • substitutes this value of $t$ into expression for $x$ to find $R$. | 1 |
| iii • writes inequality for R | 1 |
| • finds one interval for $\theta$ | 1 |
| • finds second interval for $\theta$ | 1 |

**Answer**

i. $x=40\,t\cos\theta$

$y=40\,t\sin\theta-5t^2$

ii. $y=5t(8\sin\theta-t)$

$y=0\Rightarrow t=0,\ 8\sin\theta$

Particle returns to ground level when

$x=40\,(8\sin\theta)\cos\theta$

$=160\,(2\sin\theta\cos\theta)$

$\therefore R=160\sin 2\theta$

iii. $100\le R\le 120$

$\frac{10}{16}\le\sin 2\theta\le\frac{12}{16}$

$38\cdot 68^\circ\le 2\theta\le 48\cdot 59^\circ$

*or* $131\cdot 41^\circ\le 2\theta\le 141\cdot 32$

$\therefore\ 20^\circ\le\theta\le 24^\circ$

*or* $66^\circ\le\theta\le 70^\circ$

**Question 7**

**a. Outcomes assessed : HE1, HE5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • writes $\frac{dt}{dx}$ as a function of $x$ | 1 |
| • integrates to find $t$ as a function of $x$ | 1 |
| • rearranges to find $x$ as a function of $t$ | 1 |
| • deduces $x<k$ | 1 |
| ii • finds $a$ in terms of $x$ | 1 |
| • notes that $v>0$ and $a$ and $v$ have opposite signs to make required deductions | 1 |
| iii • finds $x$ when $v=\frac{4}{100}k^2$ | 1 |
| • finds corresponding value of $t$ | 1 |

**Answer**

i. $\frac{dx}{dt}=(k-x)^2$

$\frac{dt}{dx}=(k-x)^{-2}$

$t=(k-x)^{-1}+c$

$\left.\begin{matrix}t=0\\x=0\end{matrix}\right\}\Rightarrow \begin{matrix}0=k^{-1}+c\\ \therefore c=-\frac{1}{k}\end{matrix}$

$\therefore t=\frac{1}{k-x}-\frac{1}{k}$

$t+\frac{1}{k}=\frac{1}{k-x}$

$\frac{kt+1}{k}=\frac{1}{k-x}$

$k-x=\frac{k}{kt+1}$

$x=k-\frac{k}{kt+1}$

$=\frac{k(kt+1)-k}{kt+1}$

$=\frac{k^2t}{kt+1}$

$x=k\frac{kt}{kt+1}$ where $0<\frac{kt}{kt+1}<1$

$\therefore x<k$

ii. $a=\frac{1}{2}\frac{d}{dx}v^2$

$=\frac{1}{2}\frac{d}{dx}(k-x)^4$

$=-2(k-x)^3$

$x<k\Rightarrow v>0$ and $a<0$

Hence particle is always moving right and slowing down.

iii. $v=(k-x)^2$

$=\left(\frac{k}{kt+1}\right)^2$

Initially $v=k^2$. Hence particle has 1% of initial speed when

$\frac{1}{(kt+1)^2}=\frac{1}{100}$

$kt+1=10$

$\therefore t=\frac{9}{k}$ and $x=\frac{9k}{10}$.

Hence particle has travelled $\frac{9k}{10}$ m and taken $\frac{9}{k}$ s.

b. Outcomes assessed : PE3, HE3, HE7

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • writes expression for ${}^{n+1}C_r - {}^nC_r$ | 1 |
| • rearranges to obtain required result | 1 |
| ii • uses result from i. to write required sum as difference of two sums | 1 |
| • cancels out terms to simplify and evaluate | 1 |

**Answer**

i. $${}^{n+1}C_r - {}^nC_r = \frac{(n+1)!}{r!\,(n+1-r)!} - \frac{n!}{r!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n+1-r)!}\left\{(n+1)-(n+1-r)\right\}$$

$$= \frac{n!\,.\,r}{r!\,(n+1-r)!}$$

$$= \frac{n!}{(r-1)!\,(n+1-r)!}$$

$$= {}^nC_{r-1}$$

ii. $${}^nC_2 = {}^{n+1}C_3 - {}^nC_3$$

$$\sum_{n=3}^{100} {}^nC_2 = \sum_{n=3}^{100} {}^{n+1}C_3 - \sum_{n=3}^{100} {}^nC_3$$

$$= \sum_{n=4}^{101} {}^nC_3 - \sum_{n=3}^{100} {}^nC_3$$

$$= {}^{101}C_3 - {}^3C_3$$

$$= 166\,649$$