

NSW INDEPENDENT SCHOOLS

2007
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- Write using black or blue pen
- Write your student number and/or name at the top of every page
- All necessary working should be shown in every question
- A table of standard integrals is provided

Total marks – 120

Attempt Questions 1 – 8

All questions are of equal value

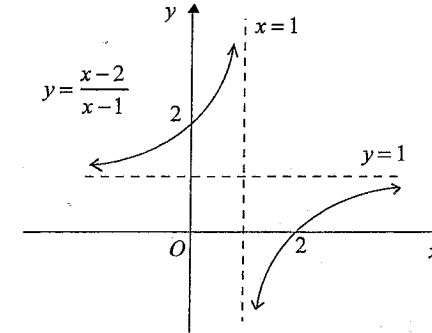
This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME.....

Question 1

Begin a new booklet

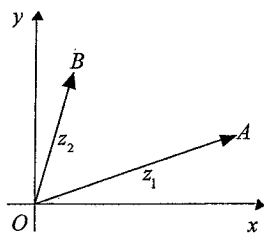
- (a) The diagram below shows the graph of the function $f(x) = \frac{x-2}{x-1}$.



On separate diagrams, sketch the following graphs, showing clearly any intercepts on the axes and the equations of any asymptotes:

- (i) $y = f(-x)$. 1
- (ii) $y = |f(x)|$. 1
- (iii) $y = f(|x|)$. 2
- (iv) $y = e^{f(x)}$. 2
- (b) The line $y = mx$ through the origin $O(0, 0)$ is tangent to the curve $y = \frac{x-2}{x-1}$, touching it at the point $P(x_1, y_1)$.
- (i) By considering the gradient of OP in two different ways, show that $x_1^2 - 4x_1 + 2 = 0$. 2
- (ii) Hence find the two possible values of m . 2
- (c) Consider the function $y = e^{-2x} \tan x$ for $0 \leq x < \frac{\pi}{2}$.
- (i) Show that $\frac{dy}{dx} = e^{-2x}(1 - \tan x)^2$. 2
- (ii) Sketch the graph of the function showing the coordinates of the endpoint, the equation of the asymptote and the coordinates of the stationary point. 3

Question 2	Begin a new booklet	Marks
(a)(i)	Find $\int \frac{1+e^x}{1+e^{-x}} dx$.	1
(ii)	Find $\int \frac{1}{\sqrt{1+x}+\sqrt{x}} dx$.	2
(b)	Use the substitution $x = \sin \theta$ to find $\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx$.	3
(c)	Evaluate $\int_0^{\sqrt{3}} \frac{x^3 - 8x^2 + 9x}{(1+x^2)(9+x^2)} dx$.	4
(d)(i)	Use the substitution $t = \tan \frac{x}{2}$ to show that $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx = 1$.	2
(ii)	Show that $\int_0^a f(x) dx = \int_0^{\frac{a}{2}} \{f(x) + f(a-x)\} dx$.	2
(iii)	Hence evaluate $\int_0^{\pi} \frac{x}{1+\sin x} dx$.	1

Question 3	Begin a new booklet	Marks
(a)(i)	Write down the expansion of $(1+ia)^4$ in ascending powers of a .	1
(ii)	Hence find the values of a such that $(1+ia)^4$ is real.	2
(b)	The equation $(\sin^2 \theta) z^2 - (\sin 2\theta) z + 1 = 0$, where $0 < \theta < \frac{\pi}{2}$, has roots α and β .	
(i)	Show that the roots of the equation are $\cot \theta + i$ and $\cot \theta - i$.	2
(ii)	Hence show that $\alpha^n + \beta^n = \frac{2 \cos n\theta}{\sin^n \theta}$.	2
(c)	In the Argand diagram below vectors \overline{OA} and \overline{OB} represent the complex numbers $z_1 = 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$ and $z_2 = \sqrt{2}\left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}\right)$ respectively.	
		
(i)	Show that $ z_2 - z_1 = \sqrt{2}$.	2
(ii)	Show that $z_2 - z_1 = i z_2$.	2
(d)(i)	On an Argand diagram shade the region where both $ z-2 \leq 1$ and $\text{Re}(z) \leq \frac{3}{2}$.	2
(ii)	Find the set of values of $\text{Arg } z$ for points in the shaded region.	2

Question 4 **Begin a new booklet**

(a)(i) On the same diagram sketch the graphs of the ellipses $E_1: \frac{x^2}{4} + \frac{y^2}{3} = 1$ 4

and $E_2: \frac{x^2}{16} + \frac{y^2}{12} = 1$, showing clearly the intercepts on the axes. Show the coordinates of the foci and the equations of the directrices of the ellipse E_1 .

(ii) $P(2\cos p, \sqrt{3}\sin p)$, where $0 < p < \frac{\pi}{2}$, is a point on the ellipse E_1 . Use differentiation to show that the tangent to the ellipse E_1 at P has equation $\frac{x \cos p}{2} + \frac{y \sin p}{\sqrt{3}} = 1$. 3

(iii) The tangent to the ellipse E_1 at P meets the ellipse E_2 at the points $Q(4\cos q, 2\sqrt{3}\sin q)$ and $R(4\cos r, 2\sqrt{3}\sin r)$, where $-\pi < q < \pi$ and $-\pi < r < \pi$. Show that q and r differ by $\frac{2\pi}{3}$. 2

(b) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with eccentricity e , has one focus S on the positive x -axis and the corresponding directrix d cuts the asymptotes to the hyperbola at points P and Q in the first and fourth quadrants respectively.

(i) Show that PS is perpendicular to the asymptote through P and that $PS = b$. 3

(ii) A circle with centre S touches the asymptotes of the hyperbola. Deduce that the points of contact are the points P and Q . 1

(iii) The circle with centre S which touches the asymptotes of the hyperbola cuts the hyperbola at points R and T . If $b = a$, show that RT is a diameter of the circle. 2

Question 5 **Begin a new booklet**

(a) When the polynomial $P(x)$ is divided by $(x^2 + 1)$ the remainder is $Ax + B$.

(i) Show that $A = \frac{P(i) - P(-i)}{2i}$ and $B = \frac{P(i) + P(-i)}{2}$. 2

(ii) If $P(x)$ is odd, find the remainder when $P(x)$ is divided by $(x^2 + 1)$. 2

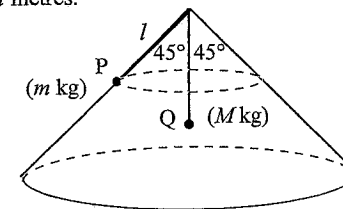
(b) The equation $x^4 - 5x + 2 = 0$ has roots α, β, γ and δ .

(i) Show that the equation $x^4 - 5x + 2 = 0$ has a real root between $x = 0$ and $x = 1$. 1

(ii) Find the monic equation with roots $\alpha^2, \beta^2, \gamma^2$ and δ^2 . Hence or otherwise show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$. 2

(iii) Find the number of non-real roots of $x^4 - 5x + 2 = 0$, giving full reasons for your answer. 2

(c) A right circular inverted cone has semi-vertical angle 45° . There is a smooth hole in the top of the cone and a light, inextensible string passes through this hole. A particle Q of mass M kg is attached to one end of this string and hangs at rest inside the cone. A second particle P of mass m kg, attached to the other end of the string, travels in a horizontal circle around the smooth outside surface of the cone with constant angular velocity ω radians per second. The length of string between P and the hole is l metres.



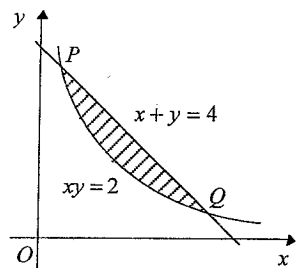
(i) If the tension in the string is T Newtons, the force exerted by the surface on P is N Newtons, and the acceleration due to gravity is g ms⁻², explain why $T + N = \sqrt{2}mg$ and $T - N = ml\omega^2$. 2

(ii) Find expressions for the product $l\omega^2$ and N in terms of M, m and g . 2

(iii) Deduce that $\frac{\sqrt{2}}{2} \leq \frac{M}{m} \leq \sqrt{2}$. 2

Question 6 **Begin a new booklet**

(a)



The region in the first quadrant bounded by the line $x + y = 4$ and the rectangular hyperbola $xy = 2$ is rotated through one complete revolution about the y -axis.

- (i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by

$$V = 2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} (4x - x^2 - 2) dx.$$

- (ii) Hence find the simplest exact numerical value of the volume of the solid formed.

(b) $I_n = \int_0^1 x^n(1-x)^n dx$, $n = 0, 1, 2, \dots$

- (i) Using the substitution $u = \frac{1}{2} - x$, show that $I_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{4} - u^2)^n du$ and hence

$$I_n = \frac{n}{2(2n+1)} I_{n-1}, \quad n = 1, 2, 3, \dots$$

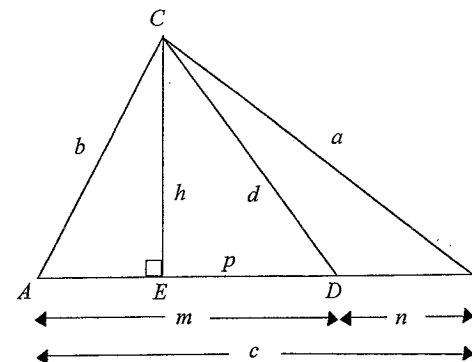
- (ii) Show that $\int_0^1 x^5(1-x)^5 dx = \frac{(5!)^2}{11!}$.

- (iii) Use the substitution $x = \sin^2(\frac{1}{2}t)$ to show that $\int_0^\pi \sin^{2n+1} t dt = 2^{2n+1} I_n$.

Hence deduce that $\int_0^\pi \sin^{2n+1} t dt = \frac{2^{2n+1} (n!)^2}{(2n+1)!}$, $n = 0, 1, 2, \dots$

Question 7 **Begin a new booklet**

(a)



In $\triangle ABC$ above, $BC = a$, $CA = b$ and $AB = c$. D is a point on AB such that $DA = m$, $DB = n$ and $DC = d$. CE is an altitude of the triangle with $CE = h$ and $ED = p$.

- (i) Use Pythagoras' theorem in $\triangle CEA$ and $\triangle CED$ to show that $b^2 = d^2 + m^2 - 2mp$. 2
- (ii) Show similarly that $a^2 = d^2 + n^2 + 2np$. 2
- (iii) Hence show that $a^2m + b^2n = c(d^2 + mn)$. 1
- (iv) In the case where CD bisects $\angle BCA$, use the sine rule in $\triangle CDA$ and $\triangle CDB$ to show that $am = bn$. Hence show that in this case, $d^2 = ab - mn$. 4

- (b) In any single play of a game, n people throw a fair coin, where $n \geq 3$. The play of the game results in an 'odd one out' if all but one of the coins show heads or all but one of the coins show tails.

- (i) Show that in any single play of the game, the probability of an 'odd one out' is $\frac{n}{2^{n-1}}$. 2
- (ii) Find the probability that there is at least one 'odd one out' in N plays of the game. 1
- (iii) Find the probability that the first 'odd one out' occurs on the N^{th} play of the game. 1
- (iv) Find the probability that the second 'odd one out' occurs on the N^{th} play of the game. 2

Marks

Question 8 **Begin a new booklet**

- (a) A sequence of numbers x_n , $n = 1, 2, 3, \dots$ is given by $x_1 = 1$ and
- $$x_{n+1} = \frac{2x_n^3 + 8}{3x_n^2}, \quad n = 1, 2, 3, \dots$$
- (i) Use Mathematical Induction to show that $x_n > 2$ for all positive integers $n \geq 2$. 3
- (ii) Hence show that $x_{n+1} < x_n$ for all positive integers $n \geq 2$. 2
- (b) Let $f(x) = \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} \right)$ where n is a positive integer. 2
- (i) Show that $f(x)$ is stationary at $x = 0$, and show that for $x > 0$, $f(x)$ is monotonic increasing if n is even, or $f(x)$ is monotonic decreasing if n is odd. 2
- (ii) Hence show that if n is a positive integer, then for all $x > 0$,
- $$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \frac{x^{2n}}{2n} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{x^{2n-1}}{2n-1}$$
- (iii) Hence find $\ln(1.2)$ correct to 2 decimal places. 1
- (c) The number e is given by the value of the limiting sum
- $$e = \sum_{r=0}^{\infty} \frac{1}{r!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$
- (i) If n is a positive integer, and $a = n! \left\{ e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right) \right\}$, show 3
- that $0 < a < \frac{1}{n}$.
- (ii) Hence deduce that e is irrational. 2

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Independent Trial HSC 2007 Mathematics Extension 2 Marking Guidelines

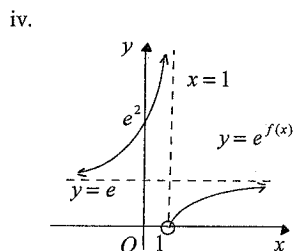
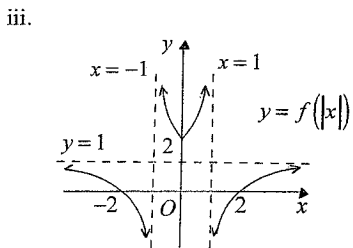
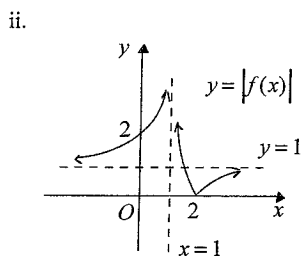
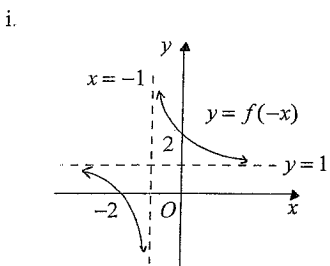
Question 1

a. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
i • sketches reflection in the y -axis showing intercepts on the axes and asymptotes	1
ii • reflects section of curve lying below the x -axis in x -axis, retaining asymptotes and intercepts	1
iii • sketches $y = f(x)$, $x \geq 0$ and its reflection in the y -axis	1
• shows all intercepts and asymptotes	1
iv • sketches left hand branch of curve showing asymptotes and intercept on y -axis	1
• sketches right hand branch of curve showing asymptote and nature of curve near $(1, 0)$	1

Answer



b. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
i • uses differentiation to find one expression for gradient of tangent OP	1
• uses coordinates of O and P to find a second expression for gradient of interval OP	1
ii • solves the quadratic equation for x_1	1
• substitutes for x_1 to find the two values for m	1

Answer

i. $y = \frac{x-2}{x-1}$
 $y = 1 - \frac{1}{x-1}$
 $\frac{dy}{dx} = \frac{1}{(x-1)^2}$

Since OP is tangent to the parabola at $P(x_1, y_1)$, gradient of OP is $\frac{1}{(x_1-1)^2}$.

Hence $\frac{1}{(x_1-1)^2} = \frac{x_1-2}{x_1(x_1-1)}$
 $(x_1-1)(x_1-2) = x_1$
 $x_1^2 - 4x_1 + 2 = 0$

Also gradient of OP is $\frac{y_1}{x_1} = \frac{x_1-2}{x_1(x_1-1)}$.

ii. $x_1^2 - 4x_1 + 2 = 0$
 $(x_1 - 2)^2 = 2$
 $x_1 - 2 = \pm\sqrt{2}$
 $x_1 - 1 = 1 \pm \sqrt{2}$

$\therefore m = \frac{1}{(1+\sqrt{2})^2} = \frac{(1-\sqrt{2})^2}{1^2} = 3 - 2\sqrt{2}$
 or $m = \frac{1}{(1-\sqrt{2})^2} = \frac{(1+\sqrt{2})^2}{1^2} = 3 + 2\sqrt{2}$

c. Outcomes assessed : E6

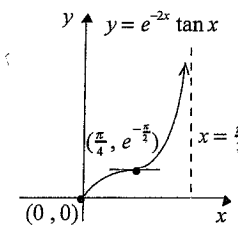
Marking Guidelines

Criteria	Marks
i • applies product rule	1
• uses trigonometric identity to simplify derivative	1
ii • sketches a rising curve with endpoint at $(0, 0)$	1
• shows vertical asymptote at $x = \frac{\pi}{2}$	1
• shows coordinates of point of horizontal inflexion	1

Answer

i. $y = e^{-2x} \tan x$, $0 \leq x < \frac{\pi}{2}$
 $\frac{dy}{dx} = -2e^{-2x} \tan x + e^{-2x} \sec^2 x$
 $= e^{-2x} \{-2 \tan x + (1 + \tan^2 x)\}$
 $= e^{-2x} (1 - \tan x)^2$

ii. $\frac{dy}{dx} = 0$ for $x = \frac{\pi}{4}$ and $\frac{dy}{dx} > 0$ for all other x values in the domain. Hence $(\frac{\pi}{4}, e^{-\frac{\pi}{2}})$ is a point of horizontal inflexion on a rising curve.



Question 2

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • rearranges integrand and finds primitive	1
ii • rationalises denominator	1
• finds primitive	1

Answer

i.
$$\int \frac{1+e^x}{1+e^{-x}} dx = \int \frac{e^x(e^{-x}+1)}{1+e^{-x}} dx$$

$$= \int e^x dx$$

$$= e^x + c$$

ii.
$$\int \frac{1}{\sqrt{1+x}+\sqrt{x}} dx = \int \frac{\sqrt{1+x}-\sqrt{x}}{(\sqrt{1+x}+\sqrt{x})(\sqrt{1+x}-\sqrt{x})} dx$$

$$= \int \frac{\sqrt{1+x}-\sqrt{x}}{(1+x)-x} dx$$

$$= \frac{2}{3}(1+x)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} + c$$

$$= \frac{2}{3}[(1+x)\sqrt{1+x} - x\sqrt{x}] + c$$

b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes dx in terms of $d\theta$ simplifies integrand in terms of θ	1
• finds primitive in terms of θ	1
• finds primitive in terms of x	1

Answer

$x = \sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$dx = \cos\theta d\theta$

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx = \int \frac{1}{(\cos^2\theta)^{\frac{3}{2}}} \cos\theta d\theta = \int \sec^2\theta d\theta$$

$$= \tan\theta + c$$

$$= \frac{x}{\sqrt{1-x^2}} + c$$

c. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• expresses integrand as a sum of partial fractions	1
• finds logarithm part of primitive	1
• finds both inverse tan parts of primitive	1
• evaluates by substitution of limits	1

Answer

$$\frac{x^3 - 8x^2 + 9x}{(1+x^2)(9+x^2)} \equiv \frac{ax+b}{1+x^2} + \frac{cx+d}{9+x^2}$$

$$x^3 - 8x^2 + 9x \equiv (ax+b)(9+x^2) + (cx+d)(1+x^2)$$

Equating coefficients of x^3 : $a+c=1$
 Equating coefficients of x^2 : $b+d=-8$
 Equating coefficients of x : $9a+c=9$
 Putting $x=0$: $9b+d=0$
 $\therefore a=1, c=0$
 $\therefore b=1, d=-9$

$$\therefore \frac{x^3 - 8x^2 + 9x}{(1+x^2)(9+x^2)} \equiv \frac{x+1}{1+x^2} + \frac{-9}{9+x^2}$$

$$\int_0^{\sqrt{3}} \frac{x^3 - 8x^2 + 9x}{(1+x^2)(9+x^2)} dx = \int_0^{\sqrt{3}} \left\{ \frac{1}{2} \left(\frac{2x}{1+x^2} \right) + \frac{1}{1+x^2} - 3 \left(\frac{3}{9+x^2} \right) \right\} dx$$

$$= \left[\frac{1}{2} \ln(1+x^2) + \tan^{-1} x - 3 \tan^{-1} \frac{x}{3} \right]_0^{\sqrt{3}}$$

$$= \frac{1}{2} (\ln 4 - \ln 1) + (\tan^{-1} \sqrt{3} - \tan^{-1} 0) - 3(\tan^{-1} \frac{1}{\sqrt{3}} - \tan^{-1} 0)$$

$$= \ln 2 + \frac{\pi}{3} - 3 \times \frac{\pi}{6}$$

$$= \ln 2 - \frac{\pi}{6}$$

d. Outcomes assessed : HE6, E8

Marking Guidelines

Criteria	Marks
i • converts dx to dt , x limits to t limits and writes $\sin x$ in terms of t	1
• finds primitive in terms of t and substitutes limits	1
ii • converts integral of $f(x)$ between $\frac{1}{2}a$ and a into integral of $f(a-x)$ between 0 and $\frac{1}{2}a$	1
• completes proof of required result	1
iii • applies result to evaluate given definite integral	1

Answer

i. $x=0 \Rightarrow t=0$
 $x=\frac{\pi}{2} \Rightarrow t=1$

$t = \tan \frac{x}{2}$
 $dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$
 $2 dt = (1+t^2) dx$
 $dx = \frac{2}{1+t^2} dt$

$1 + \sin x = 1 + \frac{2t}{1+t^2} = \frac{1+t^2+2t}{1+t^2} = \frac{(1+t)^2}{1+t^2}$
 $\frac{1}{1+\sin x} = \frac{1+t^2}{(1+t)^2}$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx = \int_0^1 \frac{1+t^2}{(1+t)^2} \cdot \frac{2}{1+t^2} dt$$

$$= 2 \left[\frac{1}{1+t} \right]_0^1$$

$$= 2 \left(\frac{1}{2} - 1 \right)$$

$$= 1$$

ii.

$u = a - x$
 $du = -dx$

$\int_{\frac{a}{2}}^a f(x) dx = \int_{\frac{a}{2}}^0 f(a-u) \cdot -du$
 $x = \frac{a}{2} \Rightarrow u = \frac{a}{2}$
 $x = a \Rightarrow u = 0$

$\therefore \int_0^a f(x) dx = \int_0^{\frac{a}{2}} f(x) dx + \int_{\frac{a}{2}}^a f(x) dx$

$$= \int_0^{\frac{a}{2}} f(a-u) du + \int_0^{\frac{a}{2}} \{f(x) + f(a-x)\} dx$$

$$= \int_0^{\frac{a}{2}} f(a-x) dx$$

iii.
$$\int_0^{\pi} \frac{x}{1+\sin x} dx = \int_0^{\frac{\pi}{2}} \left\{ \frac{x}{1+\sin x} + \frac{\pi-x}{1+\sin(\pi-x)} \right\} dx = \int_0^{\frac{\pi}{2}} \left\{ \frac{x}{1+\sin x} + \frac{\pi-x}{1+\sin x} \right\} dx$$

$$\therefore \int_0^{\pi} \frac{x}{1+\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\pi}{1+\sin x} dx = \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx = \pi$$

Question 3

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes expansion, simplifying powers of i	1
ii • writes equation for a	1
• writes three solutions for a	1

Answer

i. $(1 + ia)^4 = 1 + 4ia - 6a^2 - 4ia^3 + a^4$

ii. $(1 + ia)^4$ is real if $4a - 4a^3 = 0$. Then $a(1 - a^2) = 0$. $\therefore a = 0, 1, -1$

b. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • uses trigonometric identities to complete the square or find the discriminant	1
• solves for z using the completed square or the quadratic formula	1
ii • uses de Moivre's theorem to write an expression for $(\cot \theta + i)^n$	1
• writes a similar expression for $(\cot \theta - i)^n$ to obtain the required result by addition	1

Answer

i. $(\sin^2 \theta) z^2 - (\sin 2\theta) z + 1 = 0, 0 < \theta < \frac{\pi}{2}$

$(\sin^2 \theta) z^2 - 2(\sin \theta \cos \theta) z + \cos^2 \theta = \cos^2 \theta - 1$

$\{(\sin \theta) z - \cos \theta\}^2 = -\sin^2 \theta$

$(\sin \theta) z - \cos \theta = \pm i \sin \theta$

$\therefore z - \frac{\cos \theta}{\sin \theta} = \pm i$

$z = \cot \theta \pm i$

\therefore equation has roots $\cot \theta + i, \cot \theta - i$

ii. $\alpha = \cot \theta + i = \frac{1}{\sin \theta} (\cos \theta + i \sin \theta) \Rightarrow \alpha^n = \frac{1}{\sin^n \theta} (\cos n\theta + i \sin n\theta)$ (by De Moivre's theorem)

Then $\beta = \cot \theta - i = \bar{\alpha} \Rightarrow \beta^n = \overline{\alpha^n} = \frac{1}{\sin^n \theta} (\cos n\theta - i \sin n\theta)$

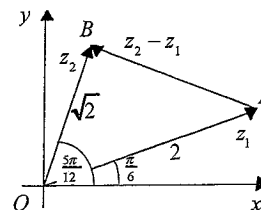
Hence $\alpha^n + \beta^n = \frac{2 \cos n\theta}{\sin^n \theta}$

c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes an expression for the square of AB using the cosine rule in triangle AOB	1
• deduces the value of $ z_2 - z_1 $	1
ii • shows that $\angle OBA = \frac{\pi}{2}$	1
• uses rotation of vectors to deduce required result.	1

Answer



i. In $\triangle AOB$, $AB^2 = 2 + 4 - 2 \times \sqrt{2} \times 2 \cos \frac{\pi}{4} = 2$

$\therefore |z_2 - z_1| = AB = \sqrt{2}$

ii. $\triangle AOB$ is isosceles, since $OB = AB = \sqrt{2}$

$\therefore \angle OAB = \angle AOB = \frac{\pi}{4}$, and hence $\angle OBA = \frac{\pi}{2}$.

$\therefore \overline{AB}$ is an anticlockwise rotation of \overline{OB} by $\frac{\pi}{2}$.

But \overline{AB} represents $z_2 - z_1$. $\therefore z_2 - z_1 = i z_2$

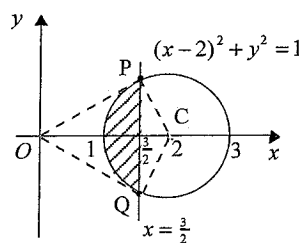
d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • shades a region inside the appropriate circle	1
• shades region that also lies to the left of appropriate vertical line	1
ii • shows that tangents to circle from O have points of contact that lie in the shaded region	1
• deduces set of values of $\text{Arg } z$ from the angles of inclination of these tangents	1

Answer

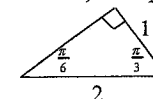
i.



ii.

$\cos \angle OCP = \frac{(\frac{1}{2})}{1} = \frac{1}{2} \therefore \angle OCP = \frac{\pi}{3}$

Hence $\triangle OCP, \triangle OCQ$ are congruent SAS to



Hence OP and OQ are tangents to the circle and P, Q represent z with max, min values of $\text{Arg } z$.

The set of values of $\text{Arg } z$ is $\{\theta : -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}\}$.

Question 4

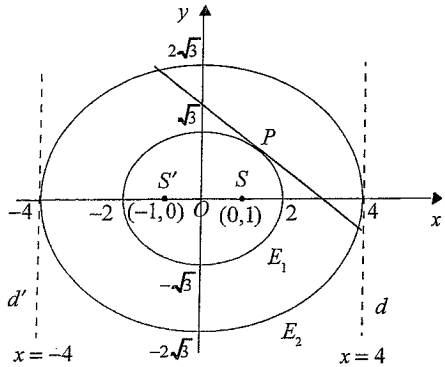
a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • sketches E_1 with correct intercepts on axes	1
• sketches E_2 with correct intercepts on axes	1
• shows foci for E_1	1
• shows directrices for E_1	1
ii • finds gradient of tangent by differentiation	1
• writes expression for equation of tangent	1
• uses trig. identity to simplify this equation into required form	1
iii • writes equation for parameter t for point on E_2 where tangent to E_1 at P cuts E_2	1
• writes parameters at Q, R in terms of p to deduce result	1

Answer

i.



ii.

$$x = 2 \cos p \quad y = \sqrt{3} \sin p$$

$$\frac{dx}{dp} = -2 \sin p \quad \frac{dy}{dp} = \sqrt{3} \cos p$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{3} \cos p}{-2 \sin p}$$

Tangent at P has gradient $-\frac{\sqrt{3} \cos p}{2 \sin p}$

and equation $(\sqrt{3} \cos p)x + (2 \sin p)y = k$ for some constant k .

P on tangent $\Rightarrow 2\sqrt{3} \cos^2 p + 2\sqrt{3} \sin^2 p = k$

$$\therefore k = 2\sqrt{3}$$

\therefore Tangent at P is $\frac{x \cos p}{2} + \frac{y \sin p}{\sqrt{3}} = 1$

iii. Tangent to E_1 at P meets E_2 at point $(4 \cos t, 2\sqrt{3} \sin t)$ where

$$\frac{4 \cos t \cos p}{2} + \frac{2\sqrt{3} \sin t \sin p}{\sqrt{3}} = 1$$

$$2(\cos t \cos p + \sin t \sin p) = 1$$

$$\cos(t - p) = \frac{1}{2}$$

Also $0 < p < \frac{\pi}{2}$ and $-\pi < t < \pi \quad \therefore t - p = \pm \frac{\pi}{3}$

Hence one of Q, R has parameter $p + \frac{\pi}{3}$, and the other has parameter $p - \frac{\pi}{3}$.

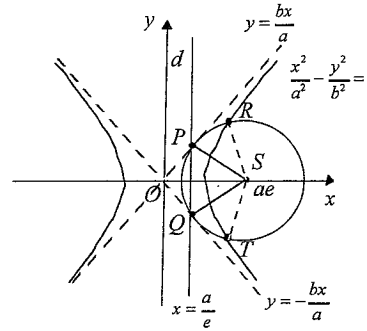
Hence q and r differ by $\frac{2\pi}{3}$.

b. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • finds the gradient of PS in terms of a, b, e	1
• uses the relationship between a, b, e to deduce that PS and OP are perpendicular	1
• uses either distance formula to show that $PS = b$	1
ii • uses a geometric argument to deduce required result	1
iii • finds the x coordinates of R, T in terms of a, b, e	1
• shows that if $a = b$ then R, S, T each have x coordinate $a\sqrt{2}$ to deduce required result	1

Answer



Also $PS^2 = (\frac{a}{e} - ae)^2 + (\frac{b}{e})^2$

$$= \frac{1}{e^2} \{ a^2(1 - e^2)^2 + b^2 \}$$

$$= \frac{1}{e^2} \{ -b^2(1 - e^2) + b^2 \}$$

$$= b^2$$

$\therefore PS = b$

ii. Since the perpendicular distance of S from the asymptote OP is b , the circle with centre S that touches this asymptote has radius b and point of contact P , since $PS = b$ and $PS \perp OP$. By symmetry, $QS \perp OQ$ and $QS = b$. Hence Q is the point of contact of this same circle (centre S , radius b) with the asymptote OQ .

iii. Since $SR = ST = b$, at R, T the locus definition of the hyperbola gives $b = e(x - \frac{a}{e}) \quad \therefore x = \frac{a+b}{e}$

If $a = b, e^2 = \frac{b^2}{a^2} + 1 = 2$.

Then S has x coordinate $a\sqrt{2}$, and at R and T , $x = 2\frac{a}{e} = a\sqrt{2}$. Hence if $a = b$, R, S and T are collinear and RT is the diameter of the circle.

i. S has coordinates $(ae, 0)$ and d has equation $x = \frac{a}{e}$. At P ,

$x = \frac{a}{e}, y = \frac{b}{a}x \quad \therefore P(\frac{a}{e}, \frac{b}{e})$ and

gradient $PS = \frac{(\frac{b}{e})}{(\frac{a}{e} - ae)} = \frac{b}{a(1 - e^2)}$

\therefore product of gradients of PS and OP

is $\frac{b^2}{a^2(1 - e^2)} = \frac{b^2}{-b^2} = -1$.

Hence $PS \perp OP$.

Question 5

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • writes division transformation and expressions for $P(i), P(-i)$ in terms of A and B	1
• solves simultaneous equations to find A and B in terms of $P(i)$ and $P(-i)$	1
ii • deduces B is zero	1
• deduces value of A and hence writes down remainder	1

Answer

i. $P(x) \equiv (x^2 + 1) \cdot Q(x) + Ax + B$ for some polynomial $Q(x)$.

$P(i) = 0 \cdot Q(i) + Ai + B \Rightarrow Ai + B = P(i) \quad (1)$

$P(-i) = 0 \cdot Q(-i) - Ai + B \Rightarrow -Ai + B = P(-i) \quad (2)$

$(1) - (2) \Rightarrow 2Ai = P(i) - P(-i) \quad \therefore A = \frac{P(i) - P(-i)}{2i}$

$(1) + (2) \Rightarrow 2B = P(i) + P(-i) \quad \therefore B = \frac{P(i) + P(-i)}{2}$

ii. $P(x)$ odd $\Rightarrow P(-i) = -P(i) \quad \therefore A = \frac{P(i)}{i}$ and $B = 0$. Hence remainder is $\frac{P(i)}{i}x$

b. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • notes that expression is continuous and shows it changes sign between $x=0$ and $x=1$	1
ii • finds required monic polynomial equation	1
• deduces the value of the sum of squares of $\alpha, \beta, \gamma, \delta$ from the coefficient of x^3	1
iii • deduces that not all the roots are real	1
• explains why there are exactly two non-real roots	1

Answer

i. Let $f(x) = x^4 - 5x + 2$

Then $f(0) = 2 > 0$ and $f(1) = -2 < 0$.

Since f is continuous, $f(x) = 0$ for some real x such that $0 < x < 1$.

Hence $x^4 - 5x + 2 = 0$ has a real root between $x = 0$ and $x = 1$.

ii. $\alpha^2, \beta^2, \gamma^2, \delta^2$ are roots of $(x^2)^4 - 5(x^2) + 2 = 0$

$$x^2 + 2 = 5x^{\frac{1}{2}}$$

$$(x^2 + 2)^2 = 25x$$

$$x^4 + 4x^2 - 25x + 4 = 0$$

Since for this equation the coefficient of x^3 is 0, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$.

iii. Let α be the real root which satisfies $0 < \alpha < 1$.

Then $\beta^2 + \gamma^2 + \delta^2 = -\alpha^2 < 0$. Hence not all of β, γ, δ are real.

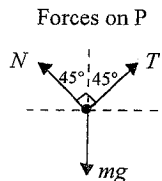
Since the coefficients of $x^4 - 5x + 2 = 0$ are real, the complex conjugate of any non-real root is also a root of the equation. Hence two of β, γ, δ are non-real complex conjugates, while the remaining root must then be real. Hence the equation has exactly two non-real roots.

c. Outcomes assessed : E5

Marking Guidelines

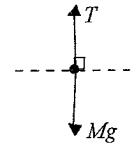
Criteria	Marks
i • resolves forces on P, using zero vertical component of resultant to obtain first equation	1
• uses horizontal component of magnitude $m\omega^2$ to obtain second equation	1
ii • writes required expression for N	1
• writes required expression for $l\omega^2$	1
iii • uses $N \geq 0$ to deduce upper limit	1
• uses $l\omega^2 \geq 0$ to deduce lower limit	1

Answer



- i. Since P is travelling with constant angular velocity in a horizontal circle, the resultant force on P is directed horizontally toward the centre of the circle with magnitude $m\omega^2$. Hence resolving horizontally and vertically,
- $$T \cos 45^\circ + N \cos 45^\circ = mg \Rightarrow T + N = mg\sqrt{2}$$
- $$T \sin 45^\circ - N \sin 45^\circ = m\omega^2 \Rightarrow T - N = m\omega^2 \quad (\text{since } r = l \sin 45^\circ)$$

Forces on Q



- ii. Since Q is in equilibrium, $T = Mg$
- $$\therefore N = mg\sqrt{2} - Mg \quad \text{and} \quad m\omega^2 = Mg - N \Rightarrow l\omega^2 = 2g\frac{M}{m} - g\sqrt{2}$$
- iii. $N = mg\left(\sqrt{2} - \frac{M}{m}\right)$ and $l\omega^2 = 2g\left(\frac{M}{m} - \frac{\sqrt{2}}{2}\right)$
- But $N \geq 0$ and $l\omega^2 \geq 0$. Hence $\frac{\sqrt{2}}{2} \leq \frac{M}{m} \leq \sqrt{2}$.

Question 6 H5, E7

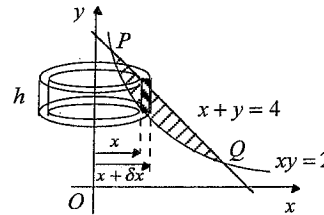
a. Outcomes assessed :

Marking Guidelines

Criteria	Marks
i • finds x coordinates at P and Q	1
• finds volume of cylindrical shell in terms of x and δx	1
• writes V as limiting sum and hence as integral	1
ii • finds a primitive function (with or without an appropriate substitution)	1
• substitutes limits	1
• simplifies exact numerical value	1

Answer

i.



At P, Q $x + y = 4$ and $xy = 2$

$$\therefore x^2 + xy = 4x$$

$$x^2 + 2 = 4x$$

$$x^2 - 4x = -2$$

$$(x-2)^2 = 2$$

$$\therefore x = 2 \pm \sqrt{2}$$

Cylindrical shell has volume

$$\delta V = \pi \left\{ (x + \delta x)^2 - x^2 \right\} h$$

$$= \pi (2x + \delta x) \delta x \left\{ (4-x) - \frac{2}{x} \right\}$$

$$= \pi \frac{(2x + \delta x) \delta x}{x} (4x - x^2 - 2)$$

Then, ignoring second order terms in $(\delta x)^2$,

$$V = \lim_{\delta x \rightarrow 0} \sum_{2-\sqrt{2}}^{2+\sqrt{2}} 2\pi (4x - x^2 - 2) \delta x$$

$$= 2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} (4x - x^2 - 2) dx$$

$$\text{ii. } V = 2\pi \int_{-\sqrt{2}}^{2+\sqrt{2}} \{2 - (x-2)^2\} dx.$$

Making the substitution $u = x - 2$,

$$du = dx$$

$$x = 2 - \sqrt{2} \Rightarrow u = -\sqrt{2}$$

$$x = 2 + \sqrt{2} \Rightarrow u = \sqrt{2}$$

$$V = 2\pi \int_{-\sqrt{2}}^{\sqrt{2}} (2 - u^2) du$$

$$= 4\pi \int_0^{\sqrt{2}} (2 - u^2) du$$

$$= 4\pi \left[2u - \frac{1}{3}u^3 \right]_0^{\sqrt{2}}$$

$$= 4\pi \left\{ 2\sqrt{2} - \frac{1}{3}(2\sqrt{2})^3 \right\}$$

$$= \frac{16\pi\sqrt{2}}{3}$$

b. Outcomes assessed : HE6, E8

Marking Guidelines

Criteria	Marks
i • makes substitution	1
• uses integration by parts	1
• rearranges new integrand into powers of $(\frac{1}{4} - u^2)$	1
• obtains required reduction formula	1
ii • uses reduction formula to find expression for I_5 in terms of I_0	1
• evaluates and rearranges into required form	1
iii • expresses dx in terms of dt , and converts t limits to x limits	1
• uses trigonometric identities to convert integrand into required form	1
• generalises expression for I_5 to obtain similar expression for I_n and hence deduce result	1

Answer

$$\text{i. } I_n = \int_0^1 x^n (1-x)^n dx, \quad n = 0, 1, 2, \dots$$

$$u = \frac{1}{2} - x$$

$$du = -dx$$

$$x = 0 \Rightarrow u = \frac{1}{2}$$

$$x = 1 \Rightarrow u = -\frac{1}{2}$$

$$x(1-x) = \left(\frac{1}{2} - u\right)\left(\frac{1}{2} + u\right) = \frac{1}{4} - u^2$$

$$\therefore I_n = \int_{\frac{1}{2}}^{-\frac{1}{2}} \left(\frac{1}{4} - u^2\right)^n \cdot -du$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - u^2\right)^n du$$

Hence for $n = 1, 2, 3, \dots$

$$I_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - u^2\right)^n du$$

$$= \left[u \left(\frac{1}{4} - u^2\right)^n \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} u \cdot n \left(\frac{1}{4} - u^2\right)^{n-1} (-2u) du$$

$$= 0 - 2n \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - u^2 - \frac{1}{4}\right) \cdot \left(\frac{1}{4} - u^2\right)^{n-1} du$$

$$= -2n \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - u^2\right)^n du - \frac{1}{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - u^2\right)^{n-1} du \right)$$

$$= -2n I_n + \frac{1}{2} n I_{n-1}$$

$$\therefore (2n+1)I_n = \frac{1}{2} n I_{n-1}$$

$$I_n = \frac{n}{2(2n+1)} I_{n-1}$$

$$\text{ii. } I_0 = \int_0^1 1 dx = 1$$

$$I_5 = \frac{5}{2 \times 11} I_4 = \frac{5}{2 \times 11} \cdot \frac{4}{2 \times 9} I_3 = \dots = \frac{5}{2 \times 11} \cdot \frac{4}{2 \times 9} \cdot \frac{3}{2 \times 7} \cdot \frac{2}{2 \times 5} \cdot \frac{1}{2 \times 3} I_0$$

$$\therefore I_5 = \frac{5^2}{11 \times (2 \times 5)} \cdot \frac{4^2}{9 \times (2 \times 4)} \cdot \frac{3^3}{7 \times (2 \times 3)} \cdot \frac{2^2}{5 \times (2 \times 2)} \cdot \frac{1^2}{3 \times 2 \times 1} = \frac{(5!)^2}{11!}$$

iii.

$$x = \sin^2\left(\frac{1}{2}t\right)$$

$$dx = \sin\left(\frac{1}{2}t\right) \cos\left(\frac{1}{2}t\right) dt$$

$$2dx = \sin t dt$$

$$t = 0 \Rightarrow x = 0$$

$$t = \pi \Rightarrow x = 1$$

$$\sin^2 t = 2^2 \sin^2\left(\frac{1}{2}t\right) \cos^2\left(\frac{1}{2}t\right)$$

$$= 2^2 \sin^2\left(\frac{1}{2}t\right) \left\{1 - \sin^2\left(\frac{1}{2}t\right)\right\}$$

$$= 2^2 x(1-x)$$

Hence for $n = 0, 1, 2, \dots$

$$\begin{aligned} \int_0^\pi \sin^{2n+1} t dt &= \int_0^\pi (\sin^2 t)^n \sin t dt \\ &= 2^{2n} \int_0^1 x^n (1-x)^n \cdot 2 dx \\ &= 2^{2n+1} I_n \end{aligned}$$

But for $n = 1, 2, \dots$

$$\begin{aligned} I_n &= \frac{n}{2(2n+1)} \cdot \frac{n-1}{2(2n-1)} \cdot \frac{n-2}{2(2n-3)} \cdots \frac{1}{2(3)} I_0 \\ &= \frac{n^2}{(2n+1)2n} \cdot \frac{(n-1)^2}{(2n-1)(2n-2)} \cdot \frac{(n-2)^2}{(2n-3)(2n-4)} \cdots \frac{1^2}{3 \times 2 \times 1} \\ &= \frac{(n!)^2}{(2n+1)!} \end{aligned}$$

and for $n = 0$

$$\begin{aligned} I_0 &= 1 \\ &= \frac{(0!)^2}{(2 \times 0 + 1)!} \end{aligned}$$

$$\therefore \int_0^\pi \sin^{2n+1} t dt = \frac{2^{2n+1} (n!)^2}{(2n+1)!}, \quad n = 0, 1, 2, \dots$$

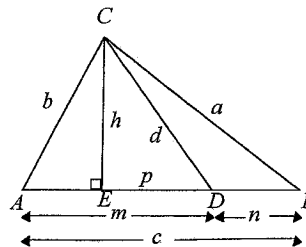
Question 7

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • writes appropriate expression from Pythagoras' theorem in each triangle	1
• eliminates h to obtain required result	1
ii • writes expression from applying Pythagoras' theorem to an appropriate third triangle	1
• eliminates h to obtain required result	1
iii • combines expressions from i. and ii., using $c = m + n$	1
iv • writes appropriate pair of equal ratios derived from sine rule for each specified triangle	1
• uses fact that $\angle CDB, \angle CDA$ are supplementary to deduce $\sin \angle CDB = \sin \angle CDA$	1
• deduces that $am = bn$ if CD bisects $\angle BCA$	1
• uses this equality and iii. to deduce required result	1

Answer



$$\text{i. In } \triangle CEA, \quad b^2 = h^2 + (m-p)^2. \quad \text{In } \triangle CED, \quad h^2 = d^2 - p^2$$

$$\therefore b^2 = (d^2 - p^2) + (m^2 - 2mp + p^2) = d^2 + m^2 - 2mp$$

$$\text{ii. In } \triangle CEB, \quad a^2 = h^2 + (p+n)^2$$

$$\therefore a^2 = (d^2 - p^2) + (n^2 + 2np + p^2) = d^2 + n^2 + 2np$$

$$\text{iii. } a^2 m + b^2 n = d^2 (m+n) + n^2 m + m^2 n$$

$$\therefore a^2 m + b^2 n = (m+n)(d^2 + mn) = c(d^2 + mn)$$

$$\text{iv. In } \triangle CDA, \frac{\sin \angle ACD}{m} = \frac{\sin \angle CDA}{b} \Rightarrow \frac{b}{m} = \frac{\sin \angle CDA}{\sin \angle ACD}$$

$$\text{In } \triangle CDB, \frac{\sin \angle BCD}{n} = \frac{\sin \angle CDB}{a} \Rightarrow \frac{a}{n} = \frac{\sin \angle CDB}{\sin \angle BCD}$$

$$\text{But } \sin \angle CDB = \sin(180^\circ - \angle CDA) = \sin \angle CDA.$$

If CD bisects $\angle BCA$, $\angle BCD = \angle ACD$.

Then

$$\begin{aligned} \frac{a}{n} &= \frac{b}{m} & c(d^2 + mn) &= a^2m + b^2n & \therefore d^2 + mn &= ab \\ \frac{a}{n} &= \frac{b}{m} & &= a(am) + b(bn) & & d^2 = ab - mn \\ \therefore am &= bn & &= a(bn) + b(am) & & \\ & & &= ab(m+n) & & \\ & & &= abc & & \end{aligned}$$

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • finds probability of either 1 head and $(n-1)$ tails, or of 1 tail and $(n-1)$ heads	1
• adds these to find the required probability	1
ii • uses the complementary event to find an expression for this probability	1
iii • finds the required probability	1
iv • states probability is zero for $N=1$, and considers how this event can occur if $N \geq 2$	1
• writes probability for $N \geq 2$	1

Answer

i. For $n = 3, 4, 5, \dots$, $P(\text{odd one out}) = P(1 \text{ tail and } n-1 \text{ heads}) + P(1 \text{ head and } n-1 \text{ tails})$

$$\therefore P(\text{odd one out}) = n \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1} + n \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1} = \frac{n}{2^{n-1}}$$

ii. $P(\text{at least one 'odd one out'}) = 1 - P(\text{none})$

$$= 1 - \left(1 - \frac{n}{2^{n-1}}\right)^n$$

iii. Require probability that 'odd one out' does not occur during first $(N-1)$ plays, then does occur

$$\text{on } N^{\text{th}} \text{ play. Hence probability is } \frac{n}{2^{n-1}} \left(1 - \frac{n}{2^{n-1}}\right)^{N-1}$$

iv. For $N=1$, this probability is clearly 0.

For $N \geq 2$, require exactly one 'odd one out' occurs during first $(N-1)$ plays, then 'odd one out' occurs again on N^{th} play.

Hence probability is

$$(N-1) \left(\frac{n}{2^{n-1}}\right) \left(1 - \frac{n}{2^{n-1}}\right)^{N-2} \left(\frac{n}{2^{n-1}}\right) = (N-1) \left(\frac{n}{2^{n-1}}\right)^2 \left(1 - \frac{n}{2^{n-1}}\right)^{N-2}$$

Question 8

a. Outcomes assessed : HE2, HE3

Marking Guidelines

Criteria	Marks
i • defines a sequence of statements and shows the first is true	1
• writes $x_{k+1} - 2$ in terms of x_k , factoring the cubic numerator	1
• deduces the truth of $S(k+1)$ conditional on the truth of $S(k)$, then completes the induction	1
ii • writes expression for $x_{n+1} + x_n$	1
• shows $x_{n+1} + x_n < 1$ for $x_n > 2$ to deduce result	1

Answer

$$\text{i. } x_1 = 1 \text{ and } x_{n+1} = \frac{2x_n^3 + 8}{3x_n^2}, \quad n = 1, 2, 3, \dots$$

Define a sequence of statements $S(n)$, $n = 2, 3, 4, \dots$ by $S(n)$: $x_n > 2$

$$\text{Consider } S(2): \quad x_2 = \frac{2x_1^3 + 8}{3x_1^2} = \frac{2+8}{3} > 2 \quad \therefore S(2) \text{ is true.}$$

$$\text{If } S(k) \text{ is true:} \quad x_k > 2 \quad **$$

$$\text{Consider } S(k+1): \quad x_{k+1} = \frac{2x_k^3 + 8}{3x_k^2}$$

$$\begin{aligned} x_{k+1} - 2 &= \frac{2x_k^3 + 8 - 6x_k^2}{3x_k^2} \\ &= \frac{2(x_k^3 - 3x_k^2 + 4)}{3x_k^2} \\ &= \frac{2(x_k + 1)(x_k^2 - 4x_k + 4)}{3x_k^2} \\ &= \frac{2(x_k + 1)(x_k - 2)^2}{3x_k^2} \\ &> 0 \end{aligned}$$

if $S(k)$ is true, using **

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(2)$ is true, hence $S(3)$ is true, and then $S(4)$ is true and so on. Hence $x_n > 2$ for $n = 2, 3, 4, \dots$

$$\text{ii. } x_{n+1} = \frac{2x_n^3 + 8}{3x_n^2}, \quad n = 1, 2, 3, \dots$$

$$\text{But } x_n > 2 \Rightarrow \frac{2}{x_n} < 1 \text{ for } n = 2, 3, 4, \dots$$

$$\frac{x_{n+1}}{x_n} = \frac{2x_n^3 + 8}{3x_n^3}$$

$$\text{Hence } \frac{x_{n+1}}{x_n} < \frac{2}{3} + \frac{1}{3} \times \frac{1}{x_n^3} = 1 \text{ for } n = 2, 3, 4, \dots$$

$$= \frac{2}{3} + \frac{1}{3} \times \left(\frac{2}{x_n}\right)^3$$

$$\therefore x_{n+1} < x_n \text{ for all positive integers } n \geq 2$$

b. Outcomes assessed : H5, HE3

Marking Guidelines

Criteria	Marks
i • uses sum formula for geometric progression to simplify expression for $f'(x)$ • deduces required properties of $f(x)$	1 1
ii • explains why for $x > 0$, $f(x) > 0$ for even n and $f(x) < 0$ for n odd • identifies $2n$ as even and $2n-1$ as odd to deduce required result	1 1
iii • selects an appropriate value of n to evaluate $\ln(1.2)$ correct to two decimal places	1

Answer

$$i. f(x) = \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n} \right)$$

$$f'(x) = \frac{1}{1+x} - (1 - x + x^2 - x^3 + \dots + (-1)^{n-1} x^{n-1})$$

$$= \frac{1}{1+x} - \frac{1 - (-x)^n}{1+x}$$

$$= \frac{(-x)^n}{1+x}$$

$\therefore f'(0) = 0$ and $f(x)$ is stationary at $x = 0$.

Also for $x > 0$, when n is even, $f'(x) > 0$ and f is monotonic increasing
when n is odd, $f'(x) < 0$ and f is monotonic decreasing

ii. $f(0) = \ln 1 - 0 = 0$

Hence for $x > 0$, if n is even, $f(x) > 0$ since f is monotonic increasing
if n is odd, $f(x) < 0$ since f is monotonic decreasing

Hence for $x > 0$, if n is even, $\ln(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n}$
if n is odd, $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n}$

But for positive integers n , $2n$ is even and $(2n-1)$ is odd.

$$\therefore x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \frac{x^{2n}}{2n} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{x^{2n-1}}{2n-1}$$

iii.

x	$x - \frac{x^2}{2}$	$x - \frac{x^2}{2} + \frac{x^3}{3}$
0.2	0.18	0.1826

$$\therefore 0.18 < \ln(1.2) < 0.1826$$

$$\therefore \ln(1.2) \approx 0.18 \quad (\text{correct to 2 decimal places})$$

c. Outcomes assessed : HE3, E9

Marking Guidelines

Criteria	Marks
i • writes a as infinite sum and deduces $a > 0$ • compares this sum with the sum of a geometric progression • finds the limiting sum of this G.P. to deduce the required inequality for a	1 1 1
ii • if e rational, uses definition of a rational number to select a value of n for which a is integral • argues by contradiction that e must be irrational	1 1

Answer

$$i. e = \sum_{r=0}^{\infty} \frac{1}{r!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$a = n! \left\{ e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right) \right\}$$

$$= n! \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \right) \quad *$$

$$= \frac{1}{n+1} + \frac{1}{(n+2)(n+1)} + \frac{1}{(n+3)(n+2)(n+1)} + \dots$$

$$< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots \quad (\text{limiting sum of a G.P.})$$

$$= \frac{\left(\frac{1}{n+1}\right)}{1 - \left(\frac{1}{n+1}\right)} \quad (\text{since } \left|\frac{1}{n+1}\right| < 1 \text{ for } n=1, 2, 3, \dots)$$

$$= \frac{1}{(n+1) - 1}$$

$$= \frac{1}{n}$$

Clearly from * above, $a > 0$. $\therefore 0 < a < \frac{1}{n}$ for $n=1, 2, 3, \dots$

ii. Suppose e is rational. Then $e = \frac{p}{q}$ for some positive integers p, q with no common factor.

$$\text{Let } n = q. \quad \text{Then } a = q! \left\{ \frac{p}{q} - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{q!} \right) \right\}$$

Now each of $q, 2!, 3!, \dots, q!$ is a factor of $q!$.

Hence a must be an integer.

But there is no integer a such that $0 < a < \frac{1}{q}$

Hence e must be irrational.

Independent HSC Trial Examination 2007 Mathematics Extension 2 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1 a i	1	Graphs	E6	E2-E3
a ii	1	Graphs	E6	E2-E3
a iii	2	Graphs	E6	E2-E3
a iv	2	Graphs	E6	E2-E3
b i	2	Graphs	E6	E2-E3
b ii	2	Graphs	E6	E2-E3
c i	2	Graphs	E6	E2-E3
c ii	3	Graphs	E6	E2-E3
2 a i	1	Integration	H5	E2-E3
a ii	2	Integration	H5	E2-E3
b	3	Integration	HE6	E2-E3
c	4	Integration	E8	E2-E3
d i	2	Integration	HE6	E2-E3
d ii	2	Integration	E8	E2-E3
d iii	1	Integration	E8	E2-E3
3 a i	1	Complex numbers	E3	E2-E3
a ii	2	Complex numbers	E3	E2-E3
b i	2	Complex numbers	E3	E2-E3
b ii	2	Complex numbers	E3	E2-E3
c i	2	Complex numbers	E3	E3-E4
c ii	2	Complex numbers	E3	E3-E4
d i	2	Complex numbers	E3	E2-E3
d ii	2	Complex numbers	E3	E2-E3
4 a i	4	Conics	E3	E2-E3
a ii	3	Conics	E3	E2-E3
a iii	2	Conics	E3	E2-E3
b i	3	Conics	E3, E4	E2-E3
b ii	1	Conics	E3, E4	E2-E3
b iii	2	Conics	E3, E4	E2-E3
5 a i	2	Polynomials	E4	E2-E3
a ii	2	Polynomials	E4	E2-E3
b i	1	Polynomials	E4	E2-E3
b ii	2	Polynomials	E4	E2-E3
b iii	2	Polynomials	E4	E2-E3
c i	2	Mechanics	E5	E3-E4
c ii	2	Mechanics	E5	E3-E4
c iii	2	Mechanics	E5	E3-E4
6 a i	3	Volumes	E7	E3-E4
a ii	3	Integration	H8	E2-E3
b i	4	Integration	HE6, E8	E3-E4
b ii	2	Integration	E8	E3-E4
b iii	3	Integration	HE6, E8	E3-E4

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
7 a i	2	Plane geometry	H5	E2-E3
a ii	2	Plane geometry	H5	E2-E3
a iii	1	Plane geometry	H5	E2-E3
a iv	4	Plane geometry	H5	E2-E3
b i	2	Probability	HE3	E3-E4
b ii	1	Probability	HE3	E3-E4
b iii	1	Probability	HE3	E3-E4
b iv	2	Probability	HE3	E3-E4
8 a i	3	Induction	HE2	E3-E4
a ii	2	Inequalities	HE3	E3-E4
b i	2	Differentiation	H5	E3-E4
b ii	2	Inequalities	HE3	E3-E4
b iii	1	Inequalities	HE3	E3-E4
c i	3	Inequalities	HE3	E3-E4
c ii	2	Inequalities	E9	E3-E4

The Trial HSC examination, marking guidelines/suggested answers and 'mapping grid' have been produced to help prepare students for the HSC to the best of our ability. Individual teachers/schools may alter parts of this product to suit their own requirement.