Independent Trial HSC Examination 2008

**2008**
**Higher School Certificate**
**Trial Examination**

# Mathematics
## Extension 2

**General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- Write using black or blue pen
- Write your student number and/or name at the top of every page
- All necessary working should be shown in every question
- A table of standard integrals is provided

**Total marks – 120**

Attempt Questions 1 – 8

All questions are of equal value

**This paper MUST NOT be removed from the examination room**

STUDENT NUMBER/NAME.................................

Student name / number ..................................

**Marks**

**Question 1** **Begin a new booklet**

(a)(i) Find $\int (x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2 \, dx$. **2**

(ii) Find $\int \frac{e^{3x}+1}{e^x+1} \, dx$ . **2**

(b) Use the substitution $u = \sin x$ to find $\int \frac{\cos x}{\sin^2 x} \, dx$ . **2**

(c) Use the substitution $t = \tan \frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x + 2\sin x} \, dx$ . **4**

(d) $I_n = \int_0^1 \frac{1}{(1+x^2)^n} \, dx$ , $n = 1, 2, 3, \ldots$

(i) Show that $I_{n+1} = \frac{2n-1}{2n} I_n + \frac{1}{n \cdot 2^{n+1}}$ , $n = 1, 2, 3, \ldots$ **3**

(ii) Hence evaluate $\int_0^1 \frac{1}{(1+x^2)^3} \, dx$. **2**

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**Marks**

**Question 2** **Begin a new booklet**

(a) $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$.

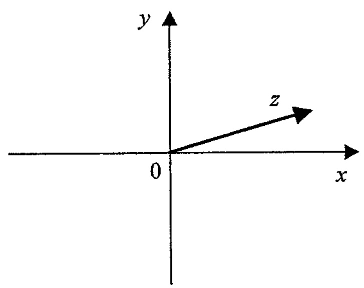
(i) Find $\frac{z_1}{z_2}$ in the form $a + ib$ where $a$ and $b$ are real. **1**

(ii) Write $z_1$ and $z_2$ in modulus - argument form. **2**

(iii) By equating equivalent expressions for $\frac{z_1}{z_2}$, write $\cos\frac{5\pi}{12}$ as a surd. **1**

(iv) Explain why there is no positive integer $n$ such that $z_1 z_2^{\,n}$ is real. **1**

(b)




(i) Copy this Argand diagram and draw vectors for $\bar{z}$, $iz$ and $\bar{z} - iz$. Show carefully any details about relative lengths and directions. **2**

(ii) Show that if $z$ has argument $\theta$, $0 < \theta < \frac{\pi}{2}$, and modulus $r$, then $\left|\bar{z} - iz\right| = 2r\left|\sin(\theta + \frac{\pi}{4})\right|$. **2**

(iii) Draw a second diagram showing vectors for $z$, $\bar{z}$ and $iz$ if $|z| = 1$, $0 < \arg z < \frac{\pi}{2}$, and $\left|\bar{z} - iz\right|$ takes its maximum value. **1**

(c)(i) If $z = \cos\theta + i\sin\theta$, explain why $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$ for positive integers $n$. **1**

(ii) By considering the Binomial expansions of $(z + z^{-1})^3$ and $(z - z^{-1})^3$, show that $4(\cos^3\theta + \sin^3\theta) = (\cos 3\theta - \sin 3\theta) + 3(\cos\theta + \sin\theta)$. **2**

(iii) Hence evaluate $\cos^3\frac{\pi}{12} + \sin^3\frac{\pi}{12}$ in simplest surd form. **2**



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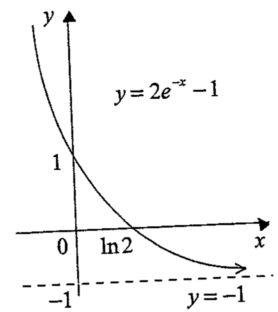
**Marks**

**Question 3** **Begin a new booklet**

(a)




The diagram shows the graph of $f(x) = 2e^{-x} - 1$. On separate diagrams sketch the following graphs, showing the intercepts on the axes and the equations of any asymptotes:

(i) $y = \left|f(x)\right|$. **1**

(ii) $y = \{f(x)\}^2$. **1**

(iii) $y = \frac{1}{f(x)}$. **2**

(iv) $y = \ln f(x)$. **1**

(b) Given that $x = \theta + \frac{1}{2}\sin 2\theta$ and $y = \theta - \frac{1}{2}\sin 2\theta$ :

(i) Show that $\frac{dy}{dx} = \tan^2\theta$. **2**

(ii) Show that $\frac{d^2y}{dx^2} = \tan\theta\sec^4\theta$. **2**

(c) Consider the function $f(x) = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$.

(i) Use differentiation to show that $e^{-x} + x - 1 \geq 0$ for all values of $x$. Hence show that $f(x)$ is an increasing function for $x \neq 0$. **3**

(ii) Show that $f(x)$ is continuous at $x = 0$. **2**

(iii) Sketch the graph of $y = f(x)$. **1**

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Marks

**Question 4** **Begin a new booklet**

(a) $P\left(cp, \frac{c}{p}\right)$, $Q\left(cq, \frac{c}{q}\right)$, $R\left(cr, \frac{c}{r}\right)$ are three points on the rectangular hyperbola $xy = c^2$ such that the parameters $p, q, r$ are in geometric progression.

(i) Explain why $P$ and $R$ must lie on the same branch of the hyperbola. Under what condition will $Q$ lie on the opposite branch to $P$ and $R$? **1**

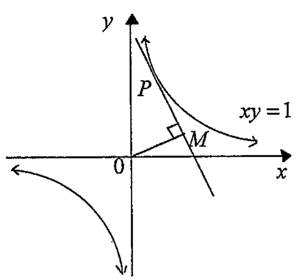
(ii) Show that the chord $PR$ is parallel to the tangent to the hyperbola at $Q$. **3**

(b) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. $PQ$ is a diameter of the ellipse. The tangent to the ellipse at $P$ meets the vertical through $Q$ at $R$.

(i) Prove that the tangent at $P$ has equation $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$. **2**

(ii) Show that the ratio of the area of $\Delta PQR$ to the area of the ellipse is $2 : \pi\left|\tan\theta\right|$. **3**

(c) $P\left(t, \frac{1}{t}\right)$ is a variable point on the rectangular hyperbola $xy = 1$. $M$ is the foot of the perpendicular from the origin to the tangent to the hyperbola at $P$.




(i) Show that the tangent to the hyperbola at $P$ has equation $x + t^2 y = 2t$. **2**

(ii) Find the equation of $OM$. **1**

(iii) Show that the locus of $M$ as $P$ varies has equation $x^2 + y^2 = 2\sqrt{xy}$. **3**



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Marks

**Question 5** **Begin a new booklet**

(a) Consider the polynomial $P(x) = x^4 - 4x^3 + 5x^2 - 2x - 2$.

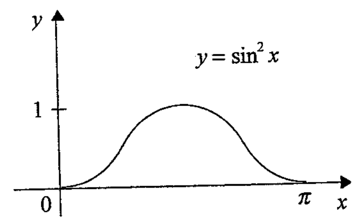
(i) Show that the curve $y = P(x)$ has a maximum turning point at $(1, -2)$ and minimum turning points at $x = 1 \pm \frac{1}{2}\sqrt{2}$. Hence deduce from a sketch of the curve that the equation $P(x) = 0$ has two real roots and two non-real roots. **3**

(ii) Explain why the real roots cannot be rational. What do you know about the nature of the non-real roots? **2**

(iii) Given that $1+i$ is a root of the equation $P(x) = 0$, factor $P(x)$ into two quadratic factors with rational coefficients. Hence find the $x$-intercepts of the curve $y = P(x)$ and show them on your graph. **3**

(b)(i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. Hence show $\int_0^\pi x\cos 2x\,dx = 0$. **3**

(ii)




The area bounded by the curve $y = \sin^2 x$ and the $x$-axis between $x = 0$ and $x = \pi$ is rotated through one revolution about the $y$-axis. By taking the limiting sum of the volumes of cylindrical shells, show that the volume of the solid of revolution is given by $V = 2\pi\int_0^\pi x\sin^2 x\,dx$. Hence find the volume of this solid. **4**

**Marks**

**Question 6** **Begin a new booklet**

(a) Consider the complex number $z$ which satisfies $|z| = 1$.

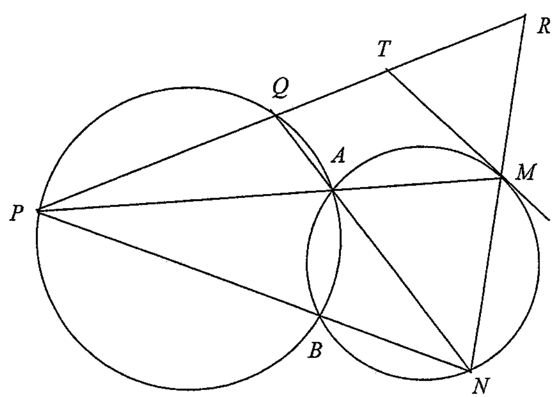
(i) Show that $1+\cos\alpha + i\sin\alpha = 2\cos\frac{1}{2}\alpha\,(\cos\frac{1}{2}\alpha + i\sin\frac{1}{2}\alpha)$. **1**

(ii) If $z = \cos\theta + i\sin\theta$, $-\pi < \theta \le \pi$, write $1+z^2$ in terms of $\cos\theta$ and $\sin\theta$. Hence deduce that if in an Argand diagram, points $A$ and $B$ represent $z$ and $1+z^2$ respectively, then $A$, $B$ and $O$ are collinear, where $O$ is the origin. State the values of $\theta$ such that $B$ lies inside the locus of $z$ in the Argand diagram. **3**

(iii) If $z = \cos\theta + i\sin\theta$, $-\pi < \theta < \pi$, show that $\left|\dfrac{1+z^2}{1+z}\right| = \left|\dfrac{\cos\theta}{\cos\frac{1}{2}\theta}\right|$. By considering graphs of $y = |\cos\theta|$ and $y = \cos\frac{1}{2}\theta$, and solving an appropriate trigonometric equation, find values of $\theta$ such that $\dfrac{1+z^2}{1+z}$ lies inside the locus of $z$. **3**

(iv) In an Argand diagram, sketch points $A$, $B$, $C$, $D$ representing $z$, $1+z^2$, $1+z$ and $\dfrac{1+z^2}{1+z}$ respectively if $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$. **2**

(b)




In the diagram, the two circles intersect at $A$ and $B$. $P$ is a point on one circle. $PA$ and $PB$ produced meet the other circle at $M$ and $N$ respectively. $NA$ produced meets the first circle at $Q$. $PQ$ and $NM$ produced meet at $R$. The tangent at $M$ to the second circle meets $PR$ at $T$.

(i) Copy the diagram. Show that $QAMR$ is a cyclic quadrilateral. **2**

(ii) Show that $TM = TR$. **4**





**Marks**

**Question 7** **Begin a new booklet**

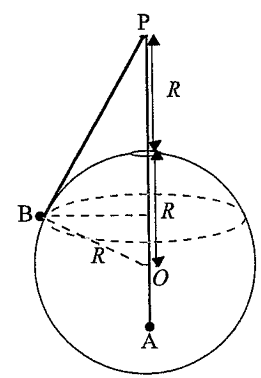
(a) A particle of mass $m$ kg falls from rest in a medium where the resistance is $\frac{1}{10}mv^2$ Newtons when the particle has speed $v$ ms$^{-1}$. The acceleration due to gravity is 10 ms$^{-2}$.

(i) If the particle has fallen a distance $x$ metres in $t$ seconds, explain why $\ddot{x} = \frac{1}{10}(100 - v^2)$. **1**

(ii) Show by integration that $v^2 = 100(1 - e^{-\frac{1}{5}x})$. **3**

(iii) Find the percentage of its terminal velocity attained by the particle in falling 10 metres. **2**

(b)




Particle A of mass $M$ kg hangs at rest inside a hollow sphere of radius $R$ metres, suspended by a light, inextensible string passing through a hole in the top of the sphere and over a smooth peg P situated $2R$ metres above the centre $O$ of the sphere. A second particle B of mass $m$ kg is fixed to the other end of the string and moves in a horizontal circle around the smooth outer surface of the sphere with constant angular velocity $\omega$ radians per second. The acceleration due to gravity is $g$ ms$^{-2}$.

(i) Explain why the section of string between P and B makes an angle of 30° with the vertical. Draw diagrams showing the forces on each of the particles A and B. **2**

(ii) Write expressions for the tension $T$ in the string and the normal reaction $N$ between the particle B and the spherical surface in terms of $M$, $m$ and $g$. **3**

(iii) Find $\omega^2$ in terms of $M$, $m$, $R$ and $g$. **2**

(iv) Deduce that $\sqrt{\dfrac{3}{4}} < \dfrac{M}{m} \le \sqrt{\dfrac{4}{3}}$. **2**

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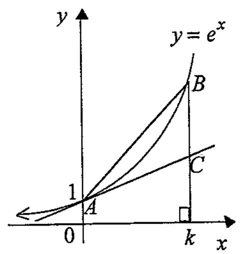
**Question 8** **Begin a new booklet**

(a)




The curve $y = e^x$ cuts the $y$-axis at $A$. $B$ is a second point on the curve such that $x = k$ at $B$, where $k > 0$. The tangent to the curve $y = e^x$ at $A$ cuts the vertical line $x = k$ at the point $C$.

(i) By considering areas, show that $\frac{1}{2}k(k+2) < e^k - 1 < \frac{1}{2}k(1+e^k)$. Hence deduce that $2\cdot 5 < e < 3$. **4**

(ii) Show that the curve $y = e^x$ bisects the area of $\Delta ABC$ for some value of $k$ such that $2 < k < 3$. Taking $k = 2\cdot 7$ as a first approximation, apply Newton's method once to obtain a second approximation. Give your answer to one decimal place. **3**

(b) A sequence of numbers $T_n$, $n = 1, 2, 3, \ldots$ is defined by $T_1 = 2$, $T_2 = 0$ and $T_n = 2T_{n-1} - 2T_{n-2}$ for $n = 3, 4, 5, \ldots$. Use Mathematical induction to show that $T_n = \left(\sqrt{2}\right)^{n+2} \cos\frac{n\pi}{4}$, $n = 1, 2, 3, \ldots$. **5**

(c) $\Delta ABC$ has sides of length $a$, $b$, $c$. If $a^2 + b^2 + c^2 = ab + bc + ca$, show that $\Delta ABC$ is equilateral. **3**

## Question 1

**a. Outcomes assessed : H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • expands the square | 1 |
| • writes the primitive function | 1 |
| ii • simplifies the integrand by factoring the sum of cubes | 1 |
| • writes the primitive | 1 |

**Answer**

i. $\int (x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2\, dx = \int (x + 2 + \frac{1}{x})\, dx$
$= \frac{1}{2}x^2 + 2x + \ln x + c$

ii. $\int \frac{e^{3x}+1}{e^x+1}\, dx = \int \frac{(e^x+1)(e^{2x}-e^x+1)}{e^x+1}\, dx$
$= \int (e^{2x} - e^x + 1)\, dx$
$= \frac{1}{2}e^{2x} - e^x + x + c$

**b. Outcomes assessed : HE6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • writes integral in terms of $u$ and primitive in terms of $u$ | 1 |
| • writes primitive as a function of $x$ | 1 |

**Answer**

$u = \sin x$
$du = \cos x\, dx$

$\int \frac{\cos x}{\sin^2 x}\, dx = \int \frac{1}{u^2}\, du = -\frac{1}{u} + c = -\operatorname{cosec} x + c$

**c. Outcomes assessed : HE6, E8**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • writes $dx$ in terms of $t$ and $dt$; changes $x$ limits to $t$ limits | 1 |
| • uses $t$ formulae to convert integrand to function of $t$ | 1 |
| • uses partial fractions to find primitive | 1 |
| • evaluates by substitution | 1 |

**Answer**

$t = \tan\frac{x}{2}$
$dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx$
$= \frac{1}{2}(1+t^2)dx$
$dx = \frac{2}{1+t^2}\, dt$

$x = 0 \Rightarrow t = 0$
$x = \frac{\pi}{2} \Rightarrow t = 1$

$2 - \cos x + 2\sin x$
$= \frac{2(1+t^2) - (1-t^2) + 4t}{1+t^2}$
$= \frac{3t^2 + 4t + 1}{1+t^2}$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x + 2\sin x}\, dx = \int_0^1 \frac{1+t^2}{(3t+1)(t+1)} \cdot \frac{2}{1+t^2}\, dt$$
$$= \int_0^1 \left\{\frac{3}{(3t+1)} - \frac{1}{(t+1)}\right\} dt$$
$$= \left[\ln(3t+1) - \ln(t+1)\right]_0^1$$
$$= (\ln 4 - \ln 1) - (\ln 2 - \ln 1)$$
$$= 2\ln 2 - \ln 2$$
$$= \ln 2$$

**d. Outcomes assessed : E8**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • applies integration by parts | 1 |
| • rearranges new integrand into form involving powers $n$ and $n$+1 | 1 |
| • obtains required recurrence relation | 1 |
| ii • applies recurrence relation to express $I_3$ in terms of $I_1$ | 1 |
| • evaluates $I_1$ and hence $I_3$ | 1 |

**Answer**

i. $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\, dx\ , \quad n = 1, 2, 3, \ldots$
$= \left[x(1+x^2)^{-n}\right]_0^1 - \int_0^1 x\,(-n)(1+x^2)^{-n-1}(2x)\, dx$
$= 2^{-n} + 2n\int_0^1 \left\{(1+x^2) - 1\right\}(1+x^2)^{-n-1}\, dx$
$= 2^{-n} + 2n\int_0^1 \left\{(1+x^2)^{-n} - (1+x^2)^{-(n+1)}\right\} dx$
$= 2^{-n} + 2n\, I_n - 2n\, I_{n+1}$
$\therefore\ 2n\, I_{n+1} = (2n-1)I_n + 2^{-n}$
$I_{n+1} = \frac{2n-1}{2n} I_n + \frac{1}{n.2^{n+1}}\ , \quad n = 1, 2, 3, \ldots$

ii. $I_3 = \frac{3}{4}I_2 + \frac{1}{16}$
$= \frac{3}{4}(\frac{1}{2}I_1 + \frac{1}{4}) + \frac{1}{16}$
$= \frac{3}{8}I_1 + \frac{1}{4}$
$I_1 = \int_0^1 \frac{1}{1+x^2}\, dx$
$= \left[\tan^{-1} x\right]_0^1$
$= \frac{\pi}{4}$
$\therefore \int_0^1 \frac{1}{(1+x^2)^3}\, dx = \frac{3\pi + 8}{32}$

## Question 2

**a. Outcomes assessed : E3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • obtains the quotient in the required form | 1 |
| ii • writes $z_1$ in modulus-argument form | 1 |
| • writes $z_2$ in modulus-argument form | 1 |
| iii • equates real parts of expressions for quotient to find required surd | 1 |
| iv • explains why product cannot be real by considering its argument | 1 |

**Answer**

i. $\frac{z_1}{z_2} = \frac{1+i}{\sqrt{3} - i}$
$= \frac{(1+i)(\sqrt{3}+i)}{3+1}$
$= \frac{(\sqrt{3}-1)}{4} + i\,\frac{(\sqrt{3}+1)}{4}$

ii. $z_1 = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
$z_2 = 2\left\{\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right\}$

iii. $\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\left\{\cos(\frac{\pi}{4} + \frac{\pi}{6}) + i\sin(\frac{\pi}{4} + \frac{\pi}{6})\right\} = \frac{\sqrt{3}-1}{4} + i\,\frac{\sqrt{3}+1}{4}$

Equating real parts $\frac{1}{\sqrt{2}}\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{4}$
$\therefore \cos\frac{5\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

iv. $\arg(z_1 z_2{}^n) = \frac{\pi}{4} - n\frac{\pi}{6} = (3-2n)\frac{\pi}{12}$
But $3 - 2n = 1 + 2(1-n)$ is an odd number for all positive integers $n$, and hence cannot be divisible by 12.
$\therefore \arg(z_1 z_2{}^n)$ cannot be a multiple of $\pi$, and hence $z_1 z_2{}^n$ cannot be real, for any positive integer $n$.

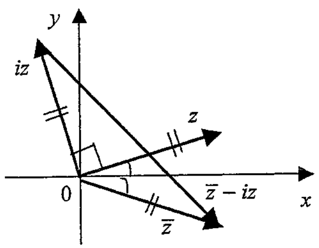
**b. Outcomes assessed : E3**

**Marking Guidelines**

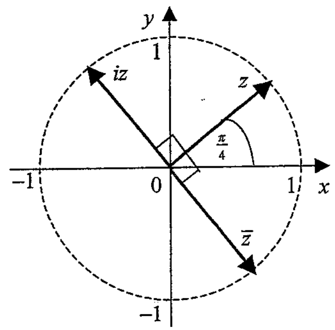
| Criteria | Marks |
|---|---|
| i • shows $iz$ as anticlockwise rotation by $\frac{\pi}{2}$, and $\bar{z}$ as reflection in $x$-axis | 1 |
| • obtains $\bar{z}-iz$ by vector subtraction | 1 |
| ii • applies cosine rule to find square of modulus | 1 |
| • uses trigonometric identity to obtain required expression for modulus | 1 |
| iii • shows vectors as required | 1 |

**Answer**

i.




iii.




ii. $|\bar{z}-iz|^2 = r^2+r^2-2r^2\cos(\frac{\pi}{2}+2\theta)$

$= 2r^2\{1-\cos(\frac{\pi}{2}+2\theta)\}$

$= 2r^2.2\sin^2\frac{1}{2}(\frac{\pi}{2}+2\theta)$

$\therefore |\bar{z}-iz| = 2r\left|\sin(\frac{\pi}{4}+\theta)\right|$

**c. Outcomes assessed : E3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • uses De Moivre's theorem to establish required results | 1 |
| ii • obtains both expansions, groups terms and substitutes appropriate trigonometric expressions | 1 |
| • simplifies and combines results to obtain required expression | 1 |
| iii • substitutes $\theta=\frac{\pi}{12}$ and obtains zero for first term | 1 |
| • expresses remaining term as single sine or cosine to obtain simplest surd form | 1 |

**Answer**

i. Using De Moivre's theorem, $z^n = \cos n\theta + i\sin n\theta$ $\qquad \therefore\ z^n + z^{-n} = 2\cos n\theta$

$z^{-n} = \cos(-n\theta)+i\sin(-n\theta) \Rightarrow z^{-n} = \cos n\theta - i\sin n\theta$ $\qquad z^n - z^{-n} = 2i\sin n\theta$

ii. $(z+z^{-1})^3 = z^3+3z^2z^{-1}+3zz^{-2}+z^{-3}$ $\qquad (z-z^{-1})^3 = z^3-3z^2z^{-1}+3zz^{-2}-z^{-3}$

$(z+z^{-1})^3 = (z^3+z^{-3})+3(z+z^{-1})$ $\qquad (z-z^{-1})^3 = (z^3-z^{-3})-3(z-z^{-1})$

$(2\cos\theta)^3 = 2\cos3\theta + 6\cos\theta$ $\qquad (2i\sin\theta)^3 = 2i\sin3\theta - 6i\sin\theta$

$4\cos^3\theta = \cos3\theta + 3\cos\theta$ $\qquad -4\sin^3\theta = \sin3\theta - 3\sin\theta$

$\therefore\ 4(\cos^3\theta+\sin^3\theta) = (\cos3\theta-\sin3\theta)+3(\cos\theta+\sin\theta)$

iii. $\cos^3\frac{\pi}{12}+\sin^3\frac{\pi}{12} = \frac{1}{4}\{(\cos\frac{\pi}{4}-\sin\frac{\pi}{4})+3(\cos\frac{\pi}{12}+\sin\frac{\pi}{12})\}$

$= 0+\frac{3\sqrt{2}}{4}(\frac{1}{\sqrt{2}}\cos\frac{\pi}{12}+\frac{1}{\sqrt{2}}\sin\frac{\pi}{12})$

$= \frac{3\sqrt{2}}{4}(\sin\frac{\pi}{4}\cos\frac{\pi}{12}+\cos\frac{\pi}{4}\sin\frac{\pi}{12})$

$\therefore\ \cos^3\frac{\pi}{12}+\sin^3\frac{\pi}{12} = \frac{3\sqrt{2}}{4}\sin(\frac{\pi}{4}+\frac{\pi}{12}) = \frac{3\sqrt{2}}{4}\sin\frac{\pi}{3} = \frac{3\sqrt{6}}{8}$

**Question 3**

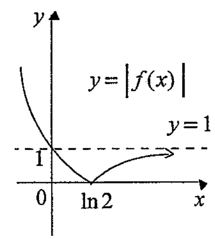
**a. Outcomes assessed : E6**

**Marking Guidelines**

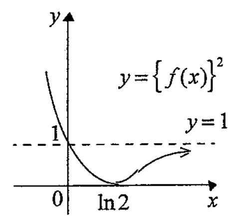
| Criteria | Marks |
|---|---|
| i • reflects section of curve for $x>\ln 2$ in $x$-axis | 1 |
| ii • shows curve with correct shape and positions of turning point and asymptote | 1 |
| iii • shows left hand branch with $y$-intercept and asymptote | 1 |
| • shows right hand branch with both asymptotes | 1 |
| iv • shows curve through origin with correct shape and asymptote | 1 |

**Answer**

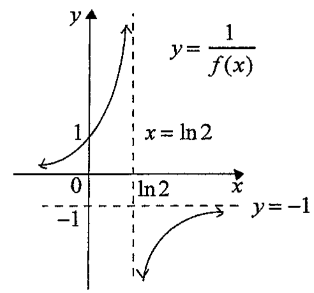
i.



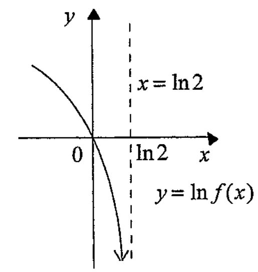

ii.




iii.




iv.




**b. Outcomes assessed : E6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • derives $x$ and $y$ with respect to $\theta$ | 1 |
| • uses double-angle trigonometric identities to show required result | 1 |
| ii • derives implicitly with respect to $x$ | 1 |
| • substitutes derivative of $\theta$ with respect to $x$ to obtain required result | 1 |

**Answer**

i. $y=\theta-\frac{1}{2}\sin 2\theta \;\Rightarrow\; \dfrac{dy}{d\theta}=1-\cos 2\theta=2\sin^2\theta$

$x=\theta+\frac{1}{2}\sin 2\theta \;\Rightarrow\; \dfrac{dx}{d\theta}=1+\cos 2\theta=2\cos^2\theta$

$\therefore \dfrac{dy}{dx}=\dfrac{dy}{d\theta}\div\dfrac{dx}{d\theta}=\tan^2\theta$

ii. $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}(\tan^2\theta)$

$=2\tan\theta\sec^2\theta\,\dfrac{d\theta}{dx}$

$=2\tan\theta\sec^2\theta\,\dfrac{1}{2\cos^2\theta}$

$=\tan\theta\sec^4\theta$

**c. Outcomes assessed : E6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • shows by differentiation that $e^{-x}+x-1$ has a minimum value of 0 when $x=0$ | 1 |
| • finds $f'(x)$ | 1 |
| • rearranges $f'(x)$ as a product of $(e^{-x}+x-1)$ and deduces $f'(x)>0$ | 1 |
| ii • expresses limiting value of $f(x)$ as $x\to 0$ as the derivative of $e^x$ at $x=0$ | 1 |
| • evaluates this derivative to show limiting value is 1 | 1 |
| iii • shows curve of correct shape with *y* intercept of 1 and asymptote | 1 |

**Answer**

i. Consider the function $g(x)=e^{-x}+x-1$.

$g(0)=0$ and $g'(x)=-e^{-x}+1 \Rightarrow g'(0)=0$

Also $g''(x)=e^{-x}>0$ for all $x$

$\therefore g(x)$ has a minimum value of 0 when $x=0$.

$\therefore e^{-x}+x-1\ge 0$ for all *x*, with equality only if $x=0$.

For $x\ne 0$, $f'(x)=\dfrac{d}{dx}\left(\dfrac{e^x-1}{x}\right)$

$=\dfrac{e^x.x-(e^x-1).1}{x^2}$

$=\dfrac{1+xe^x-e^x}{x^2}$

$=\dfrac{e^x}{x^2}\left(e^{-x}+x-1\right)$

$>0$

Hence $f(x)$ is an increasing function for $x\ne 0$.

ii. Let $h(x)=e^x$. Then $h'(x)=e^x$.

$\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=\lim\limits_{x\to 0}\dfrac{e^x-e^0}{x-0}$

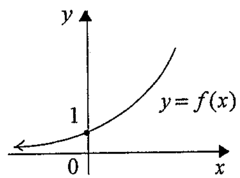
$=h'(0)$

$=1$

$=f(0)$

$\therefore f(x)$ is continuous at $x=0$.

iii:




**Question 4**

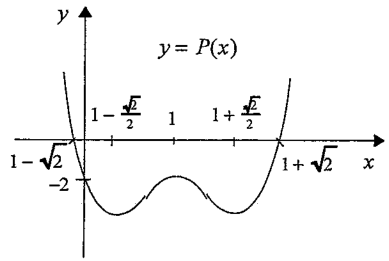
**a. Outcomes assessed : E4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • explains *p* and *r* must have the same sign; common ratio negative for *Q* on opposite branch | 1 |
| ii • finds gradient of *PR* in terms of *p* and *r* | 1 |
| • finds gradient of tangent at *Q* in terms of *q* | 1 |
| • uses relationship between consecutive terms in a GP to deduce result | 1 |



i. (cont.)




Since $y=P(x)$ has exactly two *x*-intercepts, neither of which is a stationary point, the equation $P(x)=0$ has exactly two real roots. Then the two remaining roots must be non-real.

ii. Any rational roots of $P(x)=0$ must be factors of 2. But $P(1)=-2$, $P(-1)=10$, $P(2)=-2$ and $P(-2)=70$. Hence there are no rational roots. Since the coefficients of $P(x)=0$ are real, the non-real roots must be complex conjugates.

iii. $1+i$ and $1-i$ are roots of $P(x)=0$.

$\therefore (x-1-i)(x-1+i)=x^2-2x+2$ is a factor.

If the two real roots are $\alpha$ and $\beta$, using the relationship between the coefficients and the roots, $\alpha+\beta=2$ and $\alpha\beta=-1$.

Hence $(x-\alpha)(x-\beta)=x^2-2x-1$.

$\therefore P(x)=(x^2-2x+2)(x^2-2x-1)$ and $\alpha$, $\beta$ are roots of $x^2-2x-1=0$.

$(x-1)^2=2$

Hence the real roots are $1\pm\sqrt{2}$, and these are also the *x*-intercepts of the curve $y=P(x)$.

**b. Outcomes assessed : E7, E8**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • makes an appropriate substitution to prove general result | 1 |
| • applies result to given integral | 1 |
| • rearranges, then evaluates primitive | 1 |
| ii • finds the volume of a typical cylindrical shell in terms of *x* | 1 |
| • takes the limiting sum of cylindrical shells to obtain the given integral | 1 |
| • uses an appropriate trigonometric identity to simplify the integral | 1 |
| • uses the result in (i) to evaluate the integral | 1 |

**Answer**

i.

$u=a-x \qquad x=0\Rightarrow u=a$

$du=-dx \qquad x=a\Rightarrow u=0$

$\int_0^a f(x)\,dx=\int_a^0 f(a-u).-du$

$=\int_0^a f(a-u)\,du$

$=\int_0^a f(a-x)\,dx$

$\int_0^\pi x\cos 2x\,dx=\int_0^\pi(\pi-x)\cos(2\pi-2x)\,dx$

$=\int_0^\pi(\pi-x)\cos 2x\,dx$

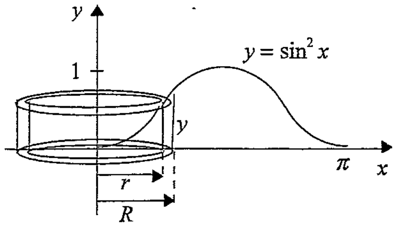
$=\pi\int_0^\pi\cos 2x\,dx-\int_0^\pi x\cos 2x\,dx$

$\therefore 2\int_0^\pi x\cos 2x\,dx=\pi\left[\tfrac{1}{2}\sin 2x\right]_0^\pi$

$=\pi(0-0)$

$\therefore \int_0^\pi x\cos 2x\,dx=0$

ii.




$R = x + \delta x, \quad r = x$

The typical cylindrical shell has volume

$\delta V = \pi(R^2 - r^2)y$

$= \pi(R+r)(R-r)y$

$= \pi(2x + \delta x)\,\delta x.\sin^2 x$

Ignoring second order terms in $(\delta x)^2$, the volume of the solid is given by

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} 2\pi x \sin^2 x \,\delta x$$

$$= 2\pi \int_0^{\pi} x \sin^2 x \, dx$$

$$= \pi \int_0^{\pi} x(1 - \cos 2x)\, dx$$

$$= \pi \int_0^{\pi} x \, dx - \pi \int_0^{\pi} x \cos 2x \, dx$$

$$= \tfrac{1}{2}\pi \left[x^2\right]_0^{\pi} - 0$$

$$= \tfrac{1}{2}\pi^3$$

Volume is $\frac{1}{2}\pi^3$ cubic units

**Question 6**

**a. Outcomes assessed : E3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • proves result using double angle formulae | 1 |
| ii • applies result to $1 + z^2$ | 1 |
| • uses result that $1 + z^2 = kz$ to deduce $A, B, O$ collinear | 1 |
| • finds required values of $\theta$ | 1 |
| iii • applies result in (i) and simplifies to obtain expression for quotient | 1 |
| • uses graph to compare values of $\lvert\cos\theta\rvert$ and $\cos\frac{1}{2}\theta$ | 1 |
| • solves appropriate trigonometric equation to find values of $\theta$ | 1 |
| iv • sketches diagram showing collinear points $O, A$ and $B$ in the correct positions | 1 |
| • shows collinear points $O, C$ and $D$ in the correct positions | 1 |

**Answer**

i. $1 + \cos\alpha + i\sin\alpha = 2\cos^2\frac{1}{2}\alpha + i\,(2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha)$

$= 2\cos\frac{1}{2}\alpha\,(\cos\frac{1}{2}\alpha + i\sin\frac{1}{2}\alpha)$

ii. $1 + z^2 = 1 + \cos 2\theta + i\sin 2\theta$

$\therefore 1 + z^2 = 2\cos\theta\,(\cos\theta + i\sin\theta)$

$\therefore 1 + z^2 = (2\cos\theta)\,z$

$\therefore \overrightarrow{OB}$ is in the same direction as $\overrightarrow{OA}$ (if $2\cos\theta > 0$), or the opposite direction to $\overrightarrow{OA}$ (if $2\cos\theta < 0$).

In either case, $A$, $B$ and $O$ are collinear.

$B$ lies inside the circle which is the locus of $z$ if $\lvert 2\cos\theta \rvert < 1$, giving

$-\frac{2\pi}{3} < \theta < -\frac{\pi}{3}$ *or* $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$

iii. $\dfrac{1+z^2}{1+z} = \dfrac{2\cos\theta\,(\cos\theta + i\sin\theta)}{2\cos\frac{1}{2}\theta\,(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta)}$

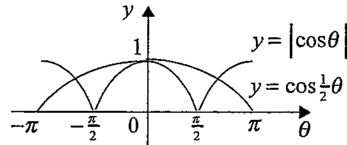
$\therefore \dfrac{1+z^2}{1+z} = \dfrac{\cos\theta}{\cos\frac{1}{2}\theta}(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta)$

$\therefore \left|\dfrac{1+z^2}{1+z}\right| = \left|\dfrac{\cos\theta}{\cos\frac{1}{2}\theta}\right|$

$\left|\dfrac{1+z^2}{1+z}\right| < 1$ for $\lvert\cos\theta\rvert < \lvert\cos\frac{1}{2}\theta\rvert$



iii. (cont)




At the intersection points other than at $\theta = 0$,

$-\cos\theta = \cos\frac{1}{2}\theta$

$-2\cos^2\frac{1}{2}\theta = \cos\frac{1}{2}\theta - 1$

$2\cos^2\frac{1}{2}\theta + \cos\frac{1}{2}\theta - 1 = 0$

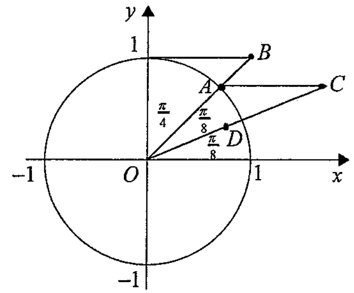
$(2\cos\frac{1}{2}\theta - 1)(\cos\frac{1}{2}\theta + 1) = 0$

$\therefore \cos\frac{1}{2}\theta = \frac{1}{2}$, since $\cos\frac{1}{2}\theta \neq -1$

$\therefore \lvert\cos\theta\rvert < \lvert\cos\frac{1}{2}\theta\rvert$ for $-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$, $\theta \neq 0$.

Hence point representing $\dfrac{1+z^2}{1+z}$ lies inside locus of $z$ for $-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$, $\theta \neq 0$.

iv.



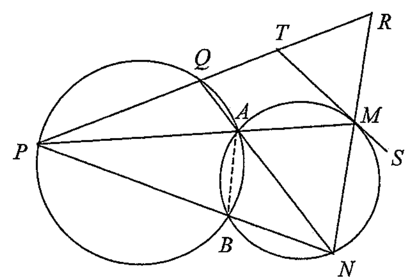

**b. Outcomes assessed : PE2, PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i • sketches diagram and quotes property of cyclic quadrilateral to deduce $\angle RMA = \angle ABN$ | 1 |
| • deduces similarly that $\angle ABN = \angle AQP$ and applies test for $QAMR$ to be cyclic | 1 |
| ii • uses alternate segment theorem to deduce equality of angles $SMN$ and $MAN$ | 1 |
| • recognises equal vertically opposite angles at $M$ and $A$ | 1 |
| • quotes property of cyclic quadrilateral $QAMR$ to deduce equality of angles $PAQ$ and $TRM$ | 1 |
| • deduces triangle $TMR$ has a pair of equal angles and hence a pair of equal sides | 1 |

**Answer**

i.




$\angle RMA = \angle ABN$ (*exterior angle of cyclic quad. ABNM is equal to interior opposite angle*)

Similarly

$\angle ABN = \angle AQP$ in cyclic quadrilateral $ABPQ$.

Hence quadrilateral $QAMR$ is cyclic.

(*exterior angle AQP is equal to interior opposite angle RMA*)

ii. Produce $TM$ to $S$. Then

$\angle TMR = \angle SMN$ (*vertically opposite angles are equal*)

$\angle SMN = \angle MAN$ (*angle between tangent and chord drawn to point of contact is equal to angle subtended by that chord in the alternate segment*)

$\angle MAN = \angle PAQ$ (*vertically opposite angles are equal*)

$\angle PAQ = \angle TRM$ (*exterior angle of cyclic quad. QAMR is equal to interior opposite angle*)

Hence in $\Delta TMR$, $\angle TMR = \angle TRM$ and hence $TM = TR$ (*sides opposite equal angles are equal*)

## Question 7

**a. Outcomes assessed : E5**

**Marking Guidelines**

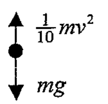
| | Criteria | Marks |
|---|---|---|
| i | • considers forces on particle and invokes Newton's second law to deduce equation of motion | 1 |
| ii | • finds primitive function for $x$ in terms of $v^2$ | 1 |
| | • uses initial conditions to evaluate constant of integration | 1 |
| | • rearranges to find $v^2$ in terms of $x$ | 1 |
| iii | • shows terminal velocity is $10\text{ms}^{-1}$ | 1 |
| | • finds v when $x = 10$ as percentage of terminal velocity | 1 |

**Answer**

i. *Initial conditions*


$t = 0$
$x = 0$
$v = 0$

*Forces on particle*




By Newton's second law

$$m\ddot{x} = mg - \tfrac{1}{10}mv^2$$
$$\ddot{x} = 10 - \tfrac{1}{10}v^2$$
$$\therefore \ddot{x} = \tfrac{1}{10}(100 - v^2)$$

$\ddot{x} \to 0$ as $v \to 10$

Terminal velocity is $10\text{ms}^{-1}$

ii.
$$\frac{1}{2}\frac{dv^2}{dx} = \tfrac{1}{10}(100 - v^2)$$
$$-\frac{1}{5}\frac{dx}{d(v^2)} = \frac{-1}{100 - v^2}$$
$$-\tfrac{1}{5}x = \ln A\,(100 - v^2), \quad A \text{ constant}$$
$$\left.\begin{array}{l} t=0 \\ x=0 \\ v=0 \end{array}\right\} \Rightarrow \quad 0 = \ln 100A \quad \therefore A = \tfrac{1}{100}$$
$$\therefore -\tfrac{1}{5}x = \ln\left(1 - \tfrac{1}{100}v^2\right)$$
$$e^{-\frac{1}{5}x} = 1 - \tfrac{1}{100}v^2$$
$$v^2 = 100\left(1 - e^{-\frac{1}{5}x}\right)$$

iii. $x = 10 \Rightarrow v^2 = 100(1 - e^{-2})$

Percentage of terminal velocity attained when particle has fallen 10 metres is

$$\frac{10\sqrt{1-e^{-2}}}{10} \times 100\% \approx 93\%$$

**b. Outcomes assessed : E5**

**Marking Guidelines**

| | Criteria | Marks |
|---|---|---|
| i | • explains why angle is 30° and shows forces on A | 1 |
| | • shows forces on B | 1 |
| ii | • identifies resultant zero force on A to write expression for $T$ | 1 |
| | • identifies zero vertical component of resultant force on B to write equation for $T$ and $N$ | 1 |
| | • finds expression for $N$ | 1 |
| iii | • considers horizontal component of resultant force on B to write equation for $T$, $N$ and $\omega^2$ | 1 |
| | • finds expression for $\omega^2$ | 1 |
| iv | • uses $N \geq 0$ to obtain upper bound | 1 |
| | • uses $\omega^2 > 0$ to obtain lower bound | 1 |



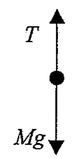
**Answer**

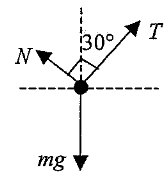
i. $\angle OBP = 90°$ (tangent $\perp$ radius)

$\therefore \sin\angle OPB = \frac{R}{2R} = \frac{1}{2}$

$\therefore \angle OPB = 30°$

*Forces on A* *Forces on B*




ii. A is in equilibrium. $\therefore T = Mg$ (1)

Resultant force on B is directed horizontally towards the centre of its circle of motion with magnitude $mr\omega^2$.

Resolving vertically and horizontally,

$$T\cos30° + N\sin30° = mg$$
$$T\sin30° - N\cos30° = m(R\sin60°)\omega^2$$

Hence
$$T\sqrt{3} + N = 2mg \qquad (2)$$
$$T\sqrt{3} - 3N = 3mR\omega^2 \qquad (3)$$

From (1) and (2) : $N = (2m - M\sqrt{3})g$

iii. From (3), substituting for $T, N$ :

$$\omega^2 = \frac{Mg\sqrt{3} - 3(2m - M\sqrt{3})g}{3mR}$$
$$\omega^2 = \frac{2g(2M\sqrt{3} - 3m)}{3mR}$$

iv. Particle in contact with the sphere $\therefore N \geq 0$.

Particle moving around the circle $\therefore \omega^2 > 0$

$$N = mg\sqrt{3}\left(\sqrt{\frac{4}{3}} - \frac{M}{m}\right) \quad \text{and}$$
$$\omega^2 = \frac{4g\sqrt{3}}{3R}\left(\frac{M}{m} - \sqrt{\frac{3}{4}}\right). \qquad \therefore \sqrt{\frac{3}{4}} < \frac{M}{m} \leq \sqrt{\frac{4}{3}}$$

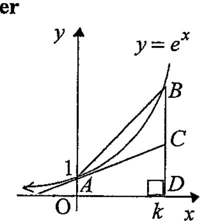
## Question 8

**a. Outcomes assessed : PE3**

**Marking Guidelines**

| | Criteria | Marks |
|---|---|---|
| i | • finds equation of tangent $AC$ and finds coordinates of $C$ | 1 |
| | • establishes lower bound for $(e^k - 1)$ by comparing definite integral with area of $AODC$ | 1 |
| | • establishes upper bound for $(e^k - 1)$ by comparing definite integral with area of $AODB$ | 1 |
| | • uses result for $k = 1$ and rearranges to obtain required inequality | 1 |
| ii | • writes equation for $k$ if triangle area is bisected | 1 |
| | • writes equation in form $f(k) = 0$ and establishes existence of root $k$ such $2 < k < 3$ | 1 |
| | • uses Newton's method to obtain second approximation | 1 |

**Answer**




i. $\frac{dy}{dx} = e^x = 1$ at $x = 0$

Hence tangent $AC$ has equation $y = x + 1$

$\therefore C(k, k+1)$. Also $B(k, e^k)$. Then

$$Area\ \ AODC < \int_0^k e^x\,dx < Area\ \ AODB$$
$$\tfrac{1}{2}k(k+2) < \left[e^x\right]_0^k < \tfrac{1}{2}k(1+e^k)$$
$$\therefore \tfrac{1}{2}k(k+2) < e^k - 1 < \tfrac{1}{2}k(1+e^k)$$

For $k = 1$, $1{\cdot}5 < e - 1 < 0{\cdot}5 + \frac{1}{2}e$

Hence $2{\cdot}5 < e$ and $\frac{1}{2}e < 1{\cdot}5$

$\therefore 2{\cdot}5 < e < 3$

ii. Area of $\Delta ABC$ is bisected if

$$(e^k - 1) - \tfrac{1}{2}k(k+2) = \tfrac{1}{2}k(1+e^k) - (e^k - 1)$$

$$(4-k)e^k - k^2 - 3k - 4 = 0$$

Let $f(k) = (4-k)e^k - k^2 - 3k - 4$

Then $f(2) \approx 0\cdot 78 > 0$ , $f(3) \approx -1\cdot 9 < 0$

and $f(k)$ is continuous.

Hence $f(k) = 0$, and the area is bisected, for some $k$ such that $2 < k < 3$

$$f(k) = (4-k)e^k - k^2 - 3k - 4$$

$$f'(k) = \left\{-e^k + (4-k)e^k\right\} - 2k - 3$$

$$= (3-k)e^k - 2k - 3$$

Taking $k_0 = 2\cdot 7$, $k_1 = 2\cdot 7 - \dfrac{f(2\cdot 7)}{f'(2\cdot 7)}$

$$\therefore k_1 \approx 2\cdot 7 - \frac{-0\cdot 4635}{3\cdot 9361}$$

$$\approx 2\cdot 688$$

Hence second approximation is $2\cdot 7$ (to one decimal place).

**b. Outcomes assessed : HE2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • shows statement true for $n=1$, $n=2$ | 1 |
| • uses recurrence relation to write $T_{k+1}$ in terms of $T_k$ and $T_{k-1}$ for $k \geq 2$ | 1 |
| • writes both $T_k$ and $T_{k-1}$ in terms of cosines, conditional on truth of $S(n)$, $n \leq k$ | 1 |
| • expands cosine expression, simplifies and regroups | 1 |
| • obtains $T_{k+1}$ in required form, conditional on truth of $S(n)$, $n \leq k$, and completes proof | 1 |

**Answer**

Define the sequence of statements $S(n)$, $n = 1, 2, 3, \ldots$ by $S(n)$: $T_n = \left(\sqrt{2}\right)^{n+2}\cos\frac{n\pi}{4}$

Consider $S(1)$, $S(2)$ :

$$\left(\sqrt{2}\right)^{1+2}\cos\tfrac{1.\pi}{4} = 2\sqrt{2}\,.\tfrac{1}{\sqrt{2}} = 2 = T_1 \qquad \therefore S(1) \text{ is true}$$

$$\left(\sqrt{2}\right)^{2+2}\cos\tfrac{2\pi}{4} = 4 \times 0 = 0 = T_2 \qquad \therefore S(2) \text{ is true}$$

If $S(n)$ is true, $n \leq k$ :

$$T_n = \left(\sqrt{2}\right)^{n+2}\cos\tfrac{n\pi}{4}, \quad n = 1, 2, 3, \ldots, k \qquad **$$

Consider $S(k+1)$, $k \geq 2$ :

$$T_{k+1} = 2T_k - 2T_{k-1} \qquad (\text{since } k+1 \geq 3)$$

$$= 2\left(\sqrt{2}\right)^{k+2}\cos\tfrac{k\pi}{4} - 2\left(\sqrt{2}\right)^{(k-1)+2}\cos\tfrac{(k-1)\pi}{4}, \qquad \text{if } S(n) \text{ is true, } n \leq k$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{\sqrt{2}\cos\tfrac{k\pi}{4} - \cos(\tfrac{k\pi}{4} - \tfrac{\pi}{4})\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{2\tfrac{1}{\sqrt{2}}\cos\tfrac{k\pi}{4} - (\cos\tfrac{k\pi}{4}\cos\tfrac{\pi}{4} + \sin\tfrac{k\pi}{4}\sin\tfrac{\pi}{4})\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{2\tfrac{1}{\sqrt{2}}\cos\tfrac{k\pi}{4} - \tfrac{1}{\sqrt{2}}\cos\tfrac{k\pi}{4} - \tfrac{1}{\sqrt{2}}\sin\tfrac{k\pi}{4})\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{\tfrac{1}{\sqrt{2}}\cos\tfrac{k\pi}{4} - \tfrac{1}{\sqrt{2}}\sin\tfrac{k\pi}{4}\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{\cos\tfrac{k\pi}{4}\cos\tfrac{\pi}{4} - \sin\tfrac{k\pi}{4}\sin\tfrac{\pi}{4}\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\cos(\tfrac{k\pi}{4} + \tfrac{\pi}{4})$$

$$= \left(\sqrt{2}\right)^{(k+1)+2}\cos\tfrac{(k+1)\pi}{4}$$

$\therefore$ if $k \geq 2$ and $S(n)$ is true for $n \leq k$, then $S(k+1)$ is true. But $S(1)$ and $S(2)$ are true, and hence $S(3)$ is true, and then $S(4)$ is true, and so on. Hence by Mathematical induction, $S(n)$ is true for all positive integers $n$. $\therefore T_n = \left(\sqrt{2}\right)^{n+2}\cos\frac{n\pi}{4}$, $n = 1, 2, 3, \ldots$

**c. Outcomes assessed : PE3, E2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • writes $2\left\{a^2+b^2+c^2-(ab+bc+ca)\right\}$ as a sum of squares | 1 |
| • explains why each of these squares is non-negative | 1 |
| • explains why $a^2+b^2+c^2 = ab+bc+ca$ requires equality of $a$, $b$ and $c$ | 1 |

**Answer**

$$a^2 + b^2 - 2ab = (a-b)^2$$

$$b^2 + c^2 - 2bc = (b-c)^2$$

$$c^2 + a^2 - 2ca = (c-a)^2$$

$$\therefore 2\left\{a^2+b^2+c^2-(ab+bc+ca)\right\} = (a-b)^2 + (b-c)^2 + (c-a)^2$$

But $a$, $b$, $c$ are positive real numbers, as they are the lengths of triangle sides.

Hence $(a-b)$, $(b-c)$ and $(c-a)$ are also real numbers.

$\therefore (a-b)^2 \geq 0$ with equality if and only if $a = b$, and similarly for $(b-c)^2$, $(c-a)^2$.

Hence if $a^2+b^2+c^2 = ab+bc+ca$, then $(a-b)^2+(b-c)^2+(c-a)^2 = 0$

$$\therefore (a-b)^2 = (b-c)^2 = (c-a)^2 = 0$$

$$\therefore a = b, \quad b = c \quad \text{and} \quad c = a$$

Hence $a = b = c$ and $\Delta ABC$ is equilateral.