NSW – INDEPENDENT SCHOOLS

2009
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

Student name / number ....

Question 1 **Begin a new booklet** **Marks**

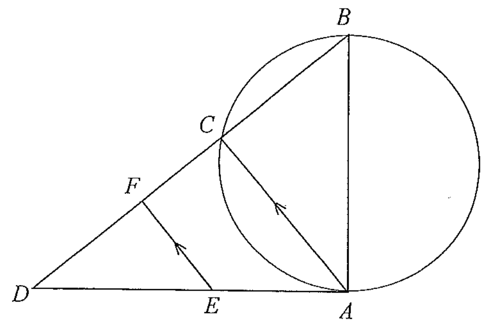
(a) Find $\lim_{x \to 0} \frac{\sin 3x}{2x}$. **2**

(b) Find the limiting sum of the geometric series $\left(\frac{e}{e+1}\right)+\left(\frac{e}{e+1}\right)^2+\left(\frac{e}{e+1}\right)^3+\ldots$. **2**

(c) The equation $x^3+2x^2+3x+6=0$ has roots α, β and γ. Find the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. **2**

(d) Find the acute angle between the lines $y=2x$ and $x+y-3=0$, giving the answer correct to the nearest degree. **2**

(e)



AB is a diameter of the circle and C is a point on the circle. The tangent to the circle at A meets BC produced at D. E is a point on AD and F is a point on CD such that EF is parallel to AC.

(i) Give a reason why $\angle EAC = \angle ABC$. **1**

(ii) Hence show that $EABF$ is a cyclic quadrilateral. **2**

(iii) Show that BE is a diameter of the circle through E, A, B and F. **1**

Student name / number ....

Marks

Question 2 **Begin a new booklet**

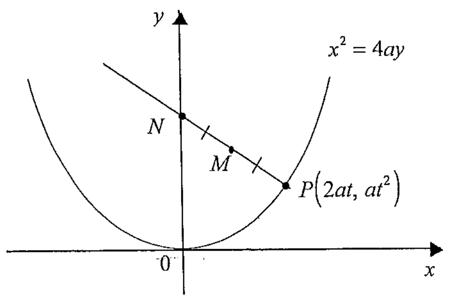
(a) Evaluate $\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx$, giving the answer in simplest exact form. **2**

(b) Find the number of ways in which 3 boys and 3 girls can be arranged in a line so that the two end positions are occupied by boys and no two boys are next to each other. **2**

(c) $A(-2, 3)$ and $B(6, -1)$ are two points. Find the coordinates of the point P that divides the interval AB internally in the ratio 3 : 2. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to show that $\dfrac{\sin x}{1-\cos x} = \cot\frac{x}{2}$. **2**

(e)



$P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$. The normal to the parabola at P cuts the y-axis at N. M is the midpoint of PN.

(i) Use differentiation to show that the normal to the parabola at P has equation $x + ty = 2at + at^3$. **2**

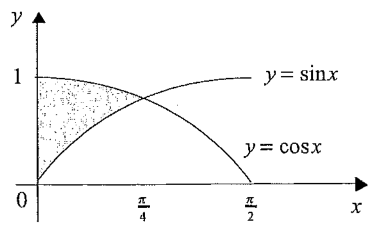
(ii) Find the equation of the locus of M as P moves on the parabola. **2**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a)



The region bounded by the curves $y = \cos x$ and $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution around the x-axis. Find the volume of the solid of revolution. **2**

(b) Use Mathematical Induction to show that for all positive integers $n \geq 2$, **4**

$$2\times1 + 3\times2 + 4\times3 + \ldots + n(n-1) = \frac{n(n^2-1)}{3}.$$

(c) Consider the function $f(x) = (x+2)^2 - 9$, $-2 \leq x \leq 2$.

(i) Find the equation of the inverse function $f^{-1}(x)$. **1**

(ii) On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the coordinates of the endpoints and the intercepts on the coordinate axes. **3**

(iii) Find the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = f^{-1}(x)$, giving the answer in simplest exact form. **2**

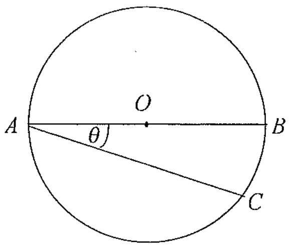
Marks

Question 4 **Begin a new booklet**

(a) Bob chooses six numbers from the numbers 1 to 40 inclusive. A machine then chooses six numbers at random from the numbers 1 to 40 inclusive. Find the probability that none of Bob's numbers match the numbers chosen by the machine, giving the answer correct to 2 decimal places. **2**

(b) Use the substitution $u = \sin^2 x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x}\,dx$, giving the answer in simplest exact form. **4**

(c)



AOB is a diameter of a circle with centre O and radius 1 metre. AC is a chord of the circle such that $\angle BAC = \theta$, where $0 < \theta < \frac{\pi}{2}$. The area of that part of the circle contained between the diameter AB and the chord AC is equal to one quarter of the area of the circle.

(i) Show that $\theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4} = 0$. **2**

(ii) Show that $0 \cdot 4 < \theta < 0 \cdot 5$. **2**

(iii) Use one application of Newton's method with an initial approximation $\theta_0 = 0 \cdot 4$ to find the next approximation to the value of θ, giving your answer correct to 2 decimal places. **2**

Marks

Question 5 **Begin a new booklet**

(a) Consider the function $f(x) = \tan^{-1}(x-1)$.

(i) Sketch the curve $y = f(x)$ showing clearly the equations of any asymptotes and the intercepts on the coordinate axes. **2**

(ii) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 1$. **2**

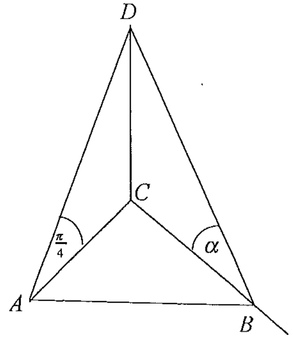
(b) A particle is moving in a straight line. After time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = \sqrt{x}$ and acceleration a ms^{-2}. Initially the particle is 1 metre to the right of O.

(i) Show that a is independent of x. **1**

(ii) Express x in terms of t. **2**

(iii) Find the distance travelled by the particle during the third second of its motion. **1**

(c)



A vertical tower CD of height 15 metres stands with its base C on horizontal ground. A is a point on the ground due South of C such that the angle of elevation of the top D of the tower from A is $\frac{\pi}{4}$ radians. B is a variable point on the ground due East of C such that the angle of elevation of the top D of the tower from B is α radians, where $0 < \alpha < \frac{\pi}{2}$. The value of α is increasing at a constant rate of $0 \cdot 01$ radians per second.

(i) show that $AB = 15\,\text{cosec}\,\alpha$. **2**

(ii) Find the rate at which the length AB is changing when $\alpha = \frac{\pi}{3}$. **2**

Student name / number ....

Question 6 **Begin a new booklet** Marks

(a) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = -12\sin(2t + \frac{\pi}{3})$ and acceleration $\ddot{x}$ ms^{-2}. Initially the particle is 5 metres to the right of O.

(i) Show that $\ddot{x} = -4(x-2)$. **3**

(ii) Find the period and the extremities of the motion. **2**

(iii) Find the time taken by the particle to return to its starting point for the first time. **1**

(b) After t hours, the number of individuals in a population is given by $N = 500 - 400e^{-0\cdot 1t}$.

(i) Sketch the graph of N as a function of t, showing clearly the initial population size and the limiting population size. **2**

(ii) Show that $\dfrac{dN}{dt} = 0\cdot 1(500 - N)$. **1**

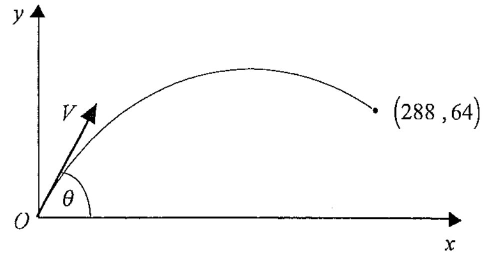
(iii) Find the population size for which the rate of growth of the population is half the initial rate of growth. **1**

(c) If $\cos^{-1} x - \sin^{-1} x = k$, where $-\frac{\pi}{2} \le k \le \frac{3\pi}{2}$, show that $x = \frac{1}{\sqrt{2}}\left(\cos\frac{k}{2} - \sin\frac{k}{2}\right)$. **2**

Student name / number ....

Question 7 **Begin a new booklet** Marks

(a)



A toy rocket is projected from a point O with speed V ms^{-1} at an angle θ above the horizontal, where $0 < \theta < \frac{\pi}{2}$. The rocket moves in a vertical plane under gravity where the acceleration due to gravity is 10 ms^{-2}. After 8 seconds the rocket hits a target at a horizontal distance 288 metres from O and at a height 64 metres above O.

(i) Use integration to show that after time t seconds, the horizontal and vertical displacements of the rocket from O, x metres and y metres respectively, are given by $x = Vt\cos\theta$ and $y = Vt\sin\theta - 5t^2$. **2**

(ii) Find the exact values of V and θ. **3**

(iii) Find the velocity of the rocket just before impact with the target, giving the speed correct to the nearest integer and the angle to the horizontal correct to the nearest degree. **3**

(b)(i) By considering the term in x^r on both sides of the identity $(1+x)^{m+n} = (1+x)^m(1+x)^n$, show that ${}^{m+n}C_r = \sum_{k=0}^{r} {}^mC_k\,{}^nC_{r-k}$, for $0 \le r \le m$ and $0 \le r \le n$. **2**

(ii) Hence show that ${}^{m+1}C_0\,{}^nC_2 + {}^{m+1}C_1\,{}^nC_1 + {}^{m+1}C_2\,{}^nC_0 = {}^mC_0\,{}^{n+1}C_2 + {}^mC_1\,{}^{n+1}C_1 + {}^mC_2\,{}^{n+1}C_0$ for $m \ge 2$ and $n \ge 2$. **2**

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• rearranges in terms of known trigonometric limit	1
• evaluates limit	1

Answer

$$\lim_{x\to 0}\frac{\sin 3x}{2x}=\tfrac{3}{2}\lim_{x\to 0}\frac{\sin 3x}{3x}=\tfrac{3}{2}\times 1=\tfrac{3}{2}$$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• identifies a and r for the G.P	1
• applies formula for limiting sum	1

Answer

$\left(\frac{e}{e+1}\right)+\left(\frac{e}{e+1}\right)^2+\left(\frac{e}{e+1}\right)^3+\dots$ is G.P. with $a=\frac{e}{e+1}$, and $r=\frac{e}{e+1}\Rightarrow 0<r<1$

$\therefore$ Limiting sum is $\frac{a}{1-r}=\frac{e}{e+1}\div\frac{1}{e+1}=e$

c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• expresses sum of reciprocals of roots in terms of sums of products	1
• evaluates using relationships between roots and coefficients	1

Answer

α, β and γ roots of $x^3+2x^2+3x+6=0$. $\quad\therefore \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=\frac{3}{-6}=-\frac{1}{2}$

d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• substitutes values of gradients into formula for tangent of acute angle between the lines	1
• evaluates required angle	1

Answer

Acute angle θ between lines $y=2x$ and $x+y-3=0$ is given by $\tan\theta=\left|\frac{2-(-1)}{1+2.(-1)}\right|=3$

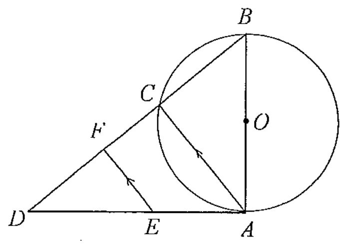
$\therefore \theta\approx 72°$ (to the nearest degree)

e. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • quotes alternate segment theorem	1
ii • gives a sequence of deductions resulting in a test for a cyclic quadrilateral	1
• justifies these deductions by quoting geometric properties and tests	1
iii • explains why BE subtends a right angle at A or at F	1

Answer



Let O be the centre of the circle.

i. The angle between the tangent at A and the chord AC is equal to the angle subtended by that chord in the alternate segment, hence $\angle EAC=\angle ABC$.

ii. $\angle EAC=\angle DEF$ *(Corresp. $\angle$'s with parallel lines AC, EF are equal)*

$\therefore \angle DEF=\angle ABC$ *(Both equal to $\angle EAC$)*

$\therefore EABF$ is cyclic *(Exterior $\angle$ equal to interior opp. $\angle$)*

iii. $\angle BAE=90°$ *(Tangent to circle ABC at A is perpendicular to radius OA drawn to point of contact)*

$\therefore BE$ is a diameter of circle $EABF$. *(subtends right $\angle$ at circumference)*

Question 2

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• finds primitive	1
• evaluates in surd form	1

Answer

$$\int_0^{\frac{\pi}{8}}\sec 2x\tan 2x\,dx=\tfrac{1}{2}\left[\sec 2x\right]_0^{\frac{\pi}{8}}=\tfrac{1}{2}\left(\sqrt{2}-1\right)$$

b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• counts arrangements for one possible pattern of B's and G's	1
• adds number of arrangements for the second possible pattern of B's and G's	1

Answer

B G B G G B or B G G B G B $\quad\therefore 2\times 3!\times 3!=72$ ways

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• finds x coordinate of P	1
• finds y coordinate of P	1

Answer

$A(-2\,,3)$ $\quad B(6\,,-1)$

$3 \quad : \quad 2$

$$P\left(\frac{3\times 6+2\times(-2)}{3+2}\,,\ \frac{3\times(-1)+2\times 3}{3+2}\right)$$

$\therefore$ P has coordinates $P\left(\frac{14}{5}\,,\frac{3}{5}\right)$

d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• simplifies $1-\cos x$ in terms of t	1
• completes simplification of given expression in terms of t to establish required result	1

Answer

$t=\tan\frac{x}{2}$

$$1-\cos x = 1-\frac{1-t^2}{1+t^2} = \frac{2t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\therefore \frac{\sin x}{1-\cos x} = \frac{2t}{1+t^2}\times\frac{1+t^2}{2t^2} = \frac{1}{t} = \cot\frac{x}{2}$$

e. Outcomes assessed : PE3, PE4

Marking Guidelines

Criteria	Marks
i • finds $\frac{dy}{dx}$ as a function of t	1
• finds equation of normal in required form	1
ii • finds coordinates of M	1
• finds equation of locus of M	1

Answer

i.

$$y=at^2 \Rightarrow \frac{dy}{dt}=2at$$

$$x=2at \Rightarrow \frac{dx}{dt}=2a$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=t$$

$\therefore$ Normal at P has gradient $-\frac{1}{t}$ and equation

$$y-at^2=-\tfrac{1}{t}(x-2at)$$

$$ty-at^3=-x+2at$$

$$x+ty=2at+at^3$$

ii. $N\left(0\,,2a+at^2\right)$ $\quad \therefore M\left(at\,,a+at^2\right)$

$P\left(2at\,,at^2\right)$

Locus of M has equation $y=a+a\left(\frac{x}{a}\right)^2$

$$x^2=a(y-a)$$

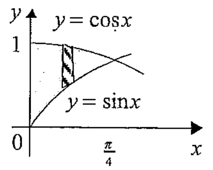
Question 3

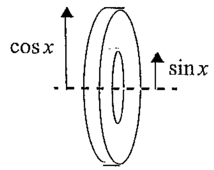
a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• writes definite integral for the volume in terms of $\cos x$ and $\sin x$	1
• evaluates the integral.	1

Answer





$$V=\pi\int_0^{\frac{\pi}{4}}(\cos^2 x-\sin^2 x)\,dx$$

$$=\pi\int_0^{\frac{\pi}{4}}\cos 2x\,dx$$

$$=\tfrac{1}{2}\pi\left[\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$=\tfrac{1}{2}\pi\,(1-0)$$

Volume is $\frac{\pi}{2}$ cubic units.

b. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• defines an appropriate sequence of statements $S(n)$ and shows the first member is true	1
• writes the LHS of $S(k+1)$ in terms of RHS of $S(k)$, conditional on truth of $S(k)$	1
• rearranges conditional expression for LHS of $S(k+1)$ to obtain RHS	1
• completes proof by Mathematical Induction	1

Answer

Let $S(n)$, $n=2,3,4,\ldots$, be the sequence of statements defined by

$$S(n):\quad 2\times1\ +\ 3\times2\ +\ 4\times3\ +\ \ldots\ +n(n-1)\ =\ \frac{n(n^2-1)}{3}$$

Consider $S(2)$: $\quad LHS=2\times1=2$; $\quad RHS=\frac{2(2^2-1)}{3}=2$. $\quad$ Hence $S(2)$ is true.

If $S(k)$ is true: $\quad 2\times1\ +\ 3\times2\ +\ 4\times3\ +\ \ldots\ +k(k-1)\ =\ \frac{k(k^2-1)}{3}\quad *$

Consider $S(k+1)$: $\quad LHS=\left\{2\times1\ +\ 3\times2\ +\ 4\times3\ +\ \ldots\ +k(k-1)\right\}+(k+1)k$

$$=\frac{k(k^2-1)}{3}+(k+1)k \qquad \text{if } S(k) \text{ is true, using } *.$$

$$=\frac{k(k+1)\{(k-1)+3\}}{3}$$

$$=\frac{(k+1)\{k^2+2k\}}{3}$$

$$=\frac{(k+1)\{(k+1)^2-1\}}{3}$$

$$=RHS$$

Hence if $S(k)$ is true then $S(k+1)$ is true. But $S(2)$ is true, and hence $S(3)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n\geq 2$.

c. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i • rearranges and interchanges x and y to obtain equation of inverse function	1
ii • sketches graph of $y = f(x)$ showing endpoints and intercepts	1
• sketches inverse function by reflection in $y = x$	1
• shows endpoints and intercepts for inverse function	1
iii • writes equation for x	1
• solves for x in simplest exact form	1

Answer

i. $f(x) = (x+2)^2 - 9, \quad -2 \le x \le 2.$

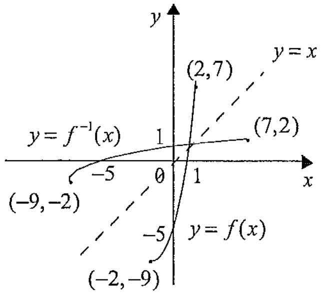
$(x+2)^2 = y+9 \quad$ and $\quad 0 \le x+2 \le 4$

$x + 2 = +\sqrt{y+9}$

$\therefore \quad x = -2 + \sqrt{y+9}\ , \quad -9 \le y \le 7$

$\therefore x \leftrightarrow y \Rightarrow f^{-1}(x) = -2 + \sqrt{x+9}\ , \ -9 \le x \le 7$

ii. Graphs of inverse functions are reflections of each other in $y = x$



iii. Graphs intersect on the line $y = x$.

Hence $(x+2)^2 - 9 = x$

$x^2 + 3x - 5 = 0$

$\therefore x > 0 \Rightarrow x = \dfrac{-3+\sqrt{29}}{2}$

Question 4

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
• writes expression for probability in terms of binomial coefficients	1
• evaluates required probability	1

Answer

$P(\text{none in common}) = \dfrac{{}^{34}C_6}{{}^{40}C_6} \approx 0\cdot35$ (to 2 decimal places)

b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes du in terms of dx and converts limits for x into limits for u	1
• finds equivalent definite integral in terms of u	1
• finds primitive and substitutes limits	1
• simplifies exact answer	1

Answer

$u = \sin^2 x$

$du = 2\sin x \cos x \, dx$

$du = \sin 2x \, dx$

$x = \frac{\pi}{4} \Rightarrow u = \frac{1}{2}$

$x = \frac{\pi}{3} \Rightarrow u = \frac{3}{4}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x}\, dx = \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{1+u}\, du$$

$$= \left[\ln(1+u)\right]_{\frac{1}{2}}^{\frac{3}{4}}$$

$$= \ln\tfrac{7}{4} - \ln\tfrac{3}{2}$$

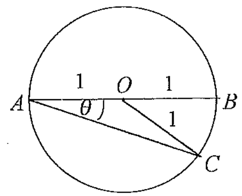
$$= \ln\tfrac{7}{6}$$

c. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • finds area of ΔAOC in terms of $\sin 2\theta$	1
• uses area information to complete equation for θ	1
ii • shows that $f(0\cdot4)$, $f(0\cdot5)$ have opposite signs	1
• notes that f is continuous, and deduces equation has one root θ, $0\cdot4 < \theta < 0\cdot5$	1
iii • applies Newton's rule to write numerical expression for next approximation	1
• evaluates this approximation	1

Answer



i. $\angle OCA = \theta \quad$ (*$\angle$'s opp. equal sides are equal in ΔAOC*)

$\angle AOC = \pi - 2\theta$ (*$\angle$ sum of Δ is π*)

$\angle BOC = 2\theta \quad$ (*adj. supp. $\angle$'s add to π*)

Area sector BOC + Area $\Delta AOC = \frac{1}{4}$ Area circle

$\therefore \frac{1}{2} \times 1^2 \times 2\theta + \frac{1}{2} \times 1^2 \times \sin(\pi - 2\theta) = \frac{1}{4} \times \pi \times 1^2$

$\theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4} = 0$

ii. Let $f(\theta) = \theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4}$

$f(0\cdot4) \approx -0\cdot03 < 0$

$f(0\cdot5) \approx 0\cdot14 > 0$

and f is continuous

Also $f'(\theta) = 1 + \cos 2\theta > 0 \Rightarrow f$ monotonic increasing

$\therefore f(\theta) = 0$ for exactly one value of θ, $0\cdot4 < \theta < 0\cdot5$

iii. Since $f'(\theta) = 1 + \cos 2\theta$,

$\theta \approx 0\cdot4 - \dfrac{f(0\cdot4)}{f'(0\cdot4)}$

$\approx 0\cdot4 - \dfrac{-0\cdot0267}{1\cdot6967}$

$\approx 0\cdot42$ (to 2 dec. pl.)

Question 5

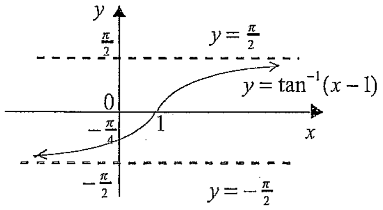
a. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i • shows correct shape and asymptotes	1
• shows intercepts on coordinate axes	1
ii • finds $\dfrac{dy}{dx}$ and evaluates for $x = 1$	1
• finds equation of tangent	1

Answer

i.



ii. $y=\tan^{-1}(x-1)$

$$\frac{dy}{dx}=\frac{1}{1+(x-1)^2}$$

$\therefore \frac{dy}{dx}=1$ when $x=1$

$\therefore$ Tangent at $(1,0)$ has gradient 1 and equation $y=x-1$

b. Outcomes assessed : HE5

Marking Guidelines

Criteria	Marks
i • shows by differentiation that a is constant	1
ii • integrates to find a primitive function for t in terms of x	1
• evaluates constant of integration using initial conditions then writes x as a function of t	1
iii • evaluates x at $t=2$ and $t=3$ to find distance travelled in third second.	1

Answer

i. $v=\sqrt{x} \Rightarrow \frac{1}{2}v^2=\frac{1}{2}x$

$\therefore a=\frac{d}{dx}(\frac{1}{2}v^2)=\frac{1}{2}$ for all x

Hence a is independent of x.

ii. $\frac{dx}{dt}=x^{\frac{1}{2}}$

$\frac{dt}{dx}=x^{-\frac{1}{2}}$

$t=2x^{\frac{1}{2}}+c$

$\left.\begin{array}{l}t=0\\x=1\end{array}\right\} \Rightarrow c=-2$

$\therefore t=2\sqrt{x}-2$

$x=\frac{1}{4}(t+2)^2$

iii. Between $t=2$ and $t=3$, particle moves right from $x=4$ to $x=\frac{25}{4}$

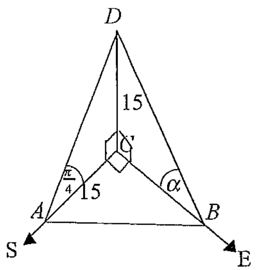
Distance travelled in third second is $2\cdot25$ m.

c. Outcomes assessed : H5, HE5, HE7

Marking Guidelines

Criteria	Marks
i • finds AC and finds BC in terms of $\cot\alpha$	1
• uses Pythagoras' theorem and an appropriate trig. identity to find AB in terms of $\operatorname{cosec}\alpha$	1
ii • differentiates AB with respect to t using chain rule or implicit differentiation	1
• substitutes given values and interprets result	1

Answer



i. In ΔACD,

$\angle DAC=\angle ADC=\frac{\pi}{4}$

$\therefore AC=15$.

In ΔBCD, $BC=15\cot\alpha$.

$\therefore$ In ΔABC,

$AB^2=15^2+15^2\cot^2\alpha$

$=15^2(1+\cot^2\alpha)$

$=15^2\operatorname{cosec}^2\alpha$

$\therefore AB=15\operatorname{cosec}\alpha$

ii. When $\alpha=\frac{\pi}{3}$,

$$\frac{dAB}{dt}=-15\operatorname{cosec}\alpha\cot\alpha\frac{d\alpha}{dt}$$

$=-15\times\frac{2}{\sqrt{3}}\times\frac{1}{\sqrt{3}}\times0\cdot01$

$=-0\cdot1$

$\therefore AB$ is decreasing at a rate of $0\cdot1\ \text{ms}^{-1}$

Question 6

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • integrates v with respect to t to find expression for x	1
• uses initial conditions to evaluate the constant of integration, giving x as a function of t	1
• differentiates v with respect to t to get $\ddot{x}$ then expresses $\ddot{x}$ in terms of x	1
ii • states period	1
• states extremities	1
iii • solves trig. equation to find time to first return	1

Answer

i. $v=-12\sin(2t+\frac{\pi}{3})$

$x=6\cos(2t+\frac{\pi}{3})+c$

$t=0,\ x=5\Rightarrow c=2$

$\therefore x=2+6\cos(2t+\frac{\pi}{3})$

$\ddot{x}=-24\cos(2t+\frac{\pi}{3})$

$\therefore \ddot{x}=-4(x-2)$

ii. Period is π seconds. $-4\le x\le 8$

iii. $x=5\Rightarrow\cos(2t+\frac{\pi}{3})=\frac{1}{2}$

$2t+\frac{\pi}{3}=\frac{\pi}{3},\ 2\pi-\frac{\pi}{3},\ldots$

$t=0,\ \frac{2\pi}{3},\ \ldots$

First return after $\frac{2\pi}{3}$ seconds.

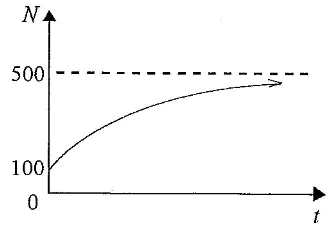
b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • sketches graph of correct shape with correct vertical intercept	1
• shows asymptote for limiting population size	1
ii • differentiates with respect to t	1
iii • writes and solves equation for N	1

Answer

i.



ii.

$N=500-400e^{-0\cdot1t}$

$\frac{dN}{dt}=0\cdot1\times400e^{-0\cdot1t}$

$=0\cdot1(500-N)$

iii.

Initial rate of growth is

$0\cdot1(500-100)=0\cdot1\times400$

$\therefore$ want N such that

$0\cdot1(500-N)=0\cdot1\times200$

$500-N=200$

$N=300$

c. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
• uses inverse trig. identity to simplify equation	1
• uses trig. expansion to evaluate x in terms of k	1

Answer

$\cos^{-1}x-\sin^{-1}x=k,\quad -\frac{\pi}{2}\le k\le\frac{3\pi}{2}$

$\cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}$

$\therefore 2\cos^{-1}x=k+\frac{\pi}{2}$

$\cos^{-1}x=\frac{k}{2}+\frac{\pi}{4}$

$x=\cos(\frac{k}{2}+\frac{\pi}{4})$

$\therefore x=\cos\frac{k}{2}\cos\frac{\pi}{4}-\sin\frac{k}{2}\sin\frac{\pi}{4}$

$=\frac{1}{\sqrt{2}}(\cos\frac{k}{2}-\sin\frac{k}{2})$

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • uses integration to find expressions for $\dot{x}$ and x	1
• uses integration to find expressions for $\dot{y}$ and y	1
ii • writes simultaneous equations for V and θ	1
• finds the value of V	1
• finds the value of θ	1
iii • finds the values of $\dot{x}$ and $\dot{y}$ just before impact	1
• uses Pythagoras' theorem to find the magnitude of v	1
• uses trigonometry to find the direction of v as an angle relative to the horizontal	1

Answer

i.

$$\ddot{x}=0 \qquad \dot{x}=c_1 \qquad \therefore \dot{x}=V\cos\theta$$

$$x=Vt\cos\theta+c_2 \qquad \left.\begin{array}{l}t=0\\x=0\end{array}\right\}\Rightarrow c_2=0 \qquad \therefore x=Vt\cos\theta$$

$$\ddot{y}=-10 \qquad \dot{y}=-10t+c_3 \qquad \left.\begin{array}{l}t=0\\\dot{y}=V\sin\theta\end{array}\right\}\Rightarrow c_3=V\sin\theta \qquad \therefore \dot{y}=-10t+V\sin\theta$$

$$y=-5t^2+Vt\sin\theta+c_4 \qquad \left.\begin{array}{l}t=0\\y=0\end{array}\right\}\Rightarrow c_4=0 \qquad \therefore y=Vt\sin\theta-5t^2$$

ii. When $t=8$

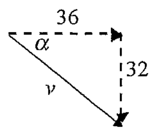
$$\left.\begin{array}{l}x=288\\y=64\end{array}\right\}\Rightarrow \begin{array}{l}8V\cos\theta=288\\8V\sin\theta=384\end{array} \qquad \therefore V^2(\cos^2\theta+\sin^2\theta)=36^2+48^2 \qquad V=60$$

$$\tan\theta=\tfrac{48}{36}=\tfrac{4}{3} \qquad \theta=\tan^{-1}\tfrac{4}{3}$$

iii. When $t=8$

$$\dot{x}=60\times\tfrac{3}{5}=36$$

$$\dot{y}=-80+60\times\tfrac{4}{5}=-32$$



$$v^2=36^2+32^2 \qquad v=4\sqrt{145}$$

$$\tan\alpha=\tfrac{8}{9} \qquad \alpha\approx 41\cdot 6°$$

Velocity of rocket just before impact is approximately 48 ms^{-1} inclined at 42° below the horizontal.

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes a typical term in x^r in the expansion of the RHS of the identity	1
• collects like terms to find coefficient of x^r, then equates to coefficient of x^r on LHS	1
ii • writes single binomial coefficient for sum on LHS	1
• writes single binomial coefficient for sum on RHS then deduces result	1

Answer

i. $(1+x)^{m+n}=(1+x)^m(1+x)^n$

For $0\le r\le m$ and $0\le r\le n$,

terms in x^r in expansion of the RHS have the form ${}^mC_k x^k\times{}^nC_{r-k}x^{r-k}$, $k=0,1,2,\ldots,r$.

Collecting such like terms gives the coefficient of x^r as $\sum_{k=0}^{r}{}^mC_k\,{}^nC_{r-k}$.

The coefficient of x^r in the expansion of the LHS is ${}^{m+n}C_r$.

Hence equating coefficients of x^r on both sides of the identity gives ${}^{m+n}C_r=\sum_{k=0}^{r}{}^mC_k\,{}^nC_{r-k}$.

ii. Using i., for $m\ge 2$ and $n\ge 2$,

$${}^{m+1}C_0\,{}^nC_2+{}^{m+1}C_1\,{}^nC_1+{}^{m+1}C_2\,{}^nC_0={}^{(m+1)+n}C_2 \quad\text{and}\quad {}^mC_0\,{}^{n+1}C_2+{}^mC_1\,{}^{n+1}C_1+{}^mC_2\,{}^{n+1}C_0={}^{m+(n+1)}C_2$$

$$\therefore {}^{m+1}C_0\,{}^nC_2+{}^{m+1}C_1\,{}^nC_1+{}^{m+1}C_2\,{}^nC_0={}^mC_0\,{}^{n+1}C_2+{}^mC_1\,{}^{n+1}C_1+{}^mC_2\,{}^{n+1}C_0={}^{m+n+1}C_2$$