2008

Higher School Certificate

Trial Examination

(OR NSW INDEPENDENT SCHOOLS)
TRIAL HSC 2008

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- Write using black or blue pen
- Write your student number and/or name at the top of every page
- All necessary working should be shown in every question
- A table of standard integrals is provided

Total marks – 84

Attempt Questions 1 – 7

All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME..

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a\neq 0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln(x+\sqrt{x^2-a^2}),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2})$$

NOTE: $\ln x = \log_e x,\ x>0$

Student name / number ....

Marks

Question 1 **Begin a new booklet**

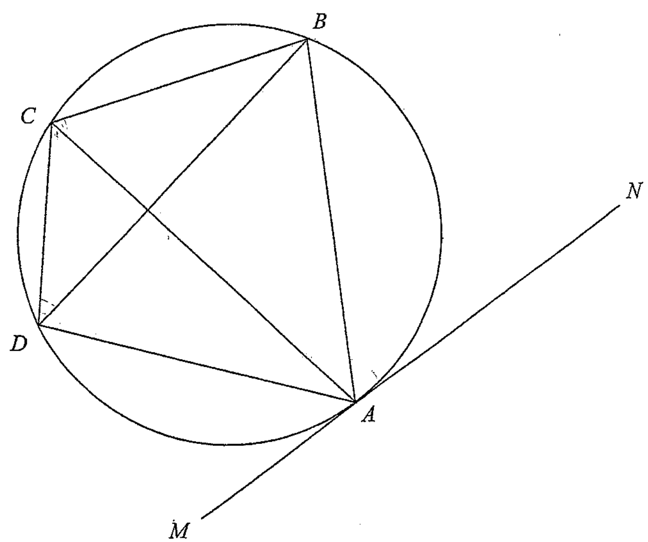
(a) Find the value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x \tan x \, dx$, giving the answer in simplest exact form. **2**

(b) Solve the inequality $\frac{x^2-4}{x} \geq 0$. **3**

(c)(i) Find the gradients at the point $P(1,1)$ of the tangents to the curves $y = x^3$ and $y = 1 - \ln x$. **2**

(ii) Hence find the acute angle between these tangents, giving the answer correct to the nearest degree. **1**

(d)



$ABCD$ is a cyclic quadrilateral. MAN is the tangent at A to the circle through A, B, C and D. CA bisects $\angle BCD$.

Copy the diagram. Show that $MAN \parallel DB$, giving reasons. **4**

Student name / number ....

Marks

Question 2 **Begin a new booklet**

(a) Find the set of values of x for which the limiting sum of the geometric series $1 + \ln x + (\ln x)^2 + \ldots$ exists. **2**

(b) $A\left(8, \sqrt{8}\right)$ and $B\left(50, \sqrt{50}\right)$ are two points. Find the coordinates of the point $P(x, y)$ which divides the interval AB internally in the ratio $2:1$, giving the answer in simplest exact form. **3**

(c)(i) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Hence solve $\cos x - \sqrt{3}\sin x = -2$ for $0 \leq x \leq 2\pi$. **1**

(d) $T\left(2t, t^2\right)$ is a point on the parabola $x^2 = 4y$.

(i) Use differentiation to show that the tangent to the parabola at T has gradient t and equation $tx - y - t^2 = 0$. **2**

(ii) Hence find the values of t such that the tangent to the parabola at T passes through the point $P(1, -2)$. **2**

Student name / number ....

Question 3 **Begin a new booklet** **Marks**

(a) Find $\frac{d}{dx}\left(x\sin^{-1}x\right)$. **2**

(b) Consider the function $f(x)=x+\frac{1}{x}$ for $x\geq 1$.

(i) Show that the function $f(x)$ is increasing and the curve $y=f(x)$ is concave up for all values of $x>1$. **2**

(ii) On the same diagram, sketch the graphs of $y=f(x)$ and the inverse function $y=f^{-1}(x)$ showing the coordinates of the endpoints and the equation of the asymptote. **2**

(iii) Find the equation of the inverse function $y=f^{-1}(x)$ in its simplest form. **2**

(c) Use Mathematical induction to show that for all positive integers $n\geq 1$, **4**

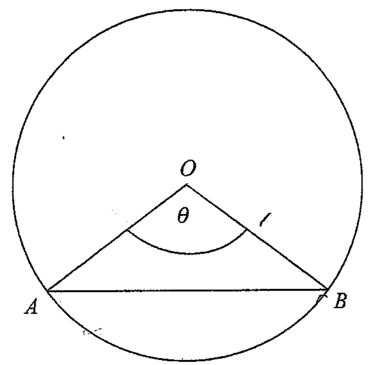
$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$$

Student name / number ....

Question 4 **Begin a new booklet** **Marks**

(a) Find $\int\cos^2 4x\,dx$. **2**

(b)



AB is a chord of a circle of radius 1 metre that subtends an angle θ at the centre of the circle, where $0<\theta<\pi$. The perimeter of the minor segment cut off by AB is equal to the diameter of the circle.

(i) Show that $\theta+2\sin\frac{1}{2}\theta-2=0$. **2**

(ii) Show that the value of θ is such that $1<\theta<2$. **2**

(iii) Use one application of Newton's method with an initial approximation of $\theta_0=1$ to find the next approximation to the value of θ, giving your answer correct to 1 decimal place. **2**

(c) Use the substitution $x=u^2$, $u\geq 0$, to evaluate $\int_1^{25}\frac{1}{x+\sqrt{x}}\,dx$, giving the answer in simplest exact form. **4**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a) Consider the polynomial $P(x) = x^3 - kx^2 + kx - 1$, where k is a real constant.

(i) Show that 1 is a root of the equation $P(x) = 0$. **1**

(ii) Given that α, $\alpha \neq 1$, is a second root of $P(x) = 0$, show that $\frac{1}{\alpha}$ is also a root of the equation. **1**

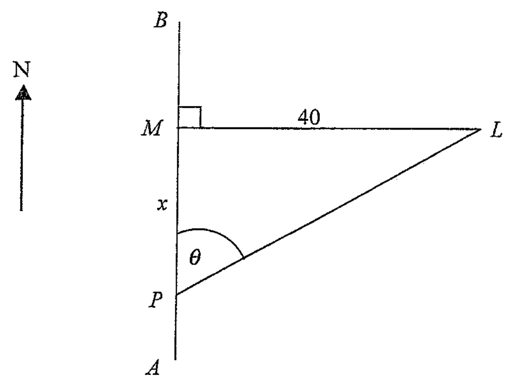
(iii) Show that $\alpha^2 + \frac{1}{\alpha^2} = k^2 - 2k - 1$. **2**

(b) Three numerals are chosen at random from the numerals 1, 2, 3, ... , 9 to form a three digit code where order is important and repetition is allowed.

(i) Find the probability that all three digits of the code are different. **2**

(ii) Find the probability that exactly two digits of the code are the same. **2**

(c)



A boat is sailing due North from point A to point B at a steady speed of $5\ \text{ms}^{-1}$. A marker buoy M on its route is situated 40 metres due West of a lighthouse L. When the boat is at point P at a distance x metres from M, the bearing of the lighthouse from the boat is θ, $0 < \theta < \frac{\pi}{2}$.

(i) Show that $\theta = \tan^{-1} \frac{40}{x}$. **1**

(ii) Hence find the rate at which θ is changing when $x = 20$. **3**

Student name / number ....

Marks

Question 6 **Begin a new booklet**

(a) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity $v\ \text{ms}^{-1}$ given by $v = 2 - x$ and acceleration $a\ \text{ms}^{-2}$. Initially the particle is 4 metres to the left of O.

(i) Find an expression for a in terms of x. **1**

(ii) Use integration to show that $x = 2 - 6e^{-t}$. **2**

(iii) Find the exact time taken by the particle to travel 4 metres from its starting point. **1**

(iv) Sketch the graph of x against t showing the intercepts on the axes and the equations of any asymptotes. **2**

(b) A particle is performing Simple Harmonic Motion on a straight line. At time t seconds it has displacement x metres from a fixed point O on the line given by $x = 2 + 2\cos 2t$.

(i) Sketch the graph of x against t showing the intercepts on the axes. **2**

(ii) Show that the acceleration of the particle is given by $\ddot{x} = -4(x - 2)$. **1**

(iii) Find the period of the motion. **1**

(iv) Find the distance travelled by the particle in the first 2 seconds of its motion, giving the answer correct to two significant figures. **2**

Student name / number ....

Marks

Question 7 **Begin a new booklet**

(a) A particle is projected from a point O with speed V ms^{-1} at an angle θ above the horizontal where $0 < \theta < \frac{\pi}{2}$. The particle moves in a vertical plane under gravity where the acceleration due to gravity is g ms^{-2}. At time t seconds its horizontal and vertical displacements from O are x metres and y metres respectively.

(i) Write down expressions for x and y in terms of V, θ, g and t. Hence show that the horizontal range R of the particle is given by $R = \dfrac{V^2 \sin 2\theta}{g}$. **2**

(ii) A lawn on horizontal ground is rectangular in shape with length 50 metres and breadth 20 metres. A garden sprinkler is located at one corner S of the lawn. It rotates horizontally, and delivers water at a speed of 20 ms^{-1} at angles of elevation between $15°$ and $45°$ above the horizontal. Taking $g = 10$, find the area of the lawn that can be watered by the sprinkler, giving the answer in simplest exact form. **4**

(b)(i) Write down the binomial expansion of $(1+x)^n$ in ascending powers of x. **1**

(ii) Show that $\sum_{r=1}^{n} {}^nC_r = 2^n - 1$. **1**

(iii) Use integration and the answer to part (i) to show that $\dfrac{1}{n+1}\sum_{r=1}^{n+1} {}^{n+1}C_r = \sum_{r=0}^{n} \dfrac{{}^nC_r}{r+1}$. **4**

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• writes primitive and substitutes for x	1
• evaluates in simplest surd form	1

Answer

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x \tan x \, dx = \left[\sec x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \sec\tfrac{\pi}{3} - \sec\tfrac{\pi}{4} = 2 - \sqrt{2}$$

b. Outcomes assessed : PE3

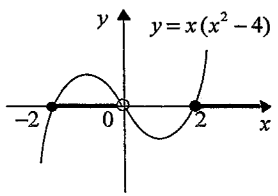
Marking Guidelines

Criteria	Marks
• writes an equivalent inequality not involving a variable denominator	1
• writes one inequality for x	1
• combines this with a second inequality for x	1

Answer

$$\frac{x^2-4}{x} \geq 0$$

$$x(x^2-4) \geq 0 \;,\quad x \neq 0$$



By inspection of the graph,
$-2 \leq x < 0$ or $x \geq 2$

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • finds gradient of tangent to $y = x^3$ at P	1
• finds gradient of tangent to $y = 1 - \ln x$ at P	1
ii • finds the acute angle between the lines correct to the nearest degree	1

Answer

i.

$y = x^3$

$\dfrac{dy}{dx} = 3x^2$

$x = 1 \Rightarrow \dfrac{dy}{dx} = 3$

Tangent at $P(1,1)$ has gradient 3

$y = 1 - \ln x$

$\dfrac{dy}{dx} = -\dfrac{1}{x}$

$x = 1 \Rightarrow \dfrac{dy}{dx} = -1$

Tangent at $P(1,1)$ has gradient -1

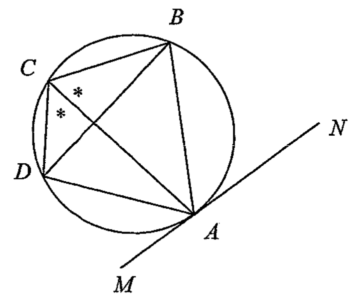
ii. $\tan\theta = \left|\dfrac{3-(-1)}{1+3\times(-1)}\right| = 2 \;\Rightarrow\; \theta \approx 63°$ (to the nearest degree)

d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
• explains why angles BAN, BCA are equal	1
• explains why angles DCA, DBA are equal	1
• uses the equality of angles BCA, DCA to complete proof that angles BAN, DBA are equal	1
• quotes test for parallel lines to deduce tangent MAN is parallel to BD	1

Answer



$\angle BAN = \angle BCA$ (*angle between tangent and chord drawn to point of contact is equal to angle subtended by the chord in the alternate segment*)

$\angle BCA = \angle DCA$ (*given that AC bisects* $\angle BCD$)

$\angle DCA = \angle DBA$ (*angles subtended at the circumference by the same arc DA are equal*)

$\therefore MAN \parallel DB$ (*equal alternate angles on transversal BA since* $\angle BAN = \angle DBA$)

Question 2

a. Outcomes assessed : H3, H5

Marking Guidelines

Criteria	Marks
• writes condition on common ratio $\ln x$ for existence of limiting sum	1
• solves this inequality for x	1

Answer

Limiting sum of geometric series $1 + \ln x + (\ln x)^2 + \ldots$ exists for $-1 < \ln x < 1$

$\therefore$ since $f(x) = e^x$ is an increasing function, $e^{-1} < e^{\ln x} < e^1$

$\therefore \frac{1}{e} < x < e$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• finds x coordinate of P	1
• finds y coordinate of P as the sum of two surds	1
• simplifies surd expression for y	1

Answer

$A(8, \sqrt{8}) \qquad B(50, \sqrt{50})$

$2 \;:\; 1$

$$\left(\frac{100+8}{2+1}, \frac{2\sqrt{50}+\sqrt{8}}{2+1}\right)$$

$$\therefore P\left(36, \frac{10\sqrt{2}+2\sqrt{2}}{3}\right)$$

$$P\left(36, 4\sqrt{2}\right)$$

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • finds value of R	1
• finds value of α	1
ii • solves equation for x	1

Answer

i. $\cos x - \sqrt{3}\sin x = 2\left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right)$

$= 2\left(\cos\frac{\pi}{3}\cos x - \sin\frac{\pi}{3}\sin x\right)$

$= 2\cos\left(x + \frac{\pi}{3}\right)$

ii. $\cos x - \sqrt{3}\sin x = -2, \quad 0 \le x \le 2\pi$

$\cos\left(x + \frac{\pi}{3}\right) = -1, \quad \frac{\pi}{3} \le x + \frac{\pi}{3} \le 2\pi + \frac{\pi}{3}$

$x + \frac{\pi}{3} = \pi$

$x = \frac{2\pi}{3}$

d. Outcomes assessed : PE3, PE4

Marking Guidelines

Criteria	Marks
i • shows by differentiation that tangent has gradient t	1
• finds the equation of the tangent	1
ii • substitutes coordinates of P to write equation for t	1
• solves equation for t	1

Answer

i. $y = t^2 \Rightarrow \frac{dy}{dt} = 2t$

$x = 2t \Rightarrow \frac{dx}{dt} = 2$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = t$

Tangent at $T\left(2t, t^2\right)$ has gradient t

and equation $y - t^2 = t(x - 2t)$

$y - t^2 = tx - 2t^2$

$tx - y - t^2 = 0$

ii. $P(1, -2)$ lies on this tangent if

$t + 2 - t^2 = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = 2$ or $t = -1$

Question 3

a. Outcomes assessed : PE5, HE4

Marking Guidelines

Criteria	Marks
• applies the product rule, obtaining first term	1
• obtains second term by deriving inverse sine	1

Answer

$\frac{d}{dx}\left(x\sin^{-1}x\right) = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$

b. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
i • shows $f(x)$ is increasing for $x > 1$	1
• shows the curve $y = f(x)$ is concave up for $x > 1$	1
ii • sketches $y = f(x)$ showing endpoint and asymptote $y = x$	1
• sketches $y = f^{-1}(x)$ showing endpoint and asymptote	1
iii • makes x the subject	1
• interchanges x and y to find equation of inverse function	1

Answer

i. $f(x) = x + \frac{1}{x}$ for $x \ge 1$

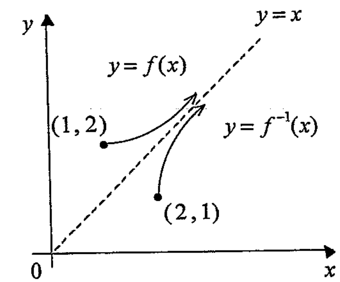
$f'(x) = 1 - \frac{1}{x^2} > 0$ for $x > 1$

$\therefore f(x)$ is increasing for $x > 1$

$f''(x) = \frac{2}{x^3} > 0$ for $x > 1$

$\therefore y = f(x)$ is concave up for $x > 1$

ii.



iii. $y = x + \frac{1}{x}, \quad x \ge 1$ and $y \ge 2$

Rearrangement gives $\quad x^2 - xy + 1 = 0, \quad x \ge 1$ and $y \ge 2$

Considering this quadratic in x: $\quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}, \quad x \ge 1$ and $y \ge 2$

Clearly the branch $x = \frac{y - \sqrt{y^2 - 4}}{2}$ contains points for which $x < 1$.

Hence expressing x as the subject of $y = f(x)$, $\quad x = \frac{y + \sqrt{y^2 - 4}}{2}, \quad y \ge 2$.

Interchanging x and y, the inverse function is $\quad f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}, \quad x \ge 2$

c. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• verifies truth of statement for $n = 1$	1
• expresses LHS of $S(k+1)$ in terms of LHS of $S(k)$	1
• expresses LHS of $S(k+1)$ in terms of RHS of $S(k)$, conditional on truth of $S(k)$	1
• completes algebraic rearrangement to show $S(k+1)$ is true if $S(k)$ is true	1

Answer

Define the sequence of statements $S(n)$, $n = 1, 2, 3, \ldots$ by $S(n): \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Consider $S(1)$: $LHS=\frac{1}{2!}=\frac{1}{2}$ $RHS=1-\frac{1}{2!}=1-\frac{1}{2}=\frac{1}{2}$ $\therefore S(1)$ is true

If $S(k)$ is true: $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}=1-\frac{1}{(k+1)!}$ **

Consider $S(k+1)$: $LHS=\left\{\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}\right\}+\frac{k+1}{(k+2)!}$

$=\left\{1-\frac{1}{(k+1)!}\right\}+\frac{k+1}{(k+2)!}$ if $S(k)$ is true, using **

$=1-\frac{k+2}{(k+2)(k+1)!}+\frac{k+1}{(k+2)!}$

$=1-\frac{k+2-(k+1)}{(k+2)!}$

$=1-\frac{1}{(k+2)!}$

$=RHS$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n\geq 1$.

Question 4

a. Outcomes assessed : H8

Marking Guidelines

Criteria	Marks
• expresses integrand in terms of $\cos 8x$	1
• finds primitive function	1

Answer

$\int \cos^2 4x\,dx=\int \frac{1}{2}(1+\cos 8x)\,dx=\frac{1}{2}x+\frac{1}{16}\sin 8x+c$

b. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • uses cosine rule and trigonometric identity to find AB in terms of $\sin\frac{1}{2}\theta$	1
• adds arc length to AB, equating sum and diameter to obtain required equation	1
ii • shows $f(1)$ and $f(2)$ have opposite signs	1
• uses continuity of $f(\theta)$ to deduce equation has root between 1 and 2.	1
iii • applies Newton's rule, substituting $\theta=1$	1
• evaluates expression to obtain next approximation, giving result correct to 1 dec. place	1

Answer

i. Using cosine rule, $AB^2=1^2+1^2-2\cos\theta$

$AB^2=2(1-\cos\theta)$

$=4\sin^2\frac{1}{2}\theta$

$\therefore AB=2\sin\frac{1}{2}\theta$

$\therefore$ Perimeter = diameter $\Rightarrow \theta+2\sin\frac{1}{2}\theta=2$

$\theta+2\sin\frac{1}{2}\theta-2=0$

ii. Let $f(\theta)=\theta+2\sin\frac{1}{2}\theta-2$

$f(1)=-1+2\sin\frac{1}{2}\approx-0\cdot04<0$

$f(2)=2\sin 1\approx 1\cdot 68>0$

Since $f(\theta)$ is continuous,

$f(\theta)=0$ for some $1<\theta<2$.

iii. $f(\theta)=\theta+2\sin\frac{1}{2}\theta-2$

$f'(\theta)=1+\cos\frac{1}{2}\theta$

$\theta_1=\theta_0-\frac{f(\theta_0)}{f'(\theta_0)}$

$\theta_1=1-\frac{-1+2\sin\frac{1}{2}}{1+\cos\frac{1}{2}}$

$\therefore \theta_1\approx 1\cdot 0$ (to 1 dec. place)

c. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes dx in terms of du	1
• writes integrand in terms of u and changes limits to u values	1
• finds primitive function	1
• evaluates in simplest exact form	1

Answer

$x=u^2,\ u\geq 0$

$dx=2u\,du$

$x=1\Rightarrow u=1$

$x=25\Rightarrow u=5$

$\int_1^{25}\frac{1}{x+\sqrt{x}}\,dx=\int_1^5\frac{1}{u(u+1)}.2u\,du$

$=2\left[\ln(u+1)\right]_1^5$

$=2(\ln 6-\ln 2)$

$=2\ln 3$

Question 5

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • shows $P(1)=0$	1
ii • uses product of roots is 1 to deduce 3rd root is reciprocal of α	1
iii • writes sum of squares of roots in terms of square of sum and sum of two-way products	1
• uses relationships between coefficients of polynomial equation and its roots	1

Answer

i. $P(x)=x^3-kx^2+kx-1$

$P(1)=1-k+k-1=0$

ii. Product of the roots is 1.

Hence if the roots are $1,\alpha,\beta$,

then $\alpha\beta=1$. $\therefore\frac{1}{\alpha}$ is the 3rd root.

iii. $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$\therefore \alpha^2+\frac{1}{\alpha^2}+1^2=k^2-2k$

$\therefore \alpha^2+\frac{1}{\alpha^2}=k^2-2k-1$

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • counts the number of codes with all three digits different	1
• divides by the total number of codes to find the probability	1
ii • counts the number of codes with exactly two digits the same	1
• writes the probability of such a code	1

Answer

i. $P(all\ \ different) = \dfrac{9\times8\times7}{9\times9\times9} = \dfrac{56}{81}$

ii. Consider code of form A, A, B or A, B, A or B, A, A
Number of such codes is $9\times8\times3$

$P(exactly\ two\ the\ same) = \dfrac{9\times8\times3}{9\times9\times9} = \dfrac{8}{27}$

c. Outcomes assessed : HE4, HE5

Marking Guidelines

Criteria	Marks
i • finds θ in terms of x	1
ii • derives θ with respect to x	1
• finds the derivative of θ with respect to t in terms of x	1
• states the rate at which θ is changing when $x=20$	1

Answer

i. $\tan\theta = \frac{40}{x}\ ,\ 0<\theta<\frac{\pi}{2}$

$\therefore\ \theta = \tan^{-1}\frac{40}{x}$

ii. $\dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}.\dfrac{dx}{dt}$

$= \dfrac{1}{1+\frac{1600}{x^2}}\cdot\dfrac{-40}{x^2}.5$

$= \dfrac{-200}{x^2+1600}$

$\therefore x=20 \Rightarrow \dfrac{d\theta}{dt} = -\dfrac{1}{10}$

θ is decreasing at a rate of $0\cdot1$ radians per second.

Question 6

a. Outcomes assessed : H3, HE3, HE5

Marking Guidelines

Criteria	Marks
i • finds a in terms of x	1
ii • finds t as a function of x by integration	1
• rearranges to find x as a function of t	1
iii • finds t when $x=0$	1
iv • shows intercepts on the axes	1
• shows asymptote $x=2$	1

Answer

i. $v=2-x$

$a = v\dfrac{dv}{dx}$

$=(2-x).(-1)$

$=x-2$

ii. $\dfrac{dx}{dt}=2-x$

$\dfrac{dt}{dx}=\dfrac{1}{2-x}$

$t=-\ln A(2-x),\ \ A$ constant

$\left.\begin{array}{l} t=0 \\ x=-4\end{array}\right\} \Rightarrow \begin{array}{l} 6A=1 \\ A=\frac{1}{6}\end{array}$

$\therefore\ -t=\ln\left(\dfrac{2-x}{6}\right)$

$e^{-t}=\dfrac{2-x}{6}$

$6\,e^{-t}=2-x$

$\therefore\ \ x=2-6e^{-t}$

iii. When $t=0,\ x=-4\ \ \therefore v>0$

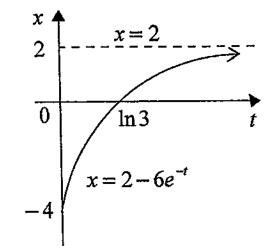
Particle is initially moving right, and it continues moving right approaching $x=2$.

Hence particle has travelled 4 metres from its starting point when $x=0$.

$x=0 \Rightarrow -t=\ln\frac{1}{3}$.

$\therefore$ particle travels first 4 metres in $\ln3$ seconds.

iv.



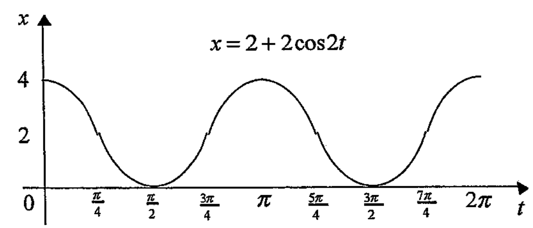
b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • sketches curve with correct shape and position showing intercept on x axis	1
• shows intercepts on t axis for at least one period	1
ii • differentiates to find $\ddot{x}$ as a function of t, and hence as a function of x	1
iii • states the period of the motion	1
iv • finds x when $t=2$	1
• states the distance travelled in the first 2 seonds	1

Answer

i.



ii. $\dot{x}=-4\sin2t$

$\ddot{x}=-4(2\cos2t)$

$=-4(x-2)$

iii. Period is π seconds

iv. $t=2\ \Rightarrow x=2+2\cos4\approx0\cdot69$

But $\frac{\pi}{2}<2<\frac{3\pi}{4}$. Hence by inspection of the graph, particle has travelled $4\cdot7$ m (correct to 2 sig. fig.)

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes expressions for x and y	1
• finds x when $y=0$ and hence required expression for R	1
ii • calculates R for $V=20$ when $\theta=15°$, $\theta=45°$	1
• identifies region that can be watered	1
• finds the area of at least part of this region	1
• finds the total area in simplest exact form	1

Answer

i. $x = Vt\cos\theta \qquad y = -\frac{1}{2}gt^2 + Vt\sin\theta$

$x = R$ when $y = 0$ and $t \neq 0$

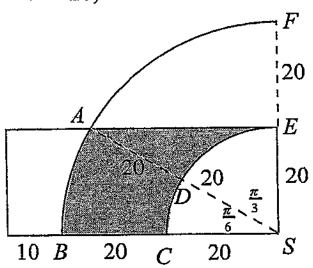
$y=0,\ t\neq 0 \Rightarrow V\sin\theta = \frac{1}{2}gt$

$t = \dfrac{2V\sin\theta}{g}$

$\therefore R = \dfrac{V^2(2\sin\theta\cos\theta)}{g} = \dfrac{V^2\sin 2\theta}{g}$

ii. $V=20,\ \theta=15° \Rightarrow R=20$

$V=20,\ \theta=45° \Rightarrow R=40$



The area of lawn that can be watered is shaded on the diagram.

Since $\cos\angle ESA = \frac{20}{40}$, $\angle ESA = \frac{\pi}{3}$ and $\angle ASB = \frac{\pi}{6}$.

Area = *Sector ABS* + Δ*AES* − *Quadrant ECS*

$= \frac{1}{2}\times 40^2 \times \frac{\pi}{6} + \frac{1}{2}\times 40\times 20\sin\frac{\pi}{3} - \frac{1}{4}\times\pi\times 20^2$

$= 100\times\frac{\pi}{3} + 200\sqrt{3}$

Area is $100\left(\frac{\pi}{3}+2\sqrt{3}\right)$ square metres.

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes binomial expansion	1
ii • substitutes $x=1$ to deduce required result	1
iii • finds primitive of LHS of i.	1
• finds primitive of RHS of i.	1
• evaluates definite integrals of LHS and RHS between limits 0 and 1	1
• uses result from ii. to deduce required result	1

Answer

i.

$(1+x)^n \equiv {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$

ii.

$x=1 \Rightarrow 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$

But ${}^nC_0 = 1 \qquad \therefore \sum_{r=1}^{n} {}^nC_r = 2^n - 1$

iii.

$\left[\frac{1}{n+1}(1+x)^{n+1}\right]_0^1 = \left[{}^nC_0x + {}^nC_1\frac{1}{2}x^2 + \ldots + {}^nC_n\frac{1}{n+1}x^{n+1}\right]_0^1$

$\frac{1}{n+1}\left(2^{n+1}-1\right) = {}^nC_0 + \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 + \ldots + \frac{1}{n+1}{}^nC_n$

$\therefore \dfrac{1}{n+1}\sum_{r=1}^{n+1} {}^{n+1}C_r = \sum_{r=0}^{n} \dfrac{{}^nC_r}{r+1}$ (using ii. with $n \to n+1$)